An inverse problem for eddy current equations

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Outline

Motivation

Electroencephalography (EEG) and magnetoencephalography (MEG) are two non-invasive techniques used to localize electric activity in the brain from measurements of external electromagnetic signals. EEG measures the scalp electric potential, while MEG measures the external magnetic flux.

Source localization is an inverse problem: knowing the value of the electric field or of the magnetic field on the surface of the head, the aim is to determine the current density that has given rise to that value.

Modelization

The frequency spectrum for electrophysiological signals in EEG and MEG is typically below 1000 Hz, most frequently between 0.1 and 100 Hz.

Hence, we can choose among different models:

- static: put the frequency equal to 0;
- eddy current: low frequency approximation (time variation of the electric field is disregarded, while time variation of the magnetic field is kept);
- full Maxwell: no term is dropped.

Maxwell equations

Let us give a closer look. We start considering the Maxwell equations

$$\begin{aligned} &\mathbf{curl}\,\mathcal{H} - \epsilon\frac{\partial\mathcal{E}}{\partial t} = \sigma\mathcal{E} + \mathcal{J}_e \quad \text{(Maxwell - Ampère)} \\ &\mathbf{curl}\,\mathcal{E} + \mu\frac{\partial\mathcal{H}}{\partial t} = \mathbf{0} \quad \text{(Faraday)} \end{aligned} \tag{1}$$

- $\mathcal{E}$, $\mathcal{H}$ are the electric and magnetic fields, respectively;
- $\mathcal{J}_e$ is the applied current density;
- ϵ is the electric permittivity, μ is the magnetic permeability and σ is the electric conductivity.

Time-harmonic Maxwell equations

For time-harmonic phenomena, we assume that

$$\begin{aligned} \mathcal{J}_e(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{J}_e(\mathbf{x})\exp(i\omega t)] \\ \mathcal{E}(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{E}(\mathbf{x})\exp(i\omega t)] \\ \mathcal{H}(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{H}(\mathbf{x})\exp(i\omega t)]\,, \end{aligned}$$

where $\omega \neq 0$ is the known angular frequency.
Substituting in (1) one has

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - i\omega\epsilon\mathbf{E} = \sigma\mathbf{E} + \mathbf{J}_e \\ \mathbf{curl}\,\mathbf{E} + i\omega\mu\mathbf{H} = \mathbf{0}\,, \end{cases}$$

namely, in terms of **E**,

$$\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{E}) - \omega^2\epsilon\mathbf{E} + i\omega\sigma\mathbf{E} = -i\omega\mathbf{J}_e\,. \quad (2)$$

Eddy current and static approximation

- The eddy current model is formally obtained by neglecting the displacement current term:

$$\begin{cases} \mathbf{curl}\, \mathbf{H} - i\omega\epsilon\mathbf{E} = \sigma\mathbf{E} + \mathbf{J}_e \\ \mathbf{curl}\, \mathbf{E} + i\omega\mu\mathbf{H} = \mathbf{0} \,. \end{cases}$$

- The static model stems form putting ω equal to 0:

$$\begin{cases} \mathbf{curl}\, \mathbf{H} - i\omega\epsilon\mathbf{E} = \sigma\mathbf{E} + \mathbf{J}_e \\ \mathbf{curl}\, \mathbf{E} + i\omega\mu\mathbf{H} = \mathbf{0} \,, \end{cases}$$

  namely, in terms of a scalar potential V such that $\mathbf{E} = -\mathbf{grad}\, V$,

$$\mathrm{div}(\sigma \mathbf{grad}\, V) = \mathrm{div} \mathbf{J}_e$$

  followed by

$$\begin{cases} \mathbf{curl}\, \mathbf{H} = -\sigma \mathbf{grad}\, V + \mathbf{J}_e \\ \mathrm{div}(\mu\mathbf{H}) = 0 \,. \end{cases}$$

Which model?

Looking back to (2), a thumb rule that drives the choice of the model could be formulated as follows: if L is a typical length (say, the diameter of the physical domain), it is possible to disregard the displacement current term provided that

$$\omega^2 \epsilon \, \mu \ll L^{-2} \quad , \quad |\omega| \, \epsilon \ll \sigma \, .$$

[Let us also recall that the wavelength can be expressed by

$$\lambda = \frac{1}{|\omega| \sqrt{\epsilon \, \mu}} \, .]$$

On the other hand, it seems reasonable to utilize the static model when, in addition,

$$|\omega| \, \mu \, \sigma \ll L^{-2} \, .$$

The parameters

For physiological problems, we have

$$\begin{aligned}
&\omega = 2\pi * 50 \ rad/s \\
&\mu = 4\pi * 10^{-7} \ H/m \\
&\sigma = 0.1 \ S/m \\
&L = 0.3 \ m \,,
\end{aligned}$$

while the electric permittivity can vary with the frequency, and a reasonable value can be

$$\epsilon \approx 10^{-6} \ F/m \,.$$

Therefore we have

$$\omega^2 \epsilon \, \mu \, L^2 \approx 10^{-8} \quad , \quad |\omega| \, \epsilon \, \sigma^{-1} \approx 3 * 10^{-3}$$

$$\lambda \approx 3 * 10^3 m \quad , \quad |\omega| \, \mu \, \sigma L^2 \approx 3.5 * 10^{-6} \,.$$

The choice of the model

- The first two values and the estimate of the wavelength say that it seems suitable to disregard the displacement current term, adopting the eddy current model.
- From the estimate of $|\omega|\,\mu\,\sigma L^2$ it seems also possible to utilize the static model. However, it is easy to construct source current densities $\mathbf{J}_e$ for which the electric field given by the static model is vanishing, while the electric field solution of the eddy current model is large (it is enough to take $\mathrm{div}\mathbf{J}_e = 0$).
- Hence the static model is not really satisfactory, and it is qualitatively different from the non-static ones. An accurate description of the problem has to be based on the eddy current model (or, possibly, for larger values of the frequency and of the electric permittivity, on the full Maxwell model).

"Gauge" and boundary conditions

Let us make complete the formulation of the eddy current problem.

Since in an insulator (where $\sigma = 0$) there are no charges, in that region one has to add

$$\mathrm{div}(\epsilon \mathbf{E}) = 0\,.$$

[A "gauge" condition necessary for having uniqueness.]

On the boundary $\partial\Omega$ we impose the magnetic boundary conditions:

$$\begin{aligned} \mathbf{H} \times \mathbf{n} = \mathbf{0} \quad &\text{on} \quad \partial\Omega \\ \epsilon \mathbf{E} \cdot \mathbf{n} = 0 \quad &\text{on} \quad \partial\Omega\,. \end{aligned}$$

[Alternatively, we can impose the electric boundary condition

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega\,.]$$

The geometry of the problem

- Let $\Omega_C \subset \mathbb{R}^3$ be a bounded open set with a Lipschitz and connected boundary Γ. [Ω_C is a conductor, the region where σ is positive; say, the human head.]
- Let $\Omega \subset \mathbb{R}^3$ be a bounded simply-connected domain with a Lipschitz boundary $\partial\Omega$, and completely containing Ω_C. [Ω is the physical domain; say, the room where the problem is studied.]
- Let us set $\Omega_I = \Omega \setminus \overline{\Omega_C}$, and assume that it is connected. [Ω_I is an insulator, where σ is vanishing; say, the air surrounding the head.]

[The assumptions that Ω is simply-connected and that Γ is connected can be dropped.]

Existence and uniqueness for the direct problem

We assume that:

- μ, ϵ and σ are symmetric and positive definite matrices, with entries in L^∞
- $\operatorname{supp} \mathbf{J}_e \subset \Omega_C$ and $\mathbf{J}_e \in L^2(\Omega_C)$.

It is known that under these conditions the eddy current problem

$$\begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{E}) + i\omega\sigma\mathbf{E} = -i\omega\mathbf{J}_e & \text{in } \Omega \\ \operatorname{div}(\epsilon\mathbf{E}) = 0 & \text{in } \Omega_I \\ (\mu^{-1}\mathbf{curl}\,\mathbf{E}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \epsilon\mathbf{E} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a unique solution $\mathbf{E}$ (and $\mathbf{H} = -(i\omega)^{-1}\mu^{-1}\mathbf{curl}\,\mathbf{E}$ in Ω).

Representation formula

Integration by parts in Ω_C easily yields ($\mathbf{n}$ pointing outward Ω_C)

$$-i\omega \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{z}} = \int_{\Omega_C} \mathbf{E} \cdot [i\omega\sigma\overline{\mathbf{z}} + \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}})] + \int_\Gamma [\mathbf{n} \times \mathbf{E} \cdot (\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) - i\omega \mathbf{n} \times \mathbf{H} \cdot \overline{\mathbf{z}}] \,.$$

Therefore, if $\mathbf{z} \in H(\mathbf{curl}\,;\Omega_C)$ satisfies

$$\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\sigma\mathbf{z} = \mathbf{0} \quad \text{in } \Omega_C \,,$$

the current density $\mathbf{J}_e$ satisfies the representation formula

$$-i\omega \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{z}} = \int_\Gamma \mathbf{n} \times \mathbf{E} \cdot (\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) - i\omega \int_\Gamma \mathbf{n} \times \mathbf{H} \cdot \overline{\mathbf{z}} \,. \qquad (4)$$

Space splitting

Let us define

$$\mathcal{W} = \{\mathbf{z} \in H(\mathbf{curl}\,;\Omega_C)\,|\,\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\sigma\mathbf{z} = \mathbf{0} \quad \text{in } \Omega_C\}$$

and W the closure of $\mathcal{W}$ in $(L^2(\Omega_C))^3$. (Note that $\mathcal{W}$ is not a trivial subspace.)

We have the orthogonal splitting

$$(L^2(\Omega_C))^3 = W \oplus W^\perp\,.$$

Let us give a more explicit description of the elements of $W^\perp$.

More on the subspace $W^\perp$

Lemma

Consider $\boldsymbol{\eta} \in C_0^\infty(\Omega_C)$ *and set* $\boldsymbol{\phi} = \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\boldsymbol{\eta}) + i\omega\sigma\boldsymbol{\eta}$*. Then* $\boldsymbol{\phi} \in W^\perp$ *(and* $W^\perp$ *is not a trivial subspace).*

Proof. Take $\mathbf{z} \in \mathcal{W}$. Then

$$\begin{aligned}\int_{\Omega_C} \boldsymbol{\phi} \cdot \overline{\mathbf{z}} &= \int_{\Omega_C} [\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\boldsymbol{\eta}) + i\omega\sigma\boldsymbol{\eta}] \cdot \overline{\mathbf{z}} \\ &= \int_{\Omega_C} \boldsymbol{\eta} \cdot [\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) + i\omega\sigma\overline{\mathbf{z}}] = 0\,.\end{aligned}$$

The result follows by a density argument. □

Unique solvability and non-radiating sources

Let us split the current density $\mathbf{J}_e$ as

$$\mathbf{J}_e = \mathbf{J}_e^\sharp + \mathbf{J}_e^\perp \quad , \quad \mathbf{J}_e^\sharp \in W \ , \ \mathbf{J}_e^\perp \in W^\perp \, .$$

We have

Theorem

(i) Let us assume that $\mathbf{J}_e = \mathbf{J}_e^\sharp \in W$ *and that* $\mathbf{E}^\sharp$ *is the corresponding solution of the eddy current problem. Then the knowledge of* $\mathbf{E}^\sharp \times \mathbf{n}$ *on* Γ *uniquely determines* $\mathbf{J}_e^\sharp$*.*
(ii) Let us assume that $\mathbf{J}_e = \mathbf{J}_e^\perp \in W^\perp$ *and that* $\mathbf{E}^\perp$ *is the corresponding solution of the eddy current problem. Then* $\mathbf{E}^\perp \times \mathbf{n} = \mathbf{0}$ *and* $\mathbf{H}^\perp \times \mathbf{n} = \mathbf{0}$ *on* Γ*, namely,* $\mathbf{J}_e^\perp$ *is a non-radiating source.*

Proof. [We assume for simplicity that $\partial\Omega$ is connected and that there are no handles on Γ.]
(i) If $\mathbf{E}^\sharp \times \mathbf{n}$ is known on Γ, we also know

$$\mathrm{div}_\tau(\mathbf{E}^\sharp \times \mathbf{n}) = \mathbf{curl}\, \mathbf{E}^\sharp \cdot \mathbf{n} = -i\omega\mu\mathbf{H}^\sharp \cdot \mathbf{n}\,,$$

hence $\mathbf{H}^\sharp$ is the solution of

$$\begin{cases} \mathbf{curl}\, \mathbf{H}^\sharp = \mathbf{0} & \text{in } \Omega_I \\ \mathrm{div}(\mu\mathbf{H}^\sharp) = 0 & \text{in } \Omega_I \\ \mu\mathbf{H}^\sharp \cdot \mathbf{n} = -(i\omega)^{-1}\mathrm{div}_\tau(\mathbf{E}^\sharp \times \mathbf{n}) & \text{on } \Gamma \\ \mathbf{H}^\sharp \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases}$$

As a consequence, if $\mathbf{E}^\sharp \times \mathbf{n} = \mathbf{0}$ on Γ it follows $\mathbf{H}^\sharp = \mathbf{0}$ in Ω_I and $\mathbf{H}^\sharp \times \mathbf{n} = \mathbf{0}$ on Γ.
Therefore from (4) we know that $\int_{\Omega_C} \mathbf{J}_e^\sharp \cdot \overline{\mathbf{z}} = 0$ for each $\mathbf{z} \in \mathcal{W}$, hence, by a density argument, for each $\mathbf{z} \in W$. Since $\mathbf{J}_e^\sharp \in W$, the thesis follows.

(ii) Since $\mathbf{J}_e^\perp \in W^\perp$, taking $\mathbf{z} \in W$ from (4) we have that

$$\int_\Gamma \mathbf{n} \times \mathbf{E}^\perp \cdot (\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) - i\omega \int_\Gamma \mathbf{n} \times \mathbf{H}^\perp \cdot \overline{\mathbf{z}} = 0\,. \qquad (5)$$

For each $\boldsymbol{\eta} \in H^{-1/2}_{\mathrm{div},\tau}(\Gamma)$ we denote by $\mathbf{Z} \in H(\mathbf{curl}\,;\Omega)$ the solution to

$$\begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{Z}) - i\omega\sigma\mathbf{Z} = \mathbf{0} & \text{in } \Omega_C \cup \Omega_I \\ \mathrm{div}(\epsilon\mathbf{Z}) = 0 & \text{in } \Omega_I \\ (\mu^{-1}\mathbf{curl}\,\mathbf{Z}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \epsilon\mathbf{Z} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ (\mu^{-1}\mathbf{curl}\,\mathbf{Z})_{|\Omega_C} \times \mathbf{n} = (\mu^{-1}\mathbf{curl}\,\mathbf{Z})_{|\Omega_I} \times \mathbf{n} + \boldsymbol{\eta} & \text{on } \Gamma\,. \end{cases}$$

We can select $\mathbf{Z}_{|\Omega_C} \in W$ as a test function in (5) and obtain

$$\begin{aligned} &\int_\Gamma \mathbf{n} \times \mathbf{E}^\perp \cdot \mu^{-1} \mathbf{curl}\, \overline{\mathbf{Z}_{|\Omega_C}} \\ &\quad = \int_\Gamma \mathbf{E}^\perp \cdot \overline{\boldsymbol{\eta}} - \int_\Gamma \mathbf{E}^\perp \cdot (\mathbf{n} \times \mu^{-1} \mathbf{curl}\, \overline{\mathbf{Z}_{|\Omega_I}}) \\ &\quad = \int_\Gamma \mathbf{E}^\perp \cdot \overline{\boldsymbol{\eta}} - \int_{\Omega_I} \mu^{-1} \mathbf{curl}\, \mathbf{E}^\perp \cdot \mathbf{curl}\, \overline{\mathbf{Z}_{|\Omega_I}} \end{aligned}$$

$$\begin{aligned} &-i\omega \int_\Gamma \mathbf{n} \times \mathbf{H}^\perp \cdot \overline{\mathbf{Z}_{|\Omega_C}} \\ &\quad = - \int_\Gamma \mu^{-1} \mathbf{curl}\, \mathbf{E}^\perp \cdot \mathbf{n} \times \overline{\mathbf{Z}_{|\Omega_I}} \\ &\quad = \int_{\Omega_I} \mu^{-1} \mathbf{curl}\, \mathbf{E}^\perp \cdot \mathbf{curl}\, \overline{\mathbf{Z}_{|\Omega_I}} \,. \end{aligned}$$

In conclusion, we have obtained

$$\int_\Gamma \mathbf{E}^\perp \cdot \overline{\boldsymbol{\eta}} = 0$$

for each $\boldsymbol{\eta} \in H^{-1/2}_{\mathrm{div},\tau}(\Gamma)$, hence $\mathbf{n} \times \mathbf{E}^\perp \times \mathbf{n} = \mathbf{0}$ on Γ.
Proceeding as in the proof of (i) we show that $\mathbf{E}^\perp \times \mathbf{n} = \mathbf{0}$ on Γ implies $\mathbf{H}^\perp \times \mathbf{n} = \mathbf{0}$ on Γ, and the proof is complete. $\square$

The direct problem for a surface current

We consider a surface current $\mathbf{J}_* \in H^{-1/2}_{\mathrm{div},\tau}(\partial B)$, where B is an open connected set with Lipschitz boundary ∂B and satisfying $\overline{B} \subset \Omega_C$. The direct problem reads

$$\begin{cases} \mathbf{curl}\,\mathbf{E} + i\omega\mu\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E} & \text{in } B \cup (\Omega \setminus \overline{B}) \\ \mathrm{div}(\epsilon\mathbf{E}) = 0 & \text{in } \Omega_I \\ \mathbf{H} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \epsilon\mathbf{E} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \mathbf{H}_{|B} \times \mathbf{n}_B - \mathbf{H}_{|\Omega\setminus\overline{B}} \times \mathbf{n}_B = \mathbf{J}_* & \text{on } \partial B\,, \end{cases} \tag{6}$$

where $\mathbf{n}_B$ is the unit normal vector on ∂B, pointing outward B.
It is easy to see that, for each given $\mathbf{J}_* \in H^{-1/2}_{\mathrm{div},\tau}(\partial B)$, this problem has a unique solution.

Representation formula and unique solvability

Theorem

Assume that the coefficients μ and σ are piecewise C^1-functions, and that the discontinuity surfaces are Lipschitz surfaces. Let $\mathbf{E}_*$ be the solution of the eddy current problem driven by the surface current $\mathbf{J}_* \in H^{-1/2}_{\mathrm{div},\tau}(\partial B)$. The knowledge of $\mathbf{E}_* \times \mathbf{n}$ on Γ uniquely determines $\mathbf{J}_*$.

Proof. As in the preceding case, by solving the problem in Ω_I we easily show that $\mathbf{E}_* \times \mathbf{n} = \mathbf{0}$ on Γ also gives $\mathbf{E}_* = \mathbf{0}$ in Ω_I, $\mathbf{H}_* = \mathbf{0}$ in Ω_I and in particular $\mathbf{H}_* \times \mathbf{n} = \mathbf{0}$ on Γ.
As a consequence of the unique continuation principle we have $\mathbf{E}_* = \mathbf{0}$ and $\mathbf{H}_* = \mathbf{0}$ in $\Omega \setminus \overline{B}$ (the assumptions on the coefficients μ and σ play a role here).

For each $\mathbf{z} \in H(\mathbf{curl}\,; B)$ with $\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{z}) \in (L^2(B))^3$ we have

$$\begin{aligned} \int_B \sigma \mathbf{E}_* \cdot \overline{\mathbf{z}} &= \int_B \mathbf{curl}\,\mathbf{H}_* \cdot \overline{\mathbf{z}} \\ &= \int_{\partial B} \mathbf{n}_B \times \mathbf{H}_{*|B} \cdot \overline{\mathbf{z}} \\ &\quad -(i\omega)^{-1} \int_{\partial B} \mathbf{n}_B \times \mathbf{E}_* \cdot (\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) \\ &\quad -(i\omega)^{-1} \int_B \mathbf{E}_* \cdot \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}})\,. \end{aligned}$$

Taking into account that $\mathbf{H}_{*|B} \times \mathbf{n}_B - \mathbf{H}_{*|\Omega\setminus\overline{B}} \times \mathbf{n}_B = \mathbf{J}_*$ on ∂B, we obtain the representation formula

$$\begin{aligned} -i\omega \int_{\partial B} \mathbf{J}_* \cdot \overline{\mathbf{z}} &= \int_{\partial B} \mathbf{n}_B \times \mathbf{E}_* \cdot (\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) \\ &\quad -i\omega \int_{\partial B} \mathbf{n}_B \times \mathbf{H}_{*|\Omega\setminus\overline{B}} \cdot \overline{\mathbf{z}} \end{aligned} \qquad (7)$$

for each $\mathbf{z} \in H(\mathbf{curl}\,; B)$ such that $\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\sigma\mathbf{z} = \mathbf{0}$ in B. Since we know that $\mathbf{E}_* = \mathbf{0}$ and $\mathbf{H}_* = \mathbf{0}$ in $\Omega \setminus \overline{B}$, it follows from (7) that $\int_{\partial B} \mathbf{J}_* \cdot \overline{\mathbf{z}} = 0$.

For each $\boldsymbol{\rho} \in H^{-1/2}_{\mathrm{curl},\tau}(\Gamma)$ we can choose $\mathbf{z} \in H(\mathbf{curl}\,; B)$, the solution to

$$\begin{cases} \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\sigma\mathbf{z} = \mathbf{0} & \text{in } B \\ \mathbf{z} \times \mathbf{n}_B = \boldsymbol{\rho} \times \mathbf{n}_B & \text{on } \partial B\,. \end{cases}$$

Hence $\int_{\partial B} \mathbf{J}_* \cdot \overline{\boldsymbol{\rho}} = 0$ for each $\boldsymbol{\rho} \in H^{-1/2}_{\mathrm{curl},\tau}(\Gamma)$, and this space is the dual space of $H^{-1/2}_{\mathrm{div},\tau}(\Gamma)$. This ends the proof. □

The direct problem for a dipole source

Suppose that the source is a finite sum of dipoles, in different positions and with non-vanishing polarizations, namely,

$$\mathbf{J}_\dagger(\mathbf{x}) = \sum_{k=1}^{M} \mathbf{p}_k \delta(\mathbf{x} - \mathbf{x}_k)\,, \qquad (8)$$

where $\mathbf{x}_k \in \Omega_C$, $\mathbf{x}_k \neq \mathbf{x}_j$ for $k \neq j$, $\mathbf{p}_k \neq \mathbf{0}$, and δ is the Dirac delta distribution.

Let us assume, for the ease of exposition, that the direct problem (3) has a unique (distributional) solution $\mathbf{E}_\dagger$ for this source $\mathbf{J}_\dagger$.

[The proof of existence and uniqueness, performed by means of the so-called "subtraction" method, contains some technical points... In particular, for obtaining this result we assume that μ and σ are smooth enough.]

Representation formula and unique solvability

Theorem

Assume that μ and σ are smooth enough. Let $\mathbf{E}_\dagger$ be the solution of the eddy current problem (3) driven by the surface current $\mathbf{J}_\dagger$ introduced in (8). The knowledge of $\mathbf{E}_\dagger \times \mathbf{n}$ on Γ uniquely determines $\mathbf{J}_\dagger$, namely, the number, the position and the polarization of the dipoles.

Proof. We start proving that the number and the position of the dipoles are uniquely determined.
By contradiction, let us denote by Q_1 and Q_2 two different sets of points where the dipoles are located, and by $\mathbf{E}_{\dagger,1}$, $\mathbf{H}_{\dagger,1}$ and $\mathbf{E}_{\dagger,2}$, $\mathbf{H}_{\dagger,2}$ the corresponding solutions, with the same value $\mathbf{E}_\dagger \times \mathbf{n}$ on Γ.
As in the preceding cases, by solving the problem in Ω_I with datum $\mathbf{E}_\dagger \times \mathbf{n}$ on Γ we obtain that $\mathbf{E}_{\dagger,1} = \mathbf{E}_{\dagger,2}$ and $\mathbf{H}_{\dagger,1} = \mathbf{H}_{\dagger,2}$ in Ω_I.

By the unique continuation principle it follows $\mathbf{E}_{\dagger,1} = \mathbf{E}_{\dagger,2}$ in $\Omega \setminus (Q_1 \cup Q_2)$ [the smoothness of the coefficients μ and σ plays a role also here]. Let $\mathbf{x}_*$ a point belonging, say, to Q_1 but not to Q_2. We have that $\mathbf{E}_{\dagger,2}$ is bounded in a neighborhood of $\mathbf{x}_*$, while $\mathbf{E}_{\dagger,1}$ is unbounded there, a contradiction since $\mathbf{E}_{\dagger,1}$ and $\mathbf{E}_{\dagger,2}$ coincide around $\mathbf{x}_*$. Therefore $Q_1 = Q_2$.
Let us prove now that the polarizations are uniquely determined. Since the problem is linear, we can assume that $\mathbf{E}_\dagger = \mathbf{0}$ in $\Omega \setminus Q_1$. Therefore, in the sense of distributions in Ω we have $\mathbf{E}_\dagger = \mathbf{0}$ and $\mathbf{curl}\,\mathbf{H}_\dagger = \mathbf{0}$, and in particular the equation

$$\sum_{k=1}^{M} \mathbf{p}_k \delta(\mathbf{x} - \mathbf{x}_k) = \mathbf{0}\,.$$

By choosing test functions in $C^\infty(\Omega)$ supported around each point $\mathbf{x}_j$ we obtain $\mathbf{p}_j = \mathbf{0}$ for each $j = 1, \dots, M$. $\square$

Explicit determination of the dipole source

For the sake of simplicity, consider a source given by only one dipole.

Assume that μ and σ are constants. Proceeding as in the proof of (4), one obtains the representation formula

$$-i\omega \mathbf{p}_1 \cdot \overline{\mathbf{z}}(\mathbf{x}_1) = \int_\Gamma \mathbf{n} \times \mathbf{E}_\dagger \cdot (\mu^{-1} \mathbf{curl}\, \overline{\mathbf{z}}) - i\omega \int_\Gamma \mathbf{n} \times \mathbf{H}_\dagger \cdot \overline{\mathbf{z}}\,, \quad (9)$$

for each $\mathbf{z} \in H(\mathbf{curl}\,; \Omega_C)$ satisfying

$$\mathbf{curl}\,(\mu^{-1} \mathbf{curl}\, \mathbf{z}) - i\omega\sigma \mathbf{z} = \mathbf{0} \quad \text{in } \Omega_C\,. \quad (10)$$

To determine the source, we have to find the polarization $\mathbf{p}_1$ and the position $\mathbf{x}_1$: therefore, six parameters. The natural idea is to choose in a suitable way some (at least six...) functions $\mathbf{z}$ in (9), and solve the corresponding nonlinear system.
The usual choice is to take $\mathbf{z}(\mathbf{x}) = \mathbf{b}e^{i\kappa\mathbf{d}\cdot\mathbf{x}}$, with $\kappa \in \mathbb{C}$, $\mathbf{b} \in \mathbb{R}^3$, $\mathbf{d} \in \mathbb{R}^3$. It is not restrictive to assume $|\mathbf{d}| = |\mathbf{b}| = 1$; in order that $\mathbf{z}$ is a solution to (10) we need

$$\kappa^2 = i\omega\mu\sigma \quad , \quad \mathbf{b} \cdot \mathbf{d} = 0 \, .$$

It can be shown that $\mathbf{p}_1$ and $\mathbf{x}_1$ are uniquely determined by solving the nonlinear system (9) obtained by suitable selections of $\mathbf{b}$ and $\mathbf{d}$.

Related results

Before finishing, let us make a few comments on some related results.

- Bleistein and Cohen (1977) have shown the existence of non-radiating sources for the Maxwell equations with constant coefficients.
- He and Romanov (1998) has solved the inverse problem for the (vector) Helmholtz equation with a dipole source.
- Ammari, Bao and Fleming (2002) has solved the inverse problem for the Maxwell equations with a dipole source.
- Albanese and Monk (2006) has solved the inverse problem for the Maxwell equations with a distributed source, a surface current and a superposition of dipole sources.

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