PDEs with a constraint on the curl or on the divergence

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Outline

Introduction

Some examples

Let us give some examples of problems where either a divergence constraint or a curl constraint appears.

- Darcy problem

$$\begin{cases} \mathcal{K}\,\mathbf{u} - \mathbf{grad}\, p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ p_{|\partial\Omega} = \varphi & \text{on } \partial\Omega\,. \end{cases}$$

- elliptic problem in mixed formulation

$$\begin{cases} \mathbf{w} - \mathcal{A}\,\mathbf{grad}\, q = \mathbf{0} & \text{in } \Omega \\ \operatorname{div} \mathbf{w} = g & \text{in } \Omega \\ q_{|\partial\Omega} = \eta & \text{on } \partial\Omega\,. \end{cases}$$

Some examples (cont'd)

- eddy current problem (with $\partial\Omega$ and $\partial\Omega_C$ connected)

$$\begin{cases} \mathbf{curl}\,\mathbf{E} + i\omega\mu\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_{e,C} & \text{in } \Omega_C \\ \mathbf{curl}\,\mathbf{H} = \mathbf{J}_{e,I} & \text{in } \Omega_I \\ [\mathbf{H} \times \mathbf{n}]_{|\partial\Omega_C \cap \partial\Omega_I} = \mathbf{0} & \text{on } \partial\Omega_C \cap \partial\Omega_I \\ \mathrm{div}\,(\epsilon\mathbf{E}) = 0 & \text{in } \Omega_I \\ \mathbf{E} \times \mathbf{n}_{|\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega \\ \int_{\partial\Omega} \epsilon\mathbf{E} \cdot \mathbf{n} = 0\,. & \end{cases}$$

- curl–div system (with $\partial\Omega$ connected)

$$\begin{cases} \mathbf{curl}\,\mathbf{u} = \mathbf{B} & \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} = G & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega\,. \end{cases}$$

Some examples (cont'd)

- Stokes system

$$
\begin{cases}
-\nu\Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f} & \text{in } \Omega \\
\operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\
\mathbf{u}_{|\partial\Omega} = \mathbf{0} & \text{on } \partial\Omega \\
\int_\Omega p = 0\,.
\end{cases}
$$

- spectral Maxwell problem (with $\partial\Omega$ connected)

$$
\begin{cases}
\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \lambda \mathbf{E} & \text{in } \Omega \\
\operatorname{div} \mathbf{E} = 0 & \text{in } \Omega \\
\mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,.
\end{cases}
$$

Some examples (cont'd)

The standard variational formulation stemming from these problems has therefore a saddle-point form:

$$\begin{cases} a(u,v) + b(v,p) = \mathcal{F}(v) & \forall\, v \in V \\ b(u,q) = \mathcal{G}(q) & \forall\, q \in Q\,, \end{cases}$$

This (very often symmetric) problem is typically indefinite, and thus needs a more complex numerical solver.

A question

A natural question is:

- can we directly formulate the problem in the constrained subspace?

This means: can we work in the space of divergence-free or curl-free vector fields?

The most common answer is:

- at the theoretical level, yes
- at the numerical level, yes for curl-free finite elements, no (or at least not easily) for divergence-free finite elements [and, anyway, the shape of the domain can require to face some complicated topological problems...].

Back to the examples

Let us show how the strategy of working in the constrained subspace could work. Let us give an example for the divergence constraint.

Consider the elliptic problem in mixed formulation:

$$\begin{cases} \mathbf{w} - \mathcal{A}\,\mathbf{grad}\,q = \mathbf{0} & \text{in } \Omega \\ \operatorname{div}\mathbf{w} = g & \text{in } \Omega \\ q_{|\partial\Omega} = \eta & \text{on } \partial\Omega\,. \end{cases}$$

If the test function $\mathbf{v}$ is divergence-free, we have

$$\begin{aligned} 0 &= \textstyle\int_\Omega \mathcal{A}^{-1}\mathbf{w}\cdot\mathbf{v} - \int_\Omega \mathbf{grad}\,q\cdot\mathbf{v} \\ &= \textstyle\int_\Omega \mathcal{A}^{-1}\mathbf{w}\cdot\mathbf{v} + \int_\Omega q\operatorname{div}\mathbf{v} + \int_{\partial\Omega} q\,\mathbf{n}\cdot\mathbf{v} \\ &= \textstyle\int_\Omega \mathcal{A}^{-1}\mathbf{w}\cdot\mathbf{v} + \int_{\partial\Omega}\eta\,\mathbf{n}\cdot\mathbf{v}\,. \end{aligned}$$

Back to the examples (cont'd)

Since $\mathbf{w}$ is not divergence-free, we need a first step in which we determine a vector field $\mathbf{w}^\star$ such that

$$\operatorname{div} \mathbf{w}^\star = g \quad \text{in } \Omega \,,$$

thus $\mathbf{W} = \mathbf{w} - \mathbf{w}^\star$ satisfies

$$\begin{cases} \mathcal{A}^{-1}\mathbf{W} - \mathbf{grad}\, q = -\mathcal{A}^{-1}\mathbf{w}^\star & \text{in } \Omega \\ \operatorname{div} \mathbf{W} = 0 & \text{in } \Omega \\ q_{|\partial\Omega} = \eta & \text{on } \partial\Omega \,, \end{cases}$$

and the final variational problem, in the subspace of divergence-free vector fields, reads:

$$\int_\Omega \mathcal{A}^{-1}\mathbf{W} \cdot \mathbf{v} = -\int_\Omega \mathcal{A}^{-1}\mathbf{w}^\star \cdot \mathbf{v} - \int_{\partial\Omega} \eta\, \mathbf{n} \cdot \mathbf{v} \,.$$

Aim

In conclusion, a two-step strategy:

- determine a potential (for the curl operator, or for the divergence operator)
- solve the variational problem in the curl-constrained or in the divergence-constrained subspace.

Aim of this talk is that we can do these two steps not only at the theoretical level, but also for finite element approximations.

Finite element potentials

First results

Determining the necessary and sufficient conditions for assuring that a function defined in a bounded domain $\Omega \subset \mathbb{R}^3$ is the gradient of a scalar potential, or the curl of a vector potential, or the divergence of a vector field is one of the most classical problem of vector analysis.

The answer is well-known, and shows an interesting interplay of differential calculus and topology (see, e.g., Cantarella et al. (2002)).

First results (cont'd)

- a vector field is the gradient of a scalar potential if and only if it is curl free and its line integral is vanishing on all the closed curves that furnish a basis of the first homology group of $\overline{\Omega}$ [essentially, all the closed curves that are not the boundary of an orientable surface contained in $\overline{\Omega}$];
- a vector field is the curl of a vector potential if and only if it is divergence free and its flux is vanishing across all the closed surfaces that furnish a basis of the second homology group of $\overline{\Omega}$ [essentially, all the closed surfaces that are not the boundary of a volume contained in $\overline{\Omega}$; equivalently, all the internal connected components of $\partial\Omega$];
- each scalar function is the divergence of a vector field.

First results (cont'd)

However, this theoretical result only clarifies when the answer is positive, and does not say how to determine an explicit and efficient procedure for constructing finite element potentials.

Our approach is based on (simple) tools from algebraic topology and graph theory. We suppose to have:

- a basis σ_n, $n = 1, \ldots, g$, of the first homology group of $\overline{\Omega}$;
- a basis $\widehat{\sigma}_n$, $n = 1, \ldots, g$, of the first homology group of $\mathbb{R}^3 \setminus \Omega$;
- a spanning tree $\mathcal{S}_h$ of the graph given by the nodes and the edges of the mesh $\mathcal{T}_h$.

[Note: an easy way for constructing σ_n and $\widehat{\sigma}_n$ is presented in Hiptmair and Ostrowski (2002); the determination of a spanning tree is a standard procedure in graph theory.]

First results (cont'd)

Let us also introduce the finite element spaces we will use:

- the space L_h of continuous piecewise-linear elements, with dimension n_v, the number of vertices in $\mathcal{T}_h$;
- the space N_h of Nédélec edge elements of degree 1, with dimension n_e, the number of edges in $\mathcal{T}_h$;
- the space RT_h of Raviart-Thomas elements of degree 1, with dimension n_f, the number of faces in $\mathcal{T}_h$;
- the space PC_h of (discontinuous) piecewise-constant elements, with dimension n_t, the number of tetrahedra in $\mathcal{T}_h$.

The grad problem

We want to solve $\mathbf{grad}\,\psi_h = \mathbf{H}_h$ in the finite element context. [This is an easy problem, and the only reason for considering it is that it is useful for understanding better the procedures needed for the other two problems.]

The "right" finite elements are: $\psi_h \in L_h$ a piecewise-linear nodal element, $\mathbf{H}_h \in N_h$ a lowest order Nédélec edge element, and we only have to impose that the line integral of $\mathbf{grad}\,\psi_h$ and $\mathbf{H}_h$ on each edge of the mesh $\mathcal{T}_h$ is the same.

The fundamental theorem of calculus says that

$$\psi_h(v_b) - \psi_h(v_a) = \int_e \mathbf{grad}\,\psi_h \cdot \boldsymbol{\tau} = \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} \qquad (1)$$

for an edge $e = [v_a, v_b]$. Hence the linear system associated to $\mathbf{grad}\,\psi_h = \mathbf{H}_h$ has exactly two non-zero values per row.

The grad problem (cont'd)

Starting from a root v_* of the spanning tree $\mathcal{S}_h$, where, for the sake of uniqueness, we impose $\psi_h(v_*) = 0$, for an edge $e' = [v_*, \widehat{v}] \in \mathcal{S}_h$ we compute

$$\psi_h(\widehat{v}) = \psi_h(v_*) + \int_{e'} \mathbf{H}_h \cdot \boldsymbol{\tau} \, ;$$

since $\mathcal{S}_h$ is a spanning tree, going on in this way we can visit all the nodes of $\mathcal{T}_h$.

The spanning tree is therefore a tool for selecting the rows for which, using the additional equation $\psi_h(v_*) = 0$, one can eliminate the unknowns one after the other.

We have thus found a nodal element ψ_h such that its gradient has line integral on all the edges of the spanning tree equal to that of $\mathbf{H}_h$. Then is easy to show that the same is true for all the other edges.

The grad problem (cont'd)

In other words, we have given a constructive way for solving the problem

$$\begin{cases} \mathbf{grad}\, \psi_h = \mathbf{H}_h & \text{in } \Omega \\ \psi_h(v_*) = 0\,. \end{cases} \tag{2}$$

Since it can be easily proved that $n_e > n_v - 1 + g$ (the n_e edges of the graph are more than the $n_v - 1$ edges in the spanning tree plus g edges, one for each homological cycle), this is a full rank overdetermined system with $n_e + 1$ equations and n_v unknowns.

Problems with a similar structure will appear in the sequel.

The curl problem

We want to solve $\mathbf{curl}\,\mathbf{A}_h = \mathbf{B}_h$ in the finite element context.

The "right" finite elements are: $\mathbf{A}_h \in N_h$ a lowest order Nédélec edge element, $\mathbf{B}_h \in RT_h$ a lowest order Raviart–Thomas face element, and we only have to impose that the flux of $\mathbf{curl}\,\mathbf{A}_h$ and $\mathbf{B}_h$ on each face of the mesh $\mathcal{T}_h$ is the same.

The Stokes theorem assures that

$$\int_{e_1} \mathbf{A}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{A}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{A}_h \cdot \boldsymbol{\tau} = \int_f \mathbf{curl}\,\mathbf{A}_h \cdot \boldsymbol{\nu}_f = \int_f \mathbf{B}_h \cdot \boldsymbol{\nu}_f \, , \quad (3)$$

where $\partial f = e_1 \cup e_2 \cup e_3$, hence the linear system associated to $\mathbf{curl}\,\mathbf{A}_h = \mathbf{B}_h$ has exactly three non-zero values for each row.

The curl problem (cont'd)

With respect to the preceding case:

- three unknowns per row instead of two.

Therefore, in order to devise an efficient elimination algorithm, it is useful to fix the value of other unknowns.

The best situation should occur when the number of the new equations agrees with the dimension of the kernel of the curl operator.

Since this kernel is given by the gradients of nodal elements plus the space generated by the basis of the first de Rham cohomology group of Ω, we see that its dimension is equal to $n_v - 1 + g$.

The curl problem (cont'd)

Having this in mind, and recalling that the number of edges e' in $\mathcal{S}_h$ is $n_v - 1$, we are led to the problem

$$\begin{cases} \mathbf{curl}\, \mathbf{A}_h = \mathbf{B}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{A}_h \cdot d\mathbf{s} = \rho_n & \forall\, n = 1, \dots, g \\ \int_{e'} \mathbf{A}_h \cdot \boldsymbol{\tau} = 0 & \forall\, e' \in \mathcal{S}_h \,, \end{cases} \tag{4}$$

for arbitrarily given constants ρ_n.

Equation $(4)_3$ can be seen as a "filter" for gradients; moreover, since homology and cohomology are in duality, equation $(4)_2$ can be seen as a "filter" for cohomology fields.

This is a full rank overdetermined system, with $n_f + g + n_v - 1$ equations and n_e unknowns [recall that the Euler–Poincaré formula says that $n_f + g + n_v - 1 = n_e + n_t + p$]. It is not difficult to prove that it has a unique solution.

Webb–Forghani algorithm

Webb and Forghani (1989) proposed this solution algorithm:

Algorithm

1. *take a face f for which at least one edge unknown has not yet been assigned*
    1. *if exactly one edge unknown is not determined, compute its value from the Stokes relation (3)*
    2. *if two or three edge unknowns are not determined, pass to another face.*

This is a simple elimination procedure for solving the linear system at hand, and it is quite efficient, as the computational cost is linearly dependent on the number of unknowns.

The weak point is that:

- it can stop without having determined all the edge unknowns (even in simple topological situations!)

An explicit formula for the vector potential

- Cure: devise an explicit formula for the solution to (4).

(We are able to do that if $\mathbf{B}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$, a quite natural condition in the most interesting physical situations, and for a suitable choice of the constants ρ_n.)

The explicit formula permits to restart the algorithm in case it stops (but it is better not using it for all the degrees of freedom, as it would be more expensive than the Webb–Forghani algorithm).

The div problem

We want to solve $\operatorname{div} \mathbf{v}_h = G_h$ in the finite element context.

The "right" finite elements are: $\mathbf{v}_h \in RT_h$ a lowest order Raviart–Thomas face element, $G_h \in PC_h$ a piecewise-constant element, and we have only to impose that the integral of $\operatorname{div} \mathbf{v}_h$ and of G_h on each element of the mesh $\mathcal{T}_h$ is the same.

The Gauss theorem says that

$$\begin{aligned} &\int_{f_1} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_2} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_3} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_4} \mathbf{v}_h \cdot \boldsymbol{\nu}_f \\ &= \int_K \operatorname{div} \mathbf{v}_h = \int_K G_h \,, \end{aligned} \tag{5}$$

where $\partial K = f_1 \cup f_2 \cup f_3 \cup f_4$, hence the linear system associated to $\operatorname{div} \mathbf{v}_h = G_h$ has exactly four unknowns per row.

The div problem (cont'd)

For having well-posedness of the system, we want to add equations by fixing the value of some unknowns. Similarly to what done before we start by analyzing the dimension of the kernel of the divergence operator.

This kernel is given by the curl of the Nédélec elements plus the space generated by the basis of the second de Rham cohomology group of Ω.

If we denote by $(\partial\Omega)_0, \dots, (\partial\Omega)_p$ the connected components of $\partial\Omega$, we know that the dimension of the second de Rham cohomology group of Ω is equal to p.

The div problem (cont'd)

On the other hand, it is easy to check that the dimension of the space of the curl of the Nédélec elements is equal to the number of the edges minus the dimension of the kernel of the curl operator: hence, it is equal to $n_e - n_v + 1 - g$.

By the Euler–Poincaré formula we have

$$n_v - n_e + n_f - n_t = 1 - g + p \,,$$

hence the dimension of the space of the curl of the Nédélec elements can be rewritten as $n_f - n_t - p$.

In conclusion, besides the topological conditions

$$\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = c_r \quad , \quad r = 1, \ldots, p \,,$$

that are a filter for the cohomology fields, we could add $n_f - n_t - p$ equations.

A dual graph

To do that, let us note that an internal face connects two tetrahedra, while a boundary face connects a tetrahedron and a connected component of $\partial\Omega$.

We can therefore consider the following (connected) dual graph $\mathcal{G}_h$: the dual vertices are $W = T \cup \Sigma$, where the elements of T are the tetrahedra of the mesh and the elements of Σ are the $p+1$ connected components of $\partial\Omega$; the set of dual arcs is F, the set of the faces of the mesh.

The number of dual vertices is equal to $n_t + p + 1$, hence a spanning tree $\mathcal{M}_h$ of $\mathcal{G}_h$ has $n_t + p$ dual arcs (and consequently its cotree has $n_f - n_t - p$ dual arcs).

A dual graph (cont'd)

Therefore the linear system

$$\begin{cases} \operatorname{div} \mathbf{v}_h = G_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = c_r & \forall\, r = 1, \dots, p \\ \int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0 & \forall\, f \notin \mathcal{M}_h \end{cases} \tag{6}$$

is a square linear system of n_f equations and unknowns.

It can be shown that this system has a unique solution, and that the solution can be determined by means of an efficient constructive procedure (essentially similar to the one used for the gradient problem, but starting from the leaves of the dual graph instead that from the root).

A summary

	from	to	unknowns	equations
grad	L_h	N_h	n_v	$n_e + 1\ (> n_v)$
curl	N_h	RT_h	n_e	$n_f + g + n_v - 1 = n_e + n_t + p$
div	RT_h	PC_h	n_f	$n_t + p + n_e - n_v + 1 - g = n_f$

Table: Finite element potentials.

Curl-free or divergence-free finite elements

Curl-free finite elements

The problem of describing in a suitable way curl-free finite elements is quite easy. In fact, it is straightforward to find a basis of the finite element space

$$\mathcal{V}_{0,h} = \{\mathbf{v}_h \in N_h \,|\, \mathbf{curl}\, \mathbf{v}_h = \mathbf{0} \text{ in } \Omega, \\ \oint_{\sigma_n} \mathbf{v}_h \cdot d\mathbf{s} = 0 \ \forall\ n = 1, \ldots, g\}\,, \tag{7}$$

as this space is coincident with $\mathbf{grad}\, L_h$ (indeed, the conditions $\oint_{\sigma_n} \mathbf{v}_h \cdot d\mathbf{s} = 0$ filter all the curl-free vector fields that are not gradients, namely, the fields belonging to the first de Rham cohomology group).

Thus we have only to identify and eliminate the kernel of the gradient operator: the constants. In conclusion, a basis for $\mathcal{V}_{0,h}$ is simply given by $\mathbf{grad}\, \Phi^i_h$, $i = 1, \ldots, n_v - 1$, where Φ^i_h, $i = 1, \ldots, n_v$, are the standard nodal basis functions of L_h.

Divergence-free finite elements

A more complicated situation arises for divergence-free finite elements. In fact, we start considering the space

$$\mathcal{W}_{0,h} = \{\mathbf{v}_h \in RT_h \,|\, \mathrm{div}\,\mathbf{v}_h = 0 \text{ in } \Omega, \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = 0 \ \forall\ r = 1,\dots,p\}\,, \tag{8}$$

and it is easy to check that $\mathcal{W}_{0,h} = \mathbf{curl}\, N_h$ (the conditions $\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = 0$ filter all the divergence-free vector fields that are not curls, namely, the fields belonging to the second de Rham cohomology group).

However, the problem is that

- the kernel of the curl operator is large: it contains the gradients of elements in L_h and the fields belonging to the first de Rham cohomology group, and has dimension equal to $n_v - 1 + g$.

Divergence-free finite elements (cont'd)

Thus we need:

- to devise a strategy for selecting $n_e - n_v + 1 - g$ edges in order that the associated edge element basis functions have linearly independent curls.

Results in this direction were obtained by Hecht (1981), Dubois (1990) and Scheichl (2002) for a simply-connected domain, and by Rapetti et al. (2003) for a κ-fold torus.

Here we present a general procedure for the determination of a set of locally-supported basis functions of $\mathcal{W}_{0,h}$, together with an easy proof of its effectiveness.

Divergence-free finite elements (cont'd)

Let us assume for a while that Ω is simply-connected (therefore we have $g = 0$). Consider all the edges not belonging to the spanning tree $\mathcal{S}_h$, namely, belonging to the cotree $\mathcal{C}_h$; their number is $n_e - n_v + 1$. The result is:

- A basis of $\mathcal{W}_{0,h}$ is given by $\mathbf{curl}\,\mathbf{w}_h^j$, for the indices j such that the corresponding edges e_j belong to the cotree $\mathcal{C}_h$ (say, $j = 1, \ldots, n_e - n_v + 1$).

Divergence-free finite elements (cont'd)

The proof is quite simple and reads as follows: from

$$0 = \sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{curl}\, \mathbf{w}_h^j = \mathbf{curl} \left(\sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j \right)$$

we can conclude that $\sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j$ is a gradient, say, $\mathbf{grad}\, \varphi_h$. This is an element of N_h for which all the degrees of freedom associated to the edges belonging to the spanning tree are vanishing. Hence φ_h is constant, and $\sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j = \mathbf{0}$. We can thus conclude that $\alpha_j = 0$ for all $j = 1, \dots, n_e - n_v + 1$, since $\{\mathbf{w}_h^j\}_{j=1}^{n_e - n_v + 1}$ are linearly independent.

Divergence-free finite elements (cont'd)

The general topological case needs the identification of g additional edges to discard: the simplest option (but not always feasible in practice...) is to select one edge for each basis element σ_n of the first homology group of $\overline{\Omega}$, having constructed the spanning tree in such a way that all the other edges of σ_n belong to it. (For definiteness, suppose these edges are associated to the indices $j = 1, \ldots, g$: the union of the spanning tree $\mathcal{S}_h$ and these additional g edges was called belted tree in Bossavit (1998), Rapetti et al. (2003).)

With this choice we have that the line integral of $\sum_{j=g+1}^{n_e-n_v+1} \alpha_j \mathbf{w}_h^j$ over σ_n vanishes for each $n = 1, \ldots, g$ (all the edges contained in σ_n belong to the belted tree, namely, they correspond to indices smaller than $g+1$ or larger than $n_e - n_v + 1$). Therefore $\sum_{j=g+1}^{n_e-n_v+1} \alpha_j \mathbf{w}_h^j$ is a gradient, and the argument develops as before.

Solving the curl–div system

Curl–div system with $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$

The complete and most general form of the problem reads

$$\begin{cases} \mathbf{curl}\,(\eta\mathbf{u}) = \mathbf{B} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = G & \text{in } \Omega \\ (\eta\mathbf{u}) \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega \\ \int_{(\partial\Omega)_r} \mathbf{u} \cdot \mathbf{n} = \alpha_r & \forall\ r = 1, \dots, p\,. \end{cases} \tag{9}$$

Curl–div system with $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The data satisfy standard regularity assumptions: $\boldsymbol{\eta}$ a symmetric matrix, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$; $\mathbf{B} \in (L^2(\Omega))^3$; $G \in L^2(\Omega)$; $\mathbf{a} \in H^{-1/2}(\mathrm{div}_\tau; \partial\Omega)$ [the space of tangential traces of vector fields belonging to $H(\mathbf{curl}\,; \Omega)$]; $\boldsymbol{\alpha} \in \mathbb{R}^p$.

Moreover, they also satisfy the necessary conditions $\mathrm{div}\,\mathbf{B} = 0$ in Ω, $\int_\Omega \mathbf{B} \cdot \boldsymbol{\rho} + \int_{\partial\Omega} \mathbf{a} \cdot \boldsymbol{\rho} = 0$ for each $\boldsymbol{\rho} \in \mathcal{H}(m)$, and $\mathbf{B} \cdot \mathbf{n} = \mathrm{div}_\tau \mathbf{a}$ on $\partial\Omega$. Here $\mathcal{H}(m)$ is the space of Neumann harmonic fields, namely,

$$\mathcal{H}(m) = \{\boldsymbol{\rho} \in (L^2(\Omega))^3 \,|\, \mathbf{curl}\,\boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \mathrm{div}\,\boldsymbol{\rho} = 0 \text{ in } \Omega, \\ \boldsymbol{\rho} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \,.$$

Curl–div system with $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The first step of the procedure is to find a vector field $\mathbf{u}^\star \in (L^2(\Omega))^3$ satisfying

$$\begin{cases} \operatorname{div} \mathbf{u}^\star = G & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{u}^\star \cdot \mathbf{n} = \alpha_r & \forall\ r = 1, \dots, p\,. \end{cases} \tag{10}$$

Then the vector field $\mathbf{W} = \mathbf{u} - \mathbf{u}^\star$ has to satisfy

$$\begin{cases} \mathbf{curl}\,(\eta \mathbf{W}) = \mathbf{B} - \mathbf{curl}\,(\eta \mathbf{u}^\star) & \text{in } \Omega \\ \operatorname{div} \mathbf{W} = 0 & \text{in } \Omega \\ (\eta \mathbf{W}) \times \mathbf{n} = \mathbf{a} - (\eta \mathbf{u}^\star) \times \mathbf{n} & \text{on } \partial\Omega \\ \int_{(\partial\Omega)_r} \mathbf{W} \cdot \mathbf{n} = 0 & \forall\ r = 1, \dots, p\,. \end{cases} \tag{11}$$

A variational formulation for the curl–div system

The second step is to devise a variational formulation of (11).

This is quite easy. Multiplying the first equation by a test function $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$, integrating in Ω and integrating by parts we find:

$$
\begin{aligned}
\int_\Omega \mathbf{B} \cdot \mathbf{v} &= \int_\Omega \mathbf{curl}\,[\eta(\mathbf{W} + \mathbf{u}^\star)] \cdot \mathbf{v} \\
&= \int_\Omega \eta(\mathbf{W} + \mathbf{u}^\star) \cdot \mathbf{curl}\,\mathbf{v} - \int_{\partial\Omega} [\eta(\mathbf{W} + \mathbf{u}^\star) \times \mathbf{n}] \cdot \mathbf{v} \\
&= \int_\Omega \eta\mathbf{W} \cdot \mathbf{curl}\,\mathbf{v} + \int_\Omega \eta\mathbf{u}^\star \cdot \mathbf{curl}\,\mathbf{v} - \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v}\,.
\end{aligned}
$$

Let us introduce the space

$$
\begin{aligned}
\mathcal{W}_0 = \{\mathbf{v} \in H(\mathrm{div}\,;\Omega) \,|\, \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega,& \\
\int_{(\partial\Omega)_r} \mathbf{v} \cdot \mathbf{n} = 0 \ \forall\ r = 1, \dots, p\}\,,&
\end{aligned}
\tag{12}
$$

in other words $\mathcal{W}_0 = \mathbf{curl}\,[H(\mathbf{curl}\,;\Omega)]$.

A variational formulation for the curl–div system (cont'd)

The vector field $\mathbf{W}$ is thus a solution to

$$\begin{aligned}&\mathbf{W} \in \mathcal{W}_0 \ : \\ &\qquad \int_\Omega \eta \mathbf{W} \cdot \mathbf{curl}\, \mathbf{v} = \int_\Omega \mathbf{B} \cdot \mathbf{v} - \int_\Omega \eta \mathbf{u}^\star \cdot \mathbf{curl}\, \mathbf{v} + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v} \\ &\forall\ \mathbf{v} \in H(\mathbf{curl}\,; \Omega)\,.\end{aligned} \tag{13}$$

More precisely, $\mathbf{W}$ is the unique solution of that problem: in fact, assuming $\mathbf{B} = \mathbf{u}^\star = \mathbf{a} = \mathbf{0}$, and taking $\mathbf{v}$ such that $\mathbf{curl}\, \mathbf{v} = \mathbf{W}$, it follows at once $\int_\Omega \eta \mathbf{W} \cdot \mathbf{W} = 0$, hence $\mathbf{W} = \mathbf{0}$.

Finite element approximation of the curl–div system

The finite element approximation follows the same steps.

The first one is finding a finite element potential $\mathbf{u}_h^\star \in RT_h$ such that

$$\begin{cases} \operatorname{div} \mathbf{u}_h^\star = G_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{u}_h^\star \cdot \mathbf{n} = \alpha_r & \forall\ r = 1, \dots, p \,, \end{cases} \tag{14}$$

where $G_h \in PC_h$ is the piecewise-constant interpolant $I_h^{PC} G$ of G. This can be done as in (6).

Finite element approximation of the curl–div system (cont'd)

The second step concerns the numerical approximation of problem (13). The natural choice for the finite element space is clearly the space $\mathcal{W}_{0,h}$ introduced in (8). Thus the finite element approximation of (13) reads as follows:

$$\begin{array}{l} \mathbf{W}_h \in \mathcal{W}_{0,h} \ : \\ \quad \int_\Omega \eta \mathbf{W}_h \cdot \mathbf{curl}\, \mathbf{v}_h \\ \qquad\qquad = \int_\Omega \mathbf{B} \cdot \mathbf{v}_h - \int_\Omega \eta \mathbf{u}_h^\star \cdot \mathbf{curl}\, \mathbf{v}_h + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v}_h \\ \forall\ \mathbf{v}_h \in N_h^\star \,, \end{array} \tag{15}$$

where

$$N_h^\star = \mathrm{span}\{\mathbf{w}_h^j\}_{j=g+1}^{n_e-n_v+1} \,. \tag{16}$$

Finite element approximation of the curl–div system (cont'd)

The corresponding algebraic problem is a square linear system of dimension $n_e - n_v + 1 - g$, and it is uniquely solvable. In fact, we note that $\mathcal{W}_{0,h} = \mathbf{curl}\, N_h^\star$, hence we can choose $\mathbf{v}_h^\star \in N_h^\star$ such that $\mathbf{curl}\, \mathbf{v}_h^\star = \mathbf{W}_h$; from (15) we find at once $\mathbf{W}_h = \mathbf{0}$, provided that $\mathbf{B} = \mathbf{u}_h^\star = \mathbf{a} = \mathbf{0}$.

The convergence of this finite element scheme is easily shown by standard arguments.

The algebraic problem

The solution $\mathbf{W}_h \in \mathcal{W}_{0,h}$ can be written in terms of the basis as $\mathbf{W}_h = \sum_{j=g+1}^{n_e-n_v+1} W_j \mathbf{curl}\,\mathbf{w}_{h,j}$. Hence the finite dimensional problem (15) can be rewritten as

$$\begin{aligned}\sum_{j=g+1}^{n_e-n_v+1} W_j \int_\Omega \eta\,\mathbf{curl}\,\mathbf{w}_{h,j} \cdot \mathbf{curl}\,\mathbf{w}_{h,m} \\ = \int_\Omega \mathbf{B} \cdot \mathbf{w}_{h,m} - \int_\Omega \eta\,\mathbf{u}_h^\star \cdot \mathbf{curl}\,\mathbf{w}_{h,m} + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{w}_{h,m}\,,\end{aligned} \tag{17}$$

for each $m = g+1, \dots, n_e - n_v + 1$.

The matrix $\mathbf{K}^\star$ with entries

$$K^\star_{mj} = \int_\Omega \eta\,\mathbf{curl}\,\mathbf{w}_{h,j} \cdot \mathbf{curl}\,\mathbf{w}_{h,m}$$

is clearly symmetric and positive definite, as the vector fields $\mathbf{curl}\,\mathbf{w}_{h,j}$ are linearly independent.

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The grad problem (back to it)

Having found a nodal element ψ_h such that its gradient has line integral on all the edges of the spanning tree equal to that of $\mathbf{H}_h$, what about the edges not belonging to the spanning tree?

For each node v_i, $v_i \neq v_*$, let us denote by C_{v_i} the set of edges in $\mathcal{S}_h$ joining v_* to v_i. Given an edge $e = [v_a, v_b]$ not belonging to $\mathcal{S}_h$, we define the cycle $D_e = C_{v_a} + e - C_{v_b}$.

Since $\mathbf{H}_h$ is a gradient (it is curl-free and its line integral on all the cycles σ_n vanishes), its line integral on D_e vanishes. Therefore we have

$$\begin{aligned} 0 &= \oint_{D_e} \mathbf{H}_h \cdot d\mathbf{s} = \psi_h(v_a) + \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} - \psi_h(v_b) \\ &= \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} - \int_e \mathbf{grad}\, \psi_h \cdot \boldsymbol{\tau} \,. \end{aligned}$$

An explicit formula for the vector potential

- Devise an explicit formula for the solution to (4).

(We are able to do that if $\mathbf{B}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$, a quite natural condition in the most interesting physical situations, and for a suitable choice of the constants ρ_n.)

The idea is the following. Define the Biot–Savart field

$$\mathbf{H}^{BS}(\mathbf{x}) = \frac{1}{4\pi} \int_\Omega \mathbf{B}_h(\mathbf{y}) \times \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{y} \,,$$

and set $\rho_n = \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s}$ in (4).
One has $\mathbf{curl}\, \mathbf{H}^{BS} = \mathbf{B}_h$ in Ω (here the condition $\mathbf{B}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$ has played a role). Hence the Nédélec interpolant $\Pi^{N_h} \mathbf{H}^{BS}$ satisfies $(4)_1$ and $(4)_2$.

An explicit formula for the vector potential (cont'd)

To find the solution to (4), we can correct $\Pi^{N_h}\mathbf{H}^{BS}$ by a gradient, namely, construct the nodal element ϕ_h whose gradient has the same line integral of $\mathbf{H}^{BS}$ on the edges of the spanning tree $\mathcal{S}_h$.

The Nédélec finite element $\mathbf{A}_h = \Pi^{N_h}\mathbf{H}^{BS} - \mathbf{grad}\,\phi_h$ is the solution to (4).

To express its degrees of freedom, we proceed as follows. For each edge $e \notin \mathcal{S}_h$, we define the cycle D_e as before (the edges from the root of the spanning tree to the first vertex of e, the edge e, the edges from the second vertex of e to the root of the spanning tree).

An explicit formula for the vector potential (cont'd)

The cycle D_e is constituted by edges all belonging to the spanning tree (except e): hence we have

$$\begin{aligned}
\int_e \mathbf{A}_h \cdot \boldsymbol{\tau} &= \int_e (\Pi^{N_h} \mathbf{H}^{BS} - \mathbf{grad}\, \phi_h) \cdot \boldsymbol{\tau} \\
&= \int_e \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - [\phi_h(v_b) - \phi_h(v_a)] \\
&= \int_e \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \left[\int_{C_{v_b}} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \int_{C_{v_a}} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} \right] \qquad (18) \\
&= \oint_{D_e} \mathbf{H}^{BS} \cdot d\mathbf{s} \\
&= \frac{1}{4\pi} \oint_{D_e} \left(\int_\Omega \mathbf{B}_h(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y} \right) \cdot d\mathbf{s}(\mathbf{x})\,.
\end{aligned}$$

Using (18), we can always restart the Webb–Forghani algorithm.

A basis of the first de Rham cohomology group

The presented approach permits to solve also the problem

$$\begin{cases} \mathbf{curl}\,\mathbf{A}_h = \mathbf{0} & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{A}_h \cdot d\mathbf{s} = \kappa_n & \forall\, n = 1, \ldots, g \\ \int_{e'} \mathbf{A}_h \cdot \boldsymbol{\tau} = 0 & \forall\, e' \in \mathcal{S}_h \,, \end{cases} \tag{19}$$

for any choice of the constants κ_n.

Taking κ_n equal to $\ell_{\mathcal{K}}(\sigma_n, \widehat{\sigma}_j)$, $j = 1, \ldots, g$, ($\ell_{\mathcal{K}}$ denotes the linking number) we find a basis $\mathbf{T}^{(j)}$ of the first de Rham cohomology group, and we have also an explicit formula like (18) for expressing the degrees of freedom of each $\mathbf{T}^{(j)}$.

The linking number

The linking number between $\widehat{\sigma}_j$ and another disjoint cycle σ is given by:

$$\ell_{\mathcal{K}}(\sigma, \widehat{\sigma}_j) = \frac{1}{4\pi} \oint_{\sigma} \left(\oint_{\widehat{\sigma}_j} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x \,.$$

- The linking number (introduced by Gauss...) is an integer that represents the number of times that each cycle winds around the other.

The explicit formula for determining the basis elements $\mathbf{T}^{(j)}$ is

$$\int_e \mathbf{T}^{(j)} \cdot \boldsymbol{\tau} = \ell_{\mathcal{K}}(D_e, \widehat{\sigma}_j) \qquad (20)$$

(where $\widehat{\sigma}_j$ has been chosen inside $\mathbb{R}^3 \setminus \overline{\Omega}$, namely, not intersecting D_e).

Well-posedness of (6)

The procedure is constructive, similar to the elimination procedure used for the grad problem but now going along the dual spanning tree, starting from the leaves. (Let us recall that a leaf of a spanning tree $\mathcal{M}_h$ is a vertex of W that has only one arc of $\mathcal{M}_h$ incident to it.)

We can reduce the problem to the faces $f \in \mathcal{M}_h$. If w (a tetrahedron or a connected component) is a leaf of $\mathcal{M}_h$, then on it there is only one face $f(w)$ belonging to the spanning tree $\mathcal{M}_h$, therefore the value of the flux of $\mathbf{v}_h$ on $f(w)$ can be computed by the Gauss theorem, if w is a tetrahedron or the connected component $(\partial\Omega)_0$, or by the equation $\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = c_r$, if w is the connected component $(\partial\Omega)_r$, $r = 1, \dots, p$ (recall that we know that $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$ for all $f \notin \mathcal{M}_h$).

Well-posedness of (6) (cont'd)

We can iterate this argument: if we remove from the spanning tree $\mathcal{M}_h$ a leaf and its corresponding incident arc, the remaining graph is still a tree. After a finite number of steps the remaining tree reduces to just on vertex, and the result is that $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f$ is known for all $f \in F$.

A basis of the second de Rham cohomology group

It can be also noted that the solutions $\mathbf{W}^{(s)}$, $s = 1, \ldots, p$, of the problems

$$\begin{cases} \operatorname{div} \mathbf{v}_h = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = \delta_{r,s} & \forall\, r = 1, \ldots, p \\ \int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0 & \forall\, f \notin \mathcal{M}_h \end{cases} \tag{21}$$

furnish a basis of the second de Rham cohomology group of Ω.

Convergence of the approximation for the first case

Theorem A. Let $\mathbf{W} \in \mathcal{W}_0$ and $\mathbf{W}_h \in \mathcal{W}_{0,h}$ be the solutions of problem (13) and (15), respectively. Set $\mathbf{u} = \mathbf{W} + \mathbf{u}^\star$ and $\mathbf{u}_h = \mathbf{W}_h + \mathbf{u}_h^\star$, where $\mathbf{u}^\star \in H(\mathrm{div}\,;\Omega)$ and $\mathbf{u}_h^\star \in RT_h$ are solutions to problem (10) and (14), respectively. Assume that $\mathbf{u}$ is regular enough, so that the interpolant $I_h^{RT}\mathbf{u}$ is defined. Then the following error estimate holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{div}\,;\Omega)} \leq c_0\big(\|\mathbf{u} - I_h^{RT}\mathbf{u}\|_{L^2(\Omega)} + \|G - I_h^{PC}G\|_{L^2(\Omega)}\big)\,. \quad (22)$$

Convergence of the approximation for the first case (cont'd)

Proof. Since $N_h^\star \subset H(\mathbf{curl}\,;\Omega)$, we can choose $\mathbf{v} = \mathbf{v}_h \in N_h^\star$ in (13). By subtracting (15) from (13) we end up with

$$\int_\Omega \eta[(\mathbf{W} + \mathbf{u}^\star) - (\mathbf{W}_h + \mathbf{u}_h^\star)] \cdot \mathbf{curl}\,\mathbf{v}_h = 0 \qquad \forall\ \mathbf{v}_h \in N_h^\star\,,$$

namely, the consistency property

$$\int_\Omega \eta(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{curl}\,\mathbf{v}_h = 0 \qquad \forall\ \mathbf{v}_h \in N_h^\star\,. \tag{23}$$

Then from $\mathcal{W}_{0,h} = \mathbf{curl}\,N_h^\star$ we can write $\mathbf{W}_h = \mathbf{curl}\,\mathbf{v}_h^\star$ for a suitable $\mathbf{v}_h^\star \in N_h^\star$, and using (23) we find

Convergence of the approximation for the first case (cont'd)

$$
\begin{aligned}
c_1 \|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(\Omega)} &\leq \int_\Omega \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{u}_h) \\
&= \int_\Omega \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{W}_h - \mathbf{u}_h^\star) \\
&= \int_\Omega \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{curl}\, \mathbf{v}_h^\star - \mathbf{u}_h^\star) \\
&= \int_\Omega \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{curl}\, \mathbf{v}_h - \mathbf{u}_h^\star) \\
&\leq c_2 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \|\mathbf{u} - \boldsymbol{\Phi}_h - \mathbf{u}_h^\star\|_{L^2(\Omega)} \quad \forall\ \boldsymbol{\Phi}_h \in \mathcal{W}_{0,h} \,.
\end{aligned}
$$

We can choose $\boldsymbol{\Phi}_h = (I_h^{RT}\mathbf{u} - \mathbf{u}_h^\star) \in \mathcal{W}_{0,h}$; in fact, $\mathrm{div}\,(I_h^{RT}\mathbf{u}) = I_h^{PC}(\mathrm{div}\,\mathbf{u}) = I_h^{PC} G = G_h$ and $\int_{(\partial\Omega)_r} I_h^{RT}\mathbf{u} \cdot \mathbf{n} = \int_{(\partial\Omega)_r} \mathbf{u} \cdot \mathbf{n} = \alpha_r$ for each $r = 1, \ldots, p$. Then it follows at once $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq \frac{c_2}{c_1} \|\mathbf{u} - I_h^{RT}\mathbf{u}\|_{L^2(\Omega)}$.
Finally, $\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h) = G - G_h = G - I_h^{PC} G$. □

Convergence of the approximation for the first case (cont'd)

Note that a sufficient condition for defining the interpolant of $\mathbf{u}$ is that $\mathbf{u} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$. This is satisfied if, e.g., $\boldsymbol{\eta}$ is a scalar Lipschitz function in $\overline{\Omega}$ and $\mathbf{a} \in (H^{\gamma}(\partial\Omega))^3$, $\gamma > 0$.

Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$

The problem at hand reads

$$\begin{cases} \mathbf{curl}\,\mathbf{u} = \mathbf{B} \\ \mathrm{div}\,(\boldsymbol{\mu}\mathbf{u}) = G \\ \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n} = b \\ \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n \quad \forall\ n = 1, \ldots, g\,, \end{cases} \tag{24}$$

where $\boldsymbol{\mu}$ is a symmetric matrix, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$, $\mathbf{B} \in (L^2(\Omega))^3$, $G \in L^2(\Omega)$, $b \in H^{-1/2}(\partial\Omega)$, $\boldsymbol{\beta} \in \mathbb{R}^g$.

Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The data satisfy the necessary conditions $\mathrm{div}\, \mathbf{B} = 0$ in Ω, $\int_\Omega G = \int_{\partial\Omega} b$; moreover, in order that the line integral of $\mathbf{u}$ on σ_n has a meaning, we also assume that $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\Omega$ (which is more restrictive than the necessary condition $\int_{(\partial\Omega)_r} \mathbf{B} \cdot \mathbf{n} = 0$ for each $r = 1, \ldots, p$).

The first step of the procedure is to find a vector field $\mathbf{u}^* \in (L^2(\Omega))^3$ satisfying

$$\begin{cases} \mathbf{curl}\, \mathbf{u}^* = \mathbf{B} & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}^* \cdot d\mathbf{s} = \beta_n & \forall\ n = 1, \ldots, g\,. \end{cases} \tag{25}$$

Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

Then the vector field $\mathbf{V} = \mathbf{u} - \mathbf{u}^*$ has to satisfy

$$\begin{cases} \mathbf{curl}\, \mathbf{V} = \mathbf{0} & \text{in } \Omega \\ \mathrm{div}\,(\mu \mathbf{V}) = G - \mathrm{div}\,(\mu \mathbf{u}^*) & \text{in } \Omega \\ (\mu \mathbf{V}) \cdot \mathbf{n} = b - (\mu \mathbf{u}^*) \cdot \mathbf{n} & \text{on } \partial\Omega \\ \oint_{\sigma_n} \mathbf{V} \cdot d\mathbf{s} = 0 & \forall\ n = 1, \ldots, g\,, \end{cases} \tag{26}$$

The second step is to devise a variational formulation of (26).

A variational formulation for the second case

Multiplying the second equation by a test function $\varphi \in H^1(\Omega)$, integrating in Ω and integrating by parts we find:

$$\begin{aligned}\int_\Omega G\,\varphi &= \int_\Omega \mathrm{div}\,[\boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)]\,\varphi \\ &= -\int_\Omega \boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)\cdot\mathbf{grad}\,\varphi + \int_{\partial\Omega}[\boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)\cdot\mathbf{n}]\,\varphi \\ &= -\int_\Omega \boldsymbol{\mu}\mathbf{V}\cdot\mathbf{grad}\,\varphi - \int_\Omega \boldsymbol{\mu}\mathbf{u}^*\cdot\mathbf{grad}\,\varphi + \int_{\partial\Omega} b\,\varphi\,.\end{aligned}$$

Let us introduce the space

$$\begin{aligned}\mathcal{V}_0 = \{\mathbf{v}\in H(\mathbf{curl}\,;\Omega)\,|\,\mathbf{curl}\,\mathbf{v} = \mathbf{0}\ \text{in}\ \Omega,& \\ \oint_{\sigma_n}\mathbf{v}\cdot d\mathbf{s} = 0\ \forall\ n=1,\dots,g\}\,.&\end{aligned} \tag{27}$$

Note that $\mathcal{V}_0 = \mathbf{grad}\,[H^1(\Omega)]$.

A variational formulation for the second case (cont'd)

The vector field $\mathbf{V}$ is thus a solution to

$$\mathbf{V} \in \mathcal{V}_0 \ : \quad \int_\Omega \boldsymbol{\mu}\mathbf{V} \cdot \mathbf{grad}\,\varphi = -\int_\Omega G\,\varphi - \int_\Omega \boldsymbol{\mu}\mathbf{u}^* \cdot \mathbf{grad}\,\varphi + \int_{\partial\Omega} b\,\varphi \qquad (28)$$
$$\forall\ \varphi \in H^1(\Omega)\,.$$

It is easily seen that $\mathbf{V}$ is indeed the unique solution of that problem: in fact, assuming $G = b = 0$, $\mathbf{u}^* = \mathbf{0}$, and taking φ such that $\mathbf{grad}\,\varphi = \mathbf{V}$, it follows at once $\int_\Omega \boldsymbol{\mu}\mathbf{V} \cdot \mathbf{V} = 0$, hence $\mathbf{V} = \mathbf{0}$.

Finite element approximation of the second case

The finite element approximation follows the same steps.

The first one is finding a finite element potential $\mathbf{u}_h^* \in N_h$ such that

$$\begin{cases} \mathbf{curl}\, \mathbf{u}_h^* = \mathbf{B}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}_h^* \cdot d\mathbf{s} = \beta_n & \forall\ n = 1, \ldots, g \,, \end{cases} \tag{29}$$

where $\mathbf{B}_h \in RT_h$ is the Raviart–Thomas interpolant $I_h^{RT}\mathbf{B}$ of $\mathbf{B}$ (we therefore assume that $\mathbf{B}$ is so regular that its interpolant $I_h^{RT}\mathbf{B}$ is defined; for instance, as already recalled, it is enough to assume $\mathbf{B} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$). The construction of $\mathbf{u}_h^*$ can be done as in (4).

Finite element approximation of the second case (cont'd)

The second step is related to the numerical approximation of problem (28). The natural choice for the finite element space is clearly the space $\mathcal{V}_{0,h}$ introduced in (7). The finite element approximation of (28) reads as follows:

$$
\begin{aligned}
&\mathbf{V}_h \in \mathcal{V}_{0,h} \ : \\
&\quad \textstyle\int_\Omega \boldsymbol{\mu} \mathbf{V}_h \cdot \mathbf{grad}\, \varphi_h = \\
&\qquad\qquad - \textstyle\int_\Omega G\, \varphi_h - \int_\Omega \boldsymbol{\mu} \mathbf{u}_h^* \cdot \mathbf{grad}\, \varphi_h + \int_{\partial\Omega} b\, \varphi_h \\
&\forall\ \varphi_h \in L_h^* \ ,
\end{aligned}
\tag{30}
$$

where

$$
L_h^* = \mathrm{span}\{\psi_{h,i}\}_{i=1}^{n_v - 1} = \{\varphi_h \in L_h \,|\, \varphi_h(v_{n_v}) = 0\}\,. \tag{31}
$$

Finite element approximation of the second case (cont'd)

The corresponding algebraic problem is a square linear system of dimension $n_v - 1$, and it is uniquely solvable. In fact, since $\mathcal{V}_{0,h} = \mathbf{grad}\, L_h^*$, we can choose $\varphi_h^* \in L_h^*$ such that $\mathbf{grad}\, \varphi_h^* = \mathbf{V}_h$; from (30) we find at once $\mathbf{V}_h = \mathbf{0}$, provided that $G = b = 0$, $\mathbf{u}_h^* = \mathbf{0}$.

The convergence of this finite element scheme is easily shown by standard arguments.

Convergence of the approximation for the second case

Theorem B. Let $\mathbf{V} \in \mathcal{V}_0$ and $\mathbf{V}_h \in \mathcal{V}_{0,h}$ be the solutions of problem (28) and (30), respectively. Set $\mathbf{u} = \mathbf{V} + \mathbf{u}^*$ and $\mathbf{u}_h = \mathbf{V}_h + \mathbf{u}_h^*$, where $\mathbf{u}^* \in H(\mathbf{curl}\,;\Omega)$ and $\mathbf{u}_h^* \in N_h$ are solutions to problem (25) and (29), respectively. Assume that $\mathbf{u}$ and $\mathbf{B}$ are regular enough, so that the interpolants $I_h^N\mathbf{u}$ and $I_h^{RT}\mathbf{B}$ are defined. Then the following error estimate holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathbf{curl}\,;\Omega)} \leq c_0\big(\|\mathbf{u} - I_h^N\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{B} - I_h^{RT}\mathbf{B}\|_{L^2(\Omega)}\big) . \quad (32)$$

Convergence of the approximation for the second case (cont'd)

Proof. Since $L_h^* \subset H^1(\Omega)$, we can choose $\varphi = \varphi_h \in L_h^*$ in (28). By subtracting (30) from (28) we end up with

$$\int_\Omega \boldsymbol{\mu}[(\mathbf{V} + \mathbf{u}^*) - (\mathbf{V}_h + \mathbf{u}_h^*)] \cdot \mathbf{grad}\, \varphi_h = 0 \qquad \forall\ \varphi_h \in L_h^*\,,$$

namely, the consistency property

$$\int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{grad}\, \varphi_h = 0 \qquad \forall\ \varphi_h \in L_h^*\,. \tag{33}$$

Then, since $\mathcal{V}_{0,h} = \mathbf{grad}\, L_h^*$ and thus $\mathbf{V}_h = \mathbf{grad}\, \varphi_h^*$ for a suitable $\varphi_h^* \in L_h^*$, from (33) we find

Convergence of the approximation for the second case (cont'd)

$$
\begin{aligned}
c_1 \|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(\Omega)} &\leq \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{u}_h) \\
&= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{V}_h - \mathbf{u}^*_h) \\
&= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{grad}\, \varphi^*_h - \mathbf{u}^*_h) \\
&= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{grad}\, \varphi_h - \mathbf{u}^*_h) \\
&\leq c_2 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \|\mathbf{u} - \boldsymbol{\Psi}_h - \mathbf{u}^*_h\|_{L^2(\Omega)} \quad \forall\ \boldsymbol{\Psi}_h \in \mathcal{V}_{0,h}\,.
\end{aligned}
$$

We can choose $\boldsymbol{\Psi}_h = (I^N_h \mathbf{u} - \mathbf{u}^*_h) \in \mathcal{V}_{0,h}$; in fact, $\mathbf{curl}\,(I^N_h \mathbf{u}) = I^{RT}_h (\mathbf{curl}\, \mathbf{u}) = I^{RT}_h \mathbf{B} = \mathbf{B}_h$ and $\oint_{\sigma_n} I^N_h \mathbf{u} \cdot d\mathbf{s} = \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n$ for each $n = 1, \dots, g$. Then we find at once $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq \frac{c_2}{c_1} \|\mathbf{u} - I^N_h \mathbf{u}\|_{L^2(\Omega)}$.

Moreover, $\mathbf{curl}\,(\mathbf{u} - \mathbf{u}_h) = \mathbf{B} - \mathbf{B}_h = \mathbf{B} - I^{RT}_h \mathbf{B}$. $\square$

Convergence of the approximation for the second case (cont'd)

Note that a sufficient condition for defining the interpolants of $\mathbf{u}$ and $\mathbf{B} = \mathbf{curl}\,\mathbf{u}$ is that they both belong to $(H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$. Thus one has to assume that $\mathbf{B} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$; moreover, $\mathbf{u}$ belongs to $(H^{\frac{1}{2}+\delta}(\Omega))^3$ if, for instance, $\boldsymbol{\mu}$ is a scalar Lipschitz function in $\overline{\Omega}$ and $b \in H^{\gamma}(\Omega)$, $\gamma > 0$.

The algebraic problem for the second case

The solution $\mathbf{V}_h \in \mathcal{V}_{0,h}$ is given by $\mathbf{V}_h = \sum_{i=1}^{n_v-1} V_i \mathbf{grad}\, \psi_{h,i}$.
Hence the finite dimensional problem (30) can be rewritten as

$$\begin{aligned}\sum_{i=1}^{n_v-1} V_i \int_\Omega \boldsymbol{\mu}\, \mathbf{grad}\, \psi_{h,i} \cdot \mathbf{grad}\, \psi_{h,l} \\ = -\textstyle\int_\Omega G\, \psi_{h,l} - \int_\Omega \boldsymbol{\mu}\, \mathbf{u}_h^* \cdot \mathbf{grad}\, \psi_{h,l} + \displaystyle\int_{\partial\Omega} b\, \psi_{h,l}\,,\end{aligned} \tag{34}$$

for each $l = 1, \ldots, n_v - 1$.
The matrix $\mathbf{K}^*$ with entries

$$K^*_{li} = \int_\Omega \boldsymbol{\mu}\, \mathbf{grad}\, \varphi_{h,i} \cdot \mathbf{grad}\, \varphi_{h,l}$$

is clearly symmetric and positive definite.

- P. Alotto and I. Perugia, *Mixed finite element methods and tree–cotree implicit condensation*, Calcolo, 36 (1999), 233–248.