

# Finite elements in electromagnetism and application to eddy current equations

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## Outline

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- 2 Maxwell and eddy current equations
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- 4 Edge finite elements
- 5 Variational formulations and numerical approximation

## What finite elements are?

Finite elements are **piecewise-polynomials** functions defined in a domain  $\Omega$ .

- The word “piecewise” indicates that  $\Omega$  has been split in a **finite number of (non-overlapping) pieces**, called **elements**. (In other words, a “triangulation”  $\mathcal{T}_h$  of  $\bar{\Omega}$  is available.)

### Simplest case:

- all the elements have the **same shape** (triangles/tetrahedra, parallelograms/parallelepipeds)
- the polynomials are of the **same type** (and therefore of a fixed degree) for all the elements.

Thus

- a space of finite elements is a **finite dimensional** vector space.

## Variational formulation of a PDE

Very often a partial differential equation (**in variational form**) can be written as:

$$u \in V : a(u, v) = \mathcal{F}(v) \quad \forall v \in V, \quad (1)$$

where  $V$  is an **infinite dimensional** vector space.

In the simplest case,  $V$  is a **Hilbert** space,  $a(\cdot, \cdot)$  is a **bilinear** form, and  $\mathcal{F}(\cdot)$  is a **linear** form.



## Example of a variational formulation of a PDE

An **example**: given the (Neumann) problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

for  $f$  such that  $\int_{\Omega} f^2 < \infty$ , multiply the equation by a **test function**  $v$ , integrate in  $\Omega$  and integrate by parts. **Using the boundary condition**, you find that  $u$  is a solution of:

$$\int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} u v = \int_{\Omega} f v \quad \forall v.$$

The space  $V$  in this case is

$$H^1(\Omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} (|\nabla v|^2 + v^2) < \infty \right\}.$$

## Discretization of a PDE

Denote by  $V_h$  a space of finite elements, where  $h$  is the **maximum diameter** of the pieces in which  $\Omega$  has been split. A **reasonable discretization** of (1) is:

$$u_h \in V_h : a(u_h, v_h) = \mathcal{F}(v_h) \quad \forall v_h \in V_h. \quad (2)$$

In the simplest case one can choose  $V_h \subset V$ , therefore the **consistency property**

$$a(u, v_h) = a(u_h, v_h) \quad \text{for each } v_h \in V_h$$

holds.

If, in some sense,  $V_h \rightarrow V$ , we **can expect** that  $u_h \rightarrow u$ .

## Discretization of a PDE (cont'd)

In fact, if

- $a(w, v) \leq \beta \|w\| \|v\|$  for all  $w, v \in V$  [continuity]
- $a(v, v) \geq \alpha \|v\|^2$  for all  $v \in V$  [coerciveness],

we obtain

$$\begin{aligned} \alpha \|u - u_h\|^2 &\leq a(u - u_h, u - u_h) = a(u - u_h, u) = a(u - u_h, u - v_h) \\ &\leq \beta \|u - u_h\| \|u - v_h\| \end{aligned}$$

for each  $v_h \in V_h$ , hence

$$\|u - u_h\| \leq \frac{\beta}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|.$$

[This is called **Céa lemma**: the basic result towards convergence.]

## Matching conditions for finite elements

Assuming that  $V_h \subset V$  **has a consequence**: we are implicitly requiring that, on the common boundary of two elements, the values of the two polynomials **have to match** in a suitable way. “Suitable” simply means that, as a function defined in  $\Omega$ ,  $v_h$  **must belong** to  $V$ .

Example:  $v_h \in H^1(\Omega)$  if and only if  $v_h$  is **continuous** across the interelements (namely, if it is continuous in  $\Omega$ ).

Summarizing: a PDE is associated to a vector space  $V$ , and this vector space is dictating the matching conditions to the finite elements. Therefore, a PDE is **dictating** which types of finite elements are suitable for its discretization.

## Examples of PDEs and variational spaces

- A **second order linear elliptic equation**

$$-\sum_{i,j} D_i(a_{ij}D_j u) + \sum_i b_i D_i u + cu = f$$

is associated to  $V = H^1(\Omega)$  (or to a closed subspace of it).

The matching condition for finite elements is **the continuity of  $u$**  across interelements.

## Examples of PDEs and variational spaces (cont'd)

- The **time-harmonic Maxwell equation**

$$\mathbf{curl} \mathbf{curl} \mathbf{u} - \omega^2 \varepsilon_0 \mu_0 \mathbf{u} = \mathbf{f}$$

is associated to

$$V = H(\mathbf{curl}; \Omega) = \left\{ \mathbf{v} : \Omega \rightarrow \mathbb{R}^3 \mid \int_{\Omega} (|\mathbf{curl} \mathbf{v}|^2 + |\mathbf{v}|^2) < \infty \right\}$$

(or to a closed subspace of it).

It is easily seen that the matching condition for finite elements is **the continuity of  $\mathbf{u} \times \mathbf{n}$**  across interelements ( $\mathbf{n}$  being the unit normal vector on the common boundary).

## Examples of PDEs and variational spaces (cont'd)

- The **Darcy system**

$$\begin{cases} K^{-1}\mathbf{u} + \nabla p = K^{-1}\mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \end{cases}$$

is associated to

$$V = H(\operatorname{div}; \Omega) = \left\{ \mathbf{v} : \Omega \rightarrow \mathbb{R}^3 \mid \int_{\Omega} ((\operatorname{div} \mathbf{v})^2 + |\mathbf{v}|^2) < \infty \right\}$$

(or to a closed subspace of it) for the velocity field  $\mathbf{u}$ , and to  $L^2(\Omega) = \left\{ q : \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} q^2 < \infty \right\}$  for the pressure  $p$ .

It is easily seen that the matching condition for finite elements is **the continuity of  $\mathbf{u} \cdot \mathbf{n}$**  across interelements (and no matching for  $p$ ).

## Maxwell equations in electromagnetism

Let us focus now on **Maxwell equations** and electromagnetism.  
The complete system reads

$$\left\{ \begin{array}{l} \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J} = \mathbf{curl} \mathcal{H} \\ \frac{\partial \mathcal{B}}{\partial t} + \mathbf{curl} \mathcal{E} = 0 \\ \mathbf{div} \mathcal{D} = \rho \\ \mathbf{div} \mathcal{B} = 0 \end{array} \right. \quad \begin{array}{l} \text{Maxwell–Ampère equation} \\ \text{Faraday equation} \\ \text{Gauss electrical equation} \\ \text{Gauss magnetic equation.} \end{array}$$

- $\mathcal{H}$  and  $\mathcal{E}$  are the **magnetic field** and **electric field**, respectively
- $\mathcal{B}$  and  $\mathcal{D}$  are the **magnetic induction** and **electric induction**, respectively
- $\mathcal{J}$  and  $\rho$  are the **electric current (surface) density** and **electric charge (volume) density**, respectively.



## Maxwell equations in electromagnetism (cont'd)

These fields are related through some **constitutive equations**: it is usually assumed a linear dependence like

$$\mathcal{D} = \varepsilon \mathcal{E} \quad , \quad \mathcal{B} = \mu \mathcal{H} \quad , \quad \mathcal{J} = \sigma \mathcal{E} + \mathcal{J}_e \quad ,$$

where  $\varepsilon$  and  $\mu$  are the **electric permittivity** and **magnetic permeability**, respectively, and  $\sigma$  is the **electric conductivity**.

[In general,  $\varepsilon$ ,  $\mu$  and  $\sigma$  are **symmetric and uniformly positive definite matrices**. Clearly, the conductivity  $\sigma$  is only present in conductors, and is identically **vanishing** in any insulator.]

- $\mathcal{J}_e$  is the **applied electric current density**.

## Eddy currents

As observed in experiments and stated by the Faraday law, a time-variation of the magnetic field generates an electric field. Therefore, in each conductor a current density  $\mathbf{J}_{\text{eddy}} = \sigma \mathbf{E}$  arises; this term expresses the presence in conducting media of the so-called **eddy currents**.

This phenomenon, and the related heating of the conductor, was observed and studied in the mid of the nineteenth century by the French physicist Léon Foucault, and in fact the generated eddy currents are also known as **Foucault currents**.



Léon Foucault  
(1819–1868).

## Eddy current approximation

In many real-life applications, the time of propagation of the electromagnetic waves is very small with respect to some characteristic time scale, or, equivalently, their **wave length** is much larger than the **diameter** of the physical domain.

Therefore one can think that the **speed of propagation is infinite**, and take into account only the **diffusion** of the electromagnetic fields, neglecting electromagnetic waves, namely, neglecting both time derivatives or one of them, either  $\frac{\partial \mathcal{D}}{\partial t}$  or  $\frac{\partial \mathcal{B}}{\partial t}$ .

Let us focus on the case in which the **displacement current** term  $\frac{\partial \mathcal{D}}{\partial t}$  can be disregarded, while the time-variation of the magnetic induction is still important, as well as the related presence of eddy currents in the conductors.

- The resulting equations are called **eddy current equations**.

## Time-harmonic Maxwell and eddy current equations

When interested in time-periodic phenomena, it is assumed that

$$\begin{aligned}\mathcal{J}_e(t, \mathbf{x}) &= \operatorname{Re}[\mathbf{J}_e(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{E}(t, \mathbf{x}) &= \operatorname{Re}[\mathbf{E}(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{H}(t, \mathbf{x}) &= \operatorname{Re}[\mathbf{H}(\mathbf{x}) \exp(i\omega t)] .\end{aligned}$$

- $\omega \neq 0$  is the (angular) **frequency**.

Inserting these relations in the Maxwell equations one obtains the so-called **time-harmonic Maxwell equations**

$$\begin{cases} \operatorname{curl} \mathbf{H} - i\omega\epsilon\mathbf{E} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \operatorname{curl} \mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega . \end{cases} \quad (3)$$

[Note that similar equations arise from the backward-Euler time-discretization of Maxwell equations: just substitute  $i\omega$  by  $\frac{1}{\Delta t} \dots$ ]

## Time-harmonic Maxwell and eddy current equations (cont'd)

As a consequence one has  $\operatorname{div}(\boldsymbol{\mu}\mathbf{H}) = 0$  in  $\Omega$  (and we can think that the electric charge in conductors is defined by  $\rho = \operatorname{div}(\boldsymbol{\epsilon}\mathbf{E})$ ).

It can be proved that the time-harmonic Maxwell equations **have a unique solution** (provided that suitable boundary conditions are added, and that the conductor is **not empty**).

On the other hand, dropping the displacement current term, the **time-harmonic eddy current equations** are

$$\begin{cases} \operatorname{curl} \mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \operatorname{curl} \mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega. \end{cases} \quad (4)$$

[Since in an insulator one has  $\boldsymbol{\sigma} = \mathbf{0}$ , it follows that  $\mathbf{E}$  is **not uniquely determined** in that region ( $\mathbf{E} + \mathbf{grad} \psi$  is still a solution). Some additional conditions are thus necessary (typically, the conditions satisfied by the solution  $\mathbf{E}$  of the Maxwell equations).]

## Boundary conditions

From now on we will denote by  $\Omega_I$  the insulator, namely, the region where  $\sigma = 0$ , by  $\Omega_C$  the conductor, and by  $\Gamma$  the interface between  $\Omega_I$  and  $\Omega_C$ . For the sake of simplicity, we also assume that  $\Omega_C$  is strictly contained in  $\Omega$ , and that  $\Omega_I$  is connected.

One of the following boundary conditions is usually imposed.

- **Electric.** One imposes  $\mathbf{E} \times \mathbf{n} = \mathbf{0}$  on  $\partial\Omega$ . [As a consequence, one also has  $\mu\mathbf{H} \cdot \mathbf{n} = 0$  on  $\partial\Omega$ .]
- **Magnetic (Maxwell).** One imposes  $\mathbf{H} \times \mathbf{n} = \mathbf{0}$  on  $\partial\Omega$ . [As a consequence, one also has  $\varepsilon\mathbf{E} \cdot \mathbf{n} = -(i\omega)^{-1}\mathbf{J}_e \cdot \mathbf{n}$  on  $\partial\Omega$ .]
- **Magnetic (eddy currents).** One imposes  $\mathbf{H} \times \mathbf{n} = \mathbf{0}$  and  $\varepsilon\mathbf{E} \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . [Note that  $\mathbf{H} \times \mathbf{n} = \mathbf{0}$  on  $\partial\Omega$  implies  $\mathbf{J}_e \cdot \mathbf{n} = 0$  on  $\partial\Omega$ .]

We will focus on the **electric** boundary condition.

## Geometry and harmonic fields

Now it is time to consider a **couple** of questions.

- If a vector field satisfies  $\mathbf{curl} \mathbf{v} = \mathbf{0}$  and  $\operatorname{div} \mathbf{v} = 0$  in a domain, together with either the boundary condition  $\mathbf{v} \times \mathbf{n} = \mathbf{0}$  or the boundary condition  $\mathbf{v} \cdot \mathbf{n} = 0$ , is it **non-trivial**, namely, not vanishing everywhere in the domain? [A field like that is called **harmonic** field.]
- If that is the case, do harmonic fields **appear** in electromagnetism?

Both questions have an affirmative answer.

## Geometry and harmonic fields (cont'd)

Let us start from the first question.

If the domain  $\mathcal{O}$  is homeomorphic to a **three-dimensional ball**, a curl-free vector field  $\mathbf{v}$  must be a gradient of a scalar function  $\psi$ , that must be harmonic due to the constraint on the divergence.

If the boundary condition is  **$\mathbf{v} \times \mathbf{n} = \mathbf{0}$  on  $\partial\mathcal{O}$** , which in this case is a connected surface, then it follows  $\psi = \text{const}$  on  $\partial\mathcal{O}$ , and therefore  $\psi = \text{const}$  in  $\mathcal{O}$  and  $\mathbf{v} = \mathbf{0}$  in  $\mathcal{O}$ .

If the boundary condition is  **$\mathbf{v} \cdot \mathbf{n} = 0$  on  $\partial\mathcal{O}$** , then  $\psi$  satisfies a homogeneous Neumann boundary condition and thus  $\psi = \text{const}$  in  $\mathcal{O}$  and  $\mathbf{v} = \mathbf{0}$  in  $\mathcal{O}$ .



## Geometry and harmonic fields (cont'd)

However, the problem is different in a **more general geometry**.

In fact, take the **magnetic field** generated in the vacuum by a current of constant intensity  $I^0$  passing along the  $x_3$ -axis: as it is well-known, for  $x_1^2 + x_2^2 > 0$  it is given by

$$\mathbf{H}(x_1, x_2, x_3) = \frac{I^0}{2\pi} \left( -\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0 \right).$$

It is easily checked that  $\mathbf{curl} \mathbf{H} = \mathbf{0}$  and  $\mathbf{div} \mathbf{H} = 0$ .

Let us consider now the **torus**  $\mathcal{T}$  obtained by rotating around the  $x_3$ -axis the disk (contained in the  $(x_1, x_3)$ -plane) of centre  $(a, 0, 0)$  and radius  $b$ , with  $0 < b < a$ . One sees at once that  $\mathbf{H} \cdot \mathbf{n} = 0$  on  $\partial\mathcal{T}$ ; hence we have found a non-trivial harmonic field  $\mathbf{H}$  in  $\mathcal{T}$  satisfying  $\mathbf{H} \cdot \mathbf{n} = 0$  on  $\partial\mathcal{T}$ .

## Geometry and harmonic fields (cont'd)

On the other hand, consider now the **electric field** generated in the vacuum by a pointwise charge  $\rho_0$  placed at the origin. For  $\mathbf{x} \neq \mathbf{0}$  it is given by

$$\mathbf{E}(x_1, x_2, x_3) = \frac{\rho_0}{4\pi\epsilon_0} \frac{\mathbf{x}}{|\mathbf{x}|^3},$$

where  $\epsilon_0$  is the electric permittivity of the vacuum.

It satisfies  $\operatorname{div} \mathbf{E} = 0$  and  $\operatorname{curl} \mathbf{E} = \mathbf{0}$ , and moreover  $\mathbf{E} \times \mathbf{n} = \mathbf{0}$  on the boundary of the **spherical shell**  $\mathcal{C} := B_{R_2} \setminus \overline{B_{R_1}}$ , where  $0 < R_1 < R_2$  and  $B_R := \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| < R\}$  is the ball of centre  $\mathbf{0}$  and radius  $R$ . We have thus found a non-trivial harmonic field  $\mathbf{E}$  in  $\mathcal{C}$  satisfying  $\mathbf{E} \times \mathbf{n} = \mathbf{0}$  on  $\partial\mathcal{C}$ .

## Geometry and harmonic fields (cont'd)

These two examples show that the **geometry** of the domain and the **type** of boundary conditions play an **essential role**.

- What are the relevant differences between the set  $\mathcal{O}$ , homeomorphic to a ball, and the sets  $\mathcal{T}$  (a torus) and  $\mathcal{C}$  (a spherical shell)?

For the former, the point is that in  $\mathcal{T}$  we have a **non-bounding closed curve**, namely, a cycle that is not the boundary of a surface contained in  $\mathcal{T}$  (take, for instance, the circle of centre  $\mathbf{0}$  and radius  $a$  in the  $(x_1, x_2)$ -plane).

In the latter case, in  $\mathcal{C}$  we have a **non-bounding closed surface**, namely, a closed surface that is not the boundary of a subdomain contained in  $\mathcal{C}$  (take, for instance,  $\partial B_{R^*}$ , with  $R_1 < R^* < R_2$ ); or, equivalently, the boundary of  $\mathcal{C}$  is **not connected**.

## The spaces of harmonic fields

Spaces of harmonic fields are often used in electromagnetism. Two examples are the following.

- For the magnetic field

$$\mathcal{H}^{(m)} = \{ \mathbf{G} \in (L^2(\Omega_I))^3 \mid \mathbf{curl} \mathbf{G} = \mathbf{0}, \operatorname{div}(\boldsymbol{\mu} \mathbf{G}) = 0, \boldsymbol{\mu} \mathbf{G} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_I \}.$$

- For the electric field

$$\mathcal{H}^{(e)} = \{ \mathbf{G} \in (L^2(\Omega_I))^3 \mid \mathbf{curl}(\boldsymbol{\varepsilon}^{-1} \mathbf{G}) = \mathbf{0}, \operatorname{div} \mathbf{G} = 0, (\boldsymbol{\varepsilon}^{-1} \mathbf{G}) \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega_I \},$$

Both are **finite dimensional**! Their dimension is a **topological invariant**! [The dimension is also independent of  $\boldsymbol{\mu}$  and  $\boldsymbol{\varepsilon}$ .]

## The spaces of harmonic fields (cont'd)

The dimension of  $\mathcal{H}^{(m)}$  is the **first Betti number** of  $\Omega_I$ , or, equivalently, the **dimension of the first homology group** of  $\overline{\Omega_I}$  (this is the quotient space between cycles in  $\overline{\Omega_I}$  and bounding cycles in  $\overline{\Omega_I}$ ). [We will call it  $g$ .]

- The number of **handles of  $\Omega_I$** !

The dimension of  $\mathcal{H}^{(e)}$  is the **second Betti number** of  $\Omega_I$ , or, equivalently, the **dimension of the second homology group** of  $\overline{\Omega_I}$  (this is the quotient space between closed surfaces in  $\overline{\Omega_I}$  and bounding surfaces in  $\overline{\Omega_I}$ ). [We will call it  $p$ .]

- The number of **connected components of  $\partial\Omega_I$  minus 1!**

## Edge finite elements

We have seen that the Maxwell equations are associated to the operator **curl**, and that therefore electromagnetic problems can be approximated by means of vector finite elements for which only the **continuity of the tangential components** on the interelements is required (and not the continuity of **all** the components).

- These elements are called **edge** elements, and have been proposed by Nédélec (1980).

## Edge finite elements (cont'd)

Let us specify them for the lowest polynomial order. For a **tetrahedral** mesh, say,  $\mathcal{T}_h$ , they are

$$N_h^1 := \{\mathbf{w}_h \in H(\mathbf{curl}; \Omega) \mid \mathbf{w}_h|_K = \mathbf{b}_K + \mathbf{a}_K \times \mathbf{x} \quad \forall K \in \mathcal{T}_h\}, \quad (5)$$

with  $\mathbf{b}_K \in \mathbb{R}^3$  and  $\mathbf{a}_K \in \mathbb{R}^3$ .

The **degrees of freedom** are associated to the **edges**  $e$  of each tetrahedron  $K$ , and are given by the **six** line integrals

$$\int_e (\mathbf{b}_K + \mathbf{a}_K \times \mathbf{x}) \cdot d\mathbf{s}. \quad (6)$$

## Edge finite elements (cont'd)

- Let us show that, if **all the six** degrees of freedom of  $\mathbf{q} = \mathbf{b}_K + \mathbf{a}_K \times \mathbf{x}$  on  $K$  are equal to 0, then  $\mathbf{q} = \mathbf{0}$ .

A direct computation shows that  $\mathbf{curl} \mathbf{q} = 2 \mathbf{a}_K$ . Moreover, from **Stokes theorem** for each face  $f$  we have

$$\begin{aligned} 0 &= \sum_e \int_e \mathbf{q} \cdot d\mathbf{s} = \int_{\partial f} \mathbf{q} \cdot d\mathbf{s} \\ &= \int_f \mathbf{curl} \mathbf{q} \cdot \mathbf{n}_f dS = 2 \mathbf{a}_K \cdot \mathbf{n}_f \text{meas}(f), \end{aligned}$$

hence  $\mathbf{a}_K \cdot \mathbf{n}_f = 0$  on  $f$ . Since three of the vectors  $\mathbf{n}_f$  are linearly independent, it follows  $\mathbf{a}_K = \mathbf{0}$ .



## Edge finite elements (cont'd)

Then for each edge  $e$

$$\begin{aligned} 0 &= \int_e \mathbf{q} \cdot d\mathbf{s} = \int_e \mathbf{b}_K \cdot d\mathbf{s} \\ &= \mathbf{b}_K \cdot \boldsymbol{\tau}_e \text{length}(e), \end{aligned}$$

and three of the unit tangent vectors  $\boldsymbol{\tau}_e$  are linearly independent, so that  $\mathbf{b}_K = \mathbf{0}$  and in conclusion  $\mathbf{q} = \mathbf{0}$ .

## Edge finite elements (cont'd)

- Another point is to prove that if the **three** edge degrees of freedom of  $\mathbf{q} = \mathbf{b}_K + \mathbf{a}_K \times \mathbf{x}$  on a **face**  $f \subset K$  are equal to 0 then  $\mathbf{q} \times \mathbf{n}_f = \mathbf{0}$  on  $f$ .

We have already seen that  $\mathbf{a}_K \cdot \mathbf{n}_f = 0$  on each  $f \subset K$ . On the other hand,

$$\begin{aligned} \mathbf{q} \times \mathbf{n}_f &= \mathbf{b}_K \times \mathbf{n}_f + (\mathbf{a}_K \times \mathbf{x}) \times \mathbf{n}_f \\ &= \mathbf{b}_K \times \mathbf{n}_f + (\mathbf{a}_K \cdot \mathbf{n}_f) \mathbf{x} - (\mathbf{x} \cdot \mathbf{n}_f) \mathbf{a}_K \\ &= \mathbf{b}_K \times \mathbf{n}_f - (\mathbf{x} \cdot \mathbf{n}_f) \mathbf{a}_K. \end{aligned}$$

Since on a face one has  $\mathbf{x} \cdot \mathbf{n}_f = \text{const}$ , it follows that  $\mathbf{q} \times \mathbf{n}_f$  is equal on  $f$  to a constant vector  $\mathbf{c}_f$  (with  $\mathbf{c}_f \cdot \mathbf{n}_f = 0$ ).

## Edge finite elements (cont'd)

Finally, for the **three** edges  $e$  of the face  $f$

$$\begin{aligned} 0 &= \int_e \mathbf{q} \cdot d\mathbf{s} = \int_e (\mathbf{n}_f \times \mathbf{q} \times \mathbf{n}_f) \cdot d\mathbf{s} \\ &= (\mathbf{n}_f \times \mathbf{c}_f) \cdot \boldsymbol{\tau}_e \text{length}(e). \end{aligned}$$

Since two of the tangent vectors  $\boldsymbol{\tau}_e$  are generating the plane containing  $f$  (and the vector  $\mathbf{n}_f \times \mathbf{c}_f$ ), it follows  $\mathbf{c}_f = \mathbf{0}$  and consequently  $\mathbf{q} \times \mathbf{n}_f = \mathbf{0}$  on  $f$ .

## Edge finite elements (cont'd)

- In particular, we have shown that the dimension of  $N_h^1$  is equal to the **total number of edges**.

The **basis functions** are defined as follows: for each edge  $e_m$  we construct the vector function  $\Phi_m$  such that

$$\int_{e_l} \Phi_m \cdot ds = \begin{cases} 1 & \text{if } m = l \\ 0 & \text{if } m \neq l. \end{cases} \quad (7)$$

These basis functions have a **“small” support**:  $\Phi_m$  is non-vanishing only in the elements  $K$  of the triangulation that contain the edge  $e_m$ .

## Finite element spaces

To have a more complete picture of the most employed finite elements, we can look at the following list (for a triangulation  $\mathcal{T}_h$  of  $\overline{\Omega}$  composed by **tetrahedra**):

- The space  $L_h^1 \subset H^1(\Omega)$  of (continuous) **piecewise-linear nodal** finite elements. Its dimension is the number of **vertices** in  $\mathcal{T}_h$ .
- The space  $N_h^1 \subset H(\mathbf{curl}; \Omega)$  of (tangentially continuous) **Nédélec edge** finite elements of degree 1 [locally:  $\mathbf{a} + \mathbf{b} \times \mathbf{x}$ ]. Its dimension is the number of **edges** in  $\mathcal{T}_h$ .
- The space  $RT_h^1 \subset H(\text{div}; \Omega)$  of (normally continuous) **Raviart–Thomas face** finite elements of degree 1 [locally:  $\mathbf{a} + b\mathbf{x}$ ]. Its dimension is the number of **faces** in  $\mathcal{T}_h$ .

We have  $\mathbf{grad} L_h^1 \subset N_h^1$  and  $\mathbf{curl} N_h^1 \subset RT_h^1$ .

## Variational formulation for Maxwell equations

The **time-harmonic Maxwell equations** (3)

$$\begin{cases} \mathbf{curl} \mathbf{H} - i\omega\epsilon\mathbf{E} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl} \mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \end{cases}$$

can be rewritten in the vacuum (where  $\boldsymbol{\sigma} = \mathbf{0}$ ) as

$$\mathbf{curl} \mathbf{curl} \mathbf{E} - \omega^2\epsilon_0\boldsymbol{\mu}_0\mathbf{E} = -i\omega\boldsymbol{\mu}_0\mathbf{J}_e. \quad (8)$$

Splitting  $\mathbf{J}_e$  into its real and imaginary parts, we can solve two real-valued problems of the same form for the real and imaginary parts of  $\mathbf{E}$ . Let us denote by  $\mathbf{F}$  a general right hand side, and, for simplicity, let us introduce the **wave number**  $\kappa = \sqrt{\boldsymbol{\mu}_0\epsilon_0}|\omega|$  (the reciprocal of the wave length).

## Variational formulation for Maxwell equations (cont'd)

Adding the electrical boundary condition  $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ , the variational formulation reads

$$\begin{aligned} \text{find } \mathbf{E} \in H_0(\mathbf{curl}; \Omega) : \\ \int_{\Omega} [\mathbf{curl} \mathbf{E} \cdot \mathbf{curl} \mathbf{v} - \kappa^2 \mathbf{E} \cdot \mathbf{v}] = \int_{\Omega} \mathbf{F} \cdot \mathbf{v} \quad (9) \\ \text{for each } \mathbf{v} \in H_0(\mathbf{curl}; \Omega), \end{aligned}$$

where  $H_0(\mathbf{curl}; \Omega) = \{\mathbf{v} \in H(\mathbf{curl}; \Omega) \mid \mathbf{v} \times \mathbf{n} = \mathbf{0}\}$ .

- Problem (9) is associated to a bilinear form that **is not coercive** in  $H(\mathbf{curl}; \Omega)$  [ $-\kappa^2$  has the “wrong” sign...]. What we can say about existence and uniqueness of a solution?

## Variational formulation for Maxwell equations (cont'd)

Consider the Maxwell **eigenvalue** problem

$$\begin{cases} \mathbf{curl} \mathbf{curl} \mathbf{E} = \eta \mathbf{E} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \quad (10)$$

The classical **Hilbert–Schmidt theory** can be applied to obtain

- besides  $\eta_0 = 0$ , there exists a sequence of positive, increasing and diverging to  $\infty$  eigenvalues  $\eta_m$  of problem (10).

**Fredholm alternative theory** can be used to prove

- when  $\kappa \neq \sqrt{\eta_m}$ ,  $m = 0, 1, 2, \dots$ , **there exists a unique solution** of problem (9).



## Numerical approximation of Maxwell equations

Numerical approximation of (9) (and of (10)) is important in order to simulate real physical devices and obtain informations for shape optimization (for instance, an electromagnetic cavity is a model for microwave ovens).

The finite element approximation problem with edge elements reads

find  $\mathbf{E}_h \in V_h$  :

$$\int_{\Omega} [\mathbf{curl} \mathbf{E}_h \cdot \mathbf{curl} \mathbf{v}_h - \kappa^2 \mathbf{E}_h \cdot \mathbf{v}_h] = \int_{\Omega} \mathbf{F} \cdot \mathbf{v}_h \quad (11)$$

$\forall \mathbf{v}_h \in V_h$ ,

where

$$V_h := N_h^1 \cap H_0(\mathbf{curl}; \Omega).$$

## Numerical approximation of Maxwell equations (cont'd)

For  $\kappa \neq \sqrt{\eta_m}$ ,  $m = 0, 1, 2, \dots$ , it is possible to prove **existence** and **uniqueness** of the solution to the discrete problem (11), as well as its **convergence** to the solution of problem (9) and the associated **error estimate**. Precisely, for  $h$  small enough we have

$$\|\mathbf{E} - \mathbf{E}_h\|_{H(\mathbf{curl}, \Omega)} \leq C \inf_{\mathbf{v}_h \in V_h} \|\mathbf{E} - \mathbf{v}_h\|_{H(\mathbf{curl}, \Omega)}, \quad (12)$$

therefore, for  $\mathbf{E}$  **smooth** enough,

$$\|\mathbf{E} - \mathbf{E}_h\|_{H(\mathbf{curl}, \Omega)} = O(h).$$

## Variational formulation for the eddy current problem

A variational formulation of the **eddy current problem**

$$\begin{cases} \operatorname{curl} \mathbf{H} - \sigma \mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \operatorname{curl} \mathbf{E} + i\omega \mu \mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

in terms of the **sole magnetic field  $\mathbf{H}$**  can be derived as follows.

Setting

$$W := \{\mathbf{w} \in H(\operatorname{curl}; \Omega) \mid \operatorname{curl} \mathbf{w}_I = \mathbf{0} \text{ in } \Omega_I\},$$

multiplying the **Faraday equation** by  $\bar{\mathbf{w}}$ , with  $\mathbf{w} \in W$ , integrating in  $\Omega$  and integrating by parts one finds

$$\int_{\Omega_C} \mathbf{E}_C \cdot \operatorname{curl} \bar{\mathbf{w}}_C + \int_{\Omega_I} \mathbf{E}_I \cdot \operatorname{curl} \bar{\mathbf{w}}_I + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \bar{\mathbf{w}} + \int_{\Omega} i\omega \mu \mathbf{H} \cdot \bar{\mathbf{w}} = 0,$$

## Variational formulation for the eddy current problem (cont'd)

thus

$$\int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl} \overline{\mathbf{w}}_C + \int_{\Omega} i\omega \mu \mathbf{H} \cdot \overline{\mathbf{w}} = 0,$$

as  $\mathbf{E} \times \mathbf{n} = \mathbf{0}$  on  $\partial\Omega$  and  $\mathbf{curl} \mathbf{w}_I = \mathbf{0}$  in  $\Omega_I$ .

Using the **Ampère equation** in  $\Omega_C$  for expressing  $\mathbf{E}_C$ , we end up with the following problem

$$\text{find } \mathbf{H}_e \in H(\mathbf{curl}; \Omega) : \mathbf{curl} \mathbf{H}_{e,I} = \mathbf{J}_{e,I} \text{ in } \Omega_I$$

$$\text{find } (\mathbf{H} - \mathbf{H}_e) \in W :$$

$$\begin{aligned} \int_{\Omega_C} \sigma^{-1} \mathbf{curl} \mathbf{H}_C \cdot \mathbf{curl} \overline{\mathbf{w}}_C + \int_{\Omega} i\omega \mu \mathbf{H} \cdot \overline{\mathbf{w}} \\ = \int_{\Omega_C} \sigma^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl} \overline{\mathbf{w}}_C \end{aligned} \tag{13}$$

for each  $\mathbf{w} \in W$ .

## Variational formulation for the eddy current problem (cont'd)

The first step simply says that  $\mathbf{J}_{e,I}$  satisfies the **necessary conditions for solvability**, namely,

$$\operatorname{div} \mathbf{J}_{e,I} = 0 \text{ in } \Omega_I, \quad \int_{(\partial\Omega_I)_j} \mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \quad \forall j = 1, \dots, p, \quad (14)$$

where  $(\partial\Omega_I)_j$ ,  $j = 0, 1, \dots, p$ , are the connected components of  $\Omega_I$ .

The second step is well-posed via the **Lax–Milgram lemma**, as the sesquilinear form

$$a_m(\mathbf{u}, \mathbf{w}) := \int_{\Omega_C} \sigma^{-1} \operatorname{curl} \mathbf{u}_C \cdot \operatorname{curl} \overline{\mathbf{w}}_C + \int_{\Omega} i\omega\mu \mathbf{u} \cdot \overline{\mathbf{w}}$$

is clearly **continuous** and **coercive** in  $W$ .

## Numerical approximation of the eddy current problem

Having denoted by  $\mathbf{J}_{e,l}^{(h)}$  a suitable (Raviart–Thomas) **finite element approximation** of  $\mathbf{J}_{e,l}$ , and having set  $W_h = N_h^1 \cap W$ , the approximate problem simply reads

$$\text{find } \mathbf{H}_e^{(h)} \in N_h^1 : \mathbf{curl} \mathbf{H}_{e,l}^{(h)} = \mathbf{J}_{e,l}^{(h)} \text{ in } \Omega_l$$

$$\text{find } (\mathbf{H}_h - \mathbf{H}_e^{(h)}) \in W_h :$$

$$\begin{aligned} \int_{\Omega_C} \sigma^{-1} \mathbf{curl} \mathbf{H}_{C,h} \cdot \mathbf{curl} \overline{\mathbf{w}_{C,h}} + \int_{\Omega} i\omega \mu \mathbf{H}_h \cdot \overline{\mathbf{w}_h} \\ = \int_{\Omega_C} \sigma^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl} \overline{\mathbf{w}_{C,h}} \end{aligned} \quad (15)$$

for each  $\mathbf{w}_h \in W_h$ .

## Numerical approximation of the eddy current problem (cont'd)

It is possible to prove **convergence** of the approximate solution  $\mathbf{H}_h$  of problem (15) to the solution  $\mathbf{H}$  of problem (13) and the associated **error estimate**. Precisely, choosing in  $\Omega_I$  as an approximation of the current density  $\mathbf{J}_{e,I}$  its Raviart–Thomas interpolant, namely,  $\mathbf{J}_{e,I}^{(h)} = \Pi^{RT_h^1} \mathbf{J}_{e,I}$ , and denoting by  $\Pi^{N_h^1}$  the Nédélec interpolation operator, one finds

$$\begin{aligned} \|\mathbf{H} - \mathbf{H}_h\|_{H(\mathbf{curl}; \Omega)} &\leq C \left( \|\mathbf{H}_C - \Pi^{N_h^1} \mathbf{H}_C\|_{H(\mathbf{curl}; \Omega_C)} \right. \\ &\quad \left. + \|\mathbf{H}_I - \Pi^{N_h^1} \mathbf{H}_I\|_{L^2(\Omega_I)} \right. \\ &\quad \left. + \|\mathbf{J}_{e,I} - \Pi^{RT_h^1} \mathbf{J}_{e,I}\|_{L^2(\Omega_I)} \right), \end{aligned} \quad (16)$$

therefore, for  $\mathbf{H}$  **smooth** enough,

$$\|\mathbf{H} - \mathbf{H}_h\|_{H(\mathbf{curl}; \Omega)} = O(h).$$

## Efficient implementation of the finite element eddy current problem

We can devise an **efficient implementation** of the finite element eddy current problem.

The ingredients are:

- compute an **edge element source field**  $\mathbf{H}_e^{(h)}$ , constructed in  $\Omega_I$  in correspondence to  $\mathbf{J}_{e,I}^{(h)}$  and extended by 0 on the edges internal to  $\Omega_C$ ;
- compute an **edge element basis**  $\mathbf{T}_\alpha$  of the first de Rham cohomology group of  $\Omega_I$  (the set of curl-free vector fields that are not gradients).

**Note:**

- The dimension of the first de Rham cohomology group of  $\Omega_I$  is equal to  $g$ !



## Efficient implementation of the finite element eddy current problem (cont'd)

Then a **basis** for the space  $W_h$  is constructed as follows:

- for all the edges internal to  $\Omega_C$  (and **not** on  $\Gamma$ ), the **Nédélec basis function** of the lowest degree, extended by 0 on the edges on  $\Gamma$  and internal to  $\Omega_I$ ;
- for all the nodes internal to  $\Omega_I$  and on  $\Gamma$  (except one), the **gradient** of the **Lagrange basis function** of degree 1, extended by 0 on the edges internal to  $\Omega_C$ ;
- the **de Rham cohomology basis functions**  $\mathbf{T}_\alpha$ ,  $\alpha = 1, \dots, g$ , extended by 0 on the edges internal to  $\Omega_C$ .

## Edge element loop fields and source fields

For determining the source field  $\mathbf{H}_e^{(h)}$  and the basis  $\mathbf{T}_\alpha$  of discrete loop fields we propose the following procedure.

### Tools:

- homology theory
  - generators of the first homology group of  $\overline{\Omega}_I$  (denoted by  $\sigma_n$ ,  $n = 1, \dots, g$ ) and  $\mathbb{R}^3 \setminus \Omega_I$  (denoted by  $\hat{\sigma}_\alpha$ ,  $\alpha = 1, \dots, g$ )
- graph theory applied to the mesh
  - a spanning tree (denoted by  $\mathcal{S}_h$ ) of the graph given by the edges of the mesh in  $\overline{\Omega}_I$
- direct elimination procedure
  - a direct algorithm of Webb and Forghani and an explicit formula.

## The fundamental discrete problem

The *main problem* now is: given  $\mathbf{J}_h \in RT_h^1$  satisfying the necessary solvability conditions (14), find  $\mathbf{Z}_h \in N_h^1$  such that

$$\begin{aligned} \operatorname{curl} \mathbf{Z}_h &= \mathbf{J}_h && \text{in } \Omega_I \\ \oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} &= \kappa_n && \forall n = 1, \dots, g \\ \int_{e'} \mathbf{Z}_h \cdot d\mathbf{s} &= 0 && \forall e' \in \mathcal{S}_h, \end{aligned} \quad (17)$$

where  $\kappa_1, \dots, \kappa_g$  are real numbers.

## Back to source fields and loop fields

Clearly,

- a **discrete source field**  $\mathbf{H}_{e,l}^{(h)}$  can be computed by solving (17), for  $\mathbf{J}_h = \mathbf{J}_{e,l}^{(h)}$  and any choice of  $\kappa_n$ .

But also

- a basis  $\mathbf{T}_\alpha$  of the **first de Rham cohomology group** of  $\Omega_l$  can be determined by solving (17) with  $\mathbf{J}_h = \mathbf{0}$  and  $\kappa_n = m_{n,\alpha}$ , for any choice of a non-singular matrix  $M = (m_{n,\alpha})$ .

A further important result is that

- if  $M$  is the matrix of the **linking numbers** of  $\sigma_n$  and  $\hat{\sigma}_\alpha$ , we have an **explicit formula** for the degrees of freedom of  $\mathbf{T}_\alpha$ .

## Webb–Forghani algorithm

When solving the problem  $\mathbf{curl} \mathbf{Z}_h = \mathbf{J}_h$  we have to match **two Raviart–Thomas elements**, hence their fluxes across each face of  $\mathcal{T}_h$  have to be the same.

Since the **Stokes theorem** assures that

$$\int_{e_1} \mathbf{Z}_h \cdot d\mathbf{s} + \int_{e_2} \mathbf{Z}_h \cdot d\mathbf{s} + \int_{e_3} \mathbf{Z}_h \cdot d\mathbf{s} = \int_f \mathbf{J}_h \cdot \mathbf{n}_f, \quad (18)$$

where  $\partial f = e_1 \cup e_2 \cup e_3$  and  $\mathbf{n}_f$  is the unit normal vector on  $f$  (with consistent orientation), we deduce that the corresponding linear system has exactly **three non-zero values** for each row.

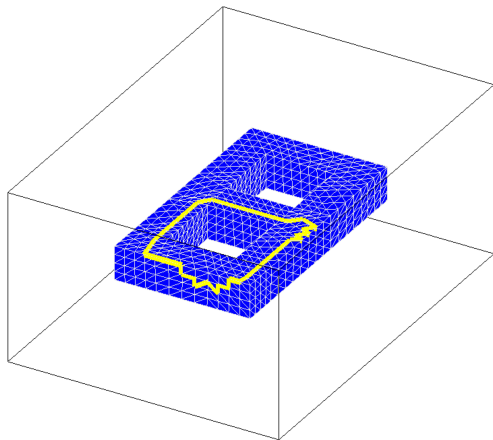
- The Webb–Forghani algorithm is a simple **elimination procedure** for solving the linear system (17), and it is quite efficient, as the computational costs is **linearly dependent** on the number of unknowns.

## Webb–Forghani algorithm (cont'd)

Let us describe the **Webb–Forghani algorithm**:

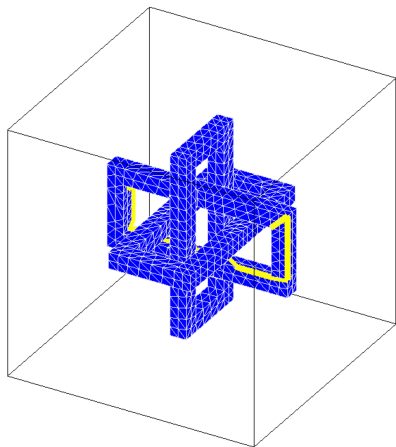
- ① set value 0 to the unknowns corresponding to an edge belonging to the **spanning tree**
- ② take a face  $f$  for which at least one edge unknown has not yet been assigned
  - ① if **exactly one edge unknown** is **not** determined, compute its value from the Stokes theorem as indicated before
  - ② otherwise, **pass** to another face
- ③ if the iterations stop before the end, **check** if one of the “homological” equations  $\oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n$  permits to restart
- ④ if the iterations stop before the end, use the **explicit formula** to restart.

## Geometries



Case A: 2-torus (one homological cycle  $\sigma_n$  is drawn). First Betti number of  $\Omega_I$ : 2.

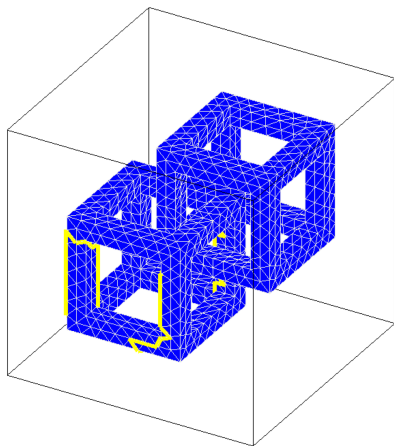
## Geometries (cont'd)



Case B: Borromean rings (one homological cycle  $\sigma_n$  is drawn). First Betti number of  $\Omega_I$ : 3.

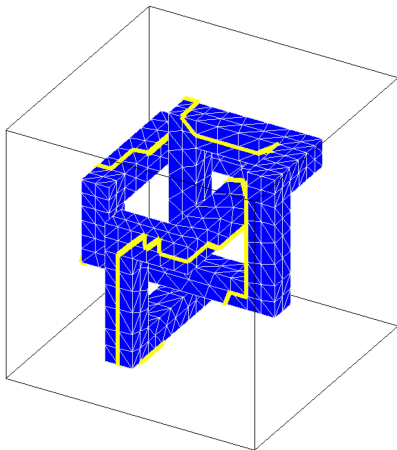


## Geometries (cont'd)



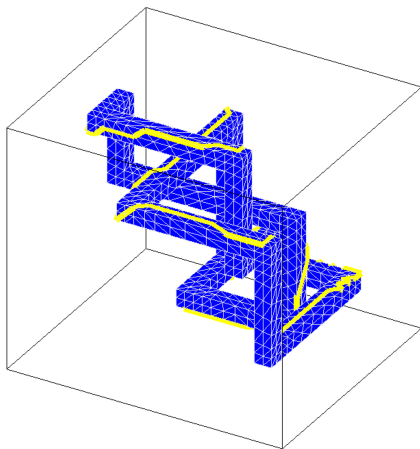
Case C: two-5-tori link (one homological cycle  $\sigma_n$  is drawn). First Betti number of  $\Omega_I$ : 10.

## Geometries (cont'd)



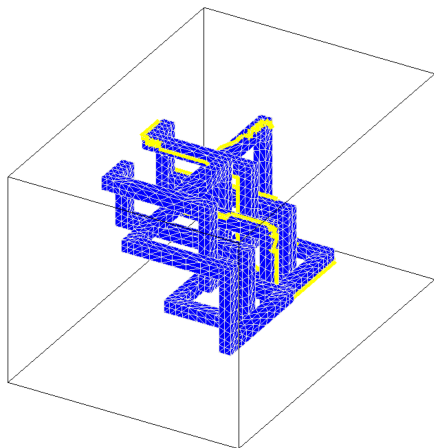
Case D: trefoil knot (one homological cycle  $\sigma_n$  is drawn). First Betti number of  $\Omega_I$ : 1.

## Geometries (cont'd)



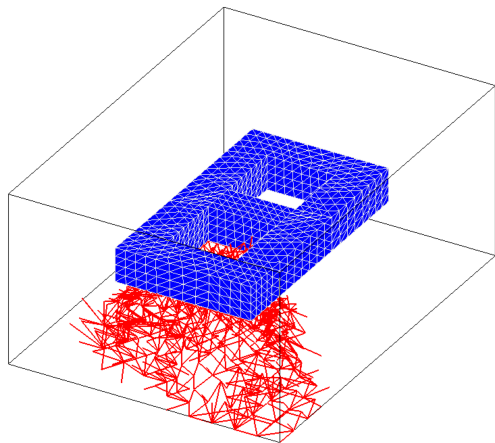
Case E: knot  $4_1$  (one homological cycle  $\sigma_n$  is drawn). First Betti number of  $\Omega_I$ : 1.

## Geometries (cont'd)



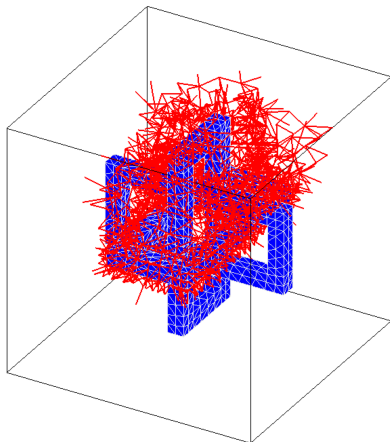
Case F: two- $4_1$ -knots link (one homological cycle  $\sigma_n$  is drawn). First Betti number of  $\Omega_I$ : 2.

## Computed loop fields



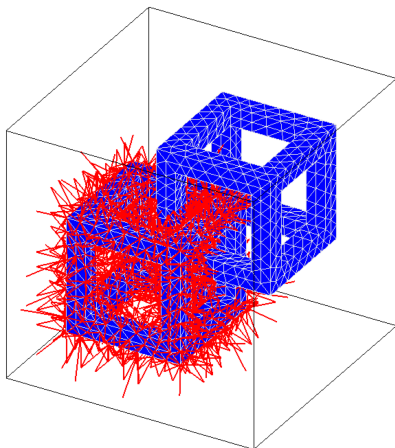
Support of a loop field. Case A: 2-torus. First Betti number of  $\Omega_I$ : 2.

## Computed loop fields (cont'd)



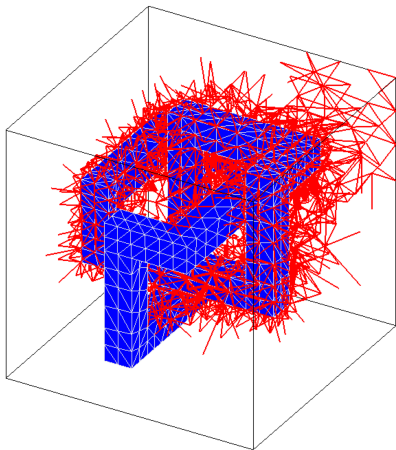
Support of a loop field. Case B: Borromean rings. First Betti number of  $\Omega_I$ : 3.

## Computed loop fields (cont'd)



Support of a loop field. Case C: two-5-tori link. First Betti number of  $\Omega_I$ : 10.

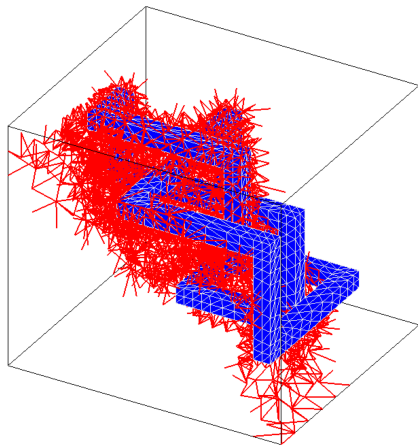
## Computed loop fields (cont'd)



Support of a loop field. Case D: trefoil knot. First Betti number of  $\Omega_I$ : 1.

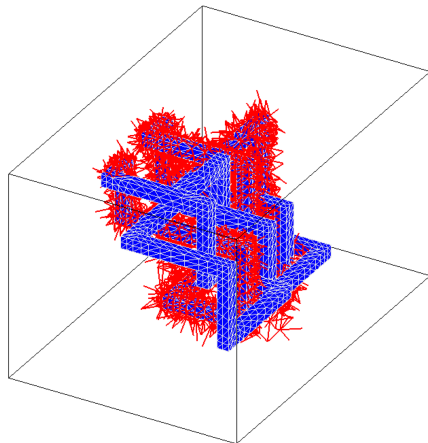


## Computed loop fields (cont'd)



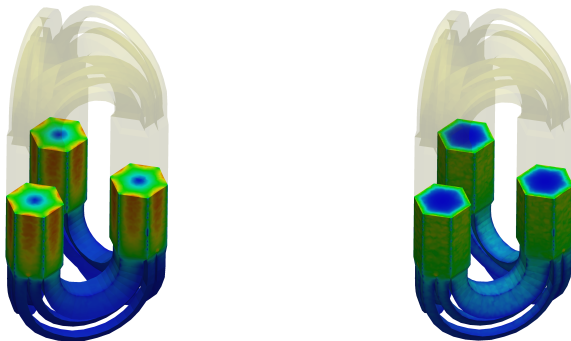
Support of a loop field. Case E: knot  $4_1$ . First Betti number of  $\Omega_I$ : 1.

## Computed loop fields (cont'd)



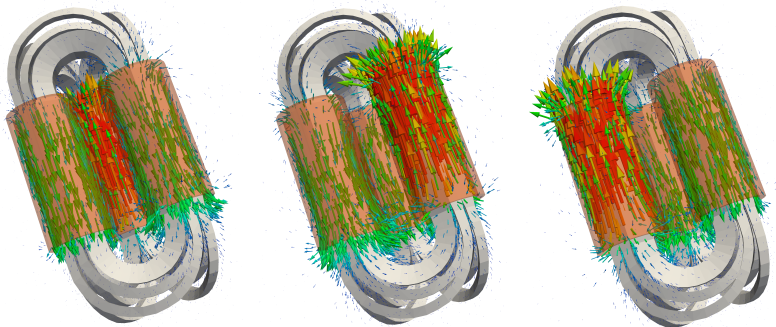
Support of a loop field. Case F: two-4<sub>1</sub>-knots link. First Betti number of  $\Omega_I$ : 2.

## Computed eddy current solution



The average eddy current along a period of time, for different magnetic permeabilities (Hexaformer transformer). First Betti number of  $\Omega_I$ : 16.

## Computed eddy current solution (cont'd)



The magnetic field at three different times (Hexaformer transformer).

First Betti number of  $\Omega_I$ : 16.

Finite elements

Maxwell and eddy current equations

Harmonic fields: where the topology comes into play

Edge finite elements

Variational formulations and numerical approximation

## Edge finite elements

- The explicit construction of a basis for the edge element space  $N_h^1$  is **easily** done.

In fact, it can be proved that the basis function  $\Phi_{i,j}$  associated to the edge  $e_{i,j}$  joining the nodes  $\mathbf{x}_i$  and  $\mathbf{x}_j$  and satisfying

$\int_{e_{i,j}} \Phi_{i,j} \cdot d\mathbf{s} = 1$  is given by

$$\Phi_{i,j} = \varphi_i \mathbf{grad} \varphi_j - \varphi_j \mathbf{grad} \varphi_i, \quad (19)$$

where  $\varphi_i$  is the **piecewise-linear nodal basis function** associated to the node  $\mathbf{x}_i$ .

For  $r \geq 1$  denote by  $\tilde{\mathbb{P}}_r$  the space of homogeneous polynomials of degree  $r$  and define

$$S_r := \{\mathbf{q} \in (\tilde{\mathbb{P}}_r)^3 \mid \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0\}, \quad R_r := (\mathbb{P}_{r-1})^3 \oplus S_r.$$

The Nédélec finite elements (for a **tetrahedral** mesh, say,  $\mathcal{T}_h$ ) are

$$N_h^r := \{\mathbf{w}_h \in H(\mathbf{curl}; \Omega) \mid \mathbf{w}_h|_K \in R_r \quad \forall K \in \mathcal{T}_h\}. \quad (20)$$

- Let us specify the form of Nédélec edge elements and their degrees of freedom for  $r = 1$ .

The condition  $\mathbf{q} \cdot \mathbf{x} = 0$  for  $\mathbf{q} \in (\tilde{\mathbb{P}}_1)^3$  says that  $\mathbf{q} = \mathbf{a} \times \mathbf{x}$  with  $\mathbf{a} \in \mathbb{R}^3$ . Hence the space  $R_1$  is given by the polynomials of the form

$$\mathbf{q}(\mathbf{x}) = \mathbf{b} + \mathbf{a} \times \mathbf{x}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3. \quad (21)$$

## An explicit formula for the loop fields

Let us show how to determine the explicit formula for the solution to (17) in the case  $\mathbf{J}_h = \mathbf{0}$ . The idea is the following: the **Biot–Savart law** gives the magnetic field generated by a unitary density current **concentrated along the edge cycle**  $\hat{\sigma}_\alpha$  (a generator of the first homology group of  $\mathbb{R}^3 \setminus \Omega_I$ ) by means of the formula:

$$\hat{\mathbf{H}}(\mathbf{x}) = \frac{1}{4\pi} \oint_{\hat{\sigma}_\alpha} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y, \quad \mathbf{x} \notin \hat{\sigma}_\alpha.$$

Since the cycle  $\hat{\sigma}_\alpha$  can be chosen **external** to  $\overline{\Omega_I}$ , one has  $\mathbf{curl} \hat{\mathbf{H}} = \mathbf{0}$  in  $\Omega_I$ . Moreover, on each cycle  $\gamma \subset \overline{\Omega_I}$  that is **linking the current** passing in  $\hat{\sigma}_\alpha$  one finds  $\oint_\gamma \hat{\mathbf{H}} \cdot d\mathbf{s} \neq 0$ , hence  $\hat{\mathbf{H}}$  is a **loop field** in  $\Omega_I$ .

[There are cycles  $\gamma$  with the required property: for instance, one of the generators of the first homology group of  $\overline{\Omega_I}$ .]



## An explicit formula for the loop fields (cont'd)

The Nédélec interpolant  $\Pi_h^{N1} \hat{\mathbf{H}}$  is a **finite element loop field** in  $\Omega_I$  (curl and line integrals are conserved). For each  $e \in \mathcal{T}_h$ , its degrees of freedom are given by

$$\hat{q}_e = \frac{1}{4\pi} \int_e \left( \oint_{\hat{\sigma}_\alpha} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x.$$

This resembles the formula for computing the **linking number** between  $\hat{\sigma}_\alpha$  and another disjoint cycle  $\sigma$ :

$$\ell_{\mathcal{K}}(\sigma, \hat{\sigma}_\alpha) = \frac{1}{4\pi} \int_\sigma \left( \oint_{\hat{\sigma}_\alpha} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x.$$

- The linking number is an **integer** that represents the number of times that each cycle **winds** around the other.

## An explicit formula for the loop fields (cont'd)

- Is it possible to **reduce** the determination of the finite element loop fields to the computation of suitable linking numbers?

Consider the **spanning tree**  $\mathcal{S}_h$ , its root  $v_*$ , and define in the vertices of  $\mathcal{T}_h \cap \overline{\Omega}_I$  the scalar function  $\phi_h \in L_h^1$  as  $\phi_h(v_*) = 0$  and

$$\phi_h(v_b) = \phi_h(v_a) + \hat{q}_{[v_a, v_b]} \quad \forall e' = [v_a, v_b] \in \mathcal{S}_h.$$

The Nédélec finite element  $\mathbf{Z}_h = \Pi_h^{N_h} \hat{\mathbf{H}} - \mathbf{grad} \phi_h$  is a **loop field** in  $\Omega_I$ , and its degrees of freedom **are equal to 0** for all the edges  $e'$  of the spanning tree  $\mathcal{S}_h$ .

For each  $e \in \mathcal{T}_h \cap \overline{\Omega}_I$ , define now by  $D_e$  the edge cycle constituted by: the edges from the **root** of the spanning tree  $\mathcal{S}_h$  to the **first vertex**  $v_e^-$  of  $e$ , the edge  $e$ , the edges from the **second vertex**  $v_e^+$  of  $e$  to the **root** of the spanning tree  $\mathcal{S}_h$ . In particular,  $D_{e'}$  is a trivial cycle if  $e' \in \mathcal{S}_h$ .

## An explicit formula for the loop fields (cont'd)

When  $e \notin \mathcal{S}_h$  the cycle  $D_e$  is constituted by edges **all belonging to the spanning tree** (except  $e$ ): hence we have

$$\begin{aligned} & \frac{1}{4\pi} \oint_{D_e} \left( \oint_{\hat{\sigma}_\alpha} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x \\ &= \hat{\mathbf{q}}_e + \sum_{e' \in D_e \cap \mathcal{S}_h} \hat{\mathbf{q}}_{e'} \\ &= \hat{\mathbf{q}}_e + \sum_{e' \in D_e \cap \mathcal{S}_h} (\phi_h(\mathbf{v}_{e'}^+) - \phi_h(\mathbf{v}_{e'}^-)) \\ &= \hat{\mathbf{q}}_e + (\phi_h(\mathbf{v}_e^-) - \phi_h(\mathbf{v}_e^+)) = \int_e \mathbf{Z}_h \cdot d\mathbf{s}, \end{aligned}$$

and thus the degrees of freedom of  $\mathbf{Z}_h$  are given by

$$\int_e \mathbf{Z}_h \cdot d\mathbf{s} = \ell_\kappa(D_e, \hat{\sigma}_\alpha).$$

In particular, the loop field  $\mathbf{Z}_h$  thus defined satisfies problem (17) with  $\kappa_n = m_{n,\alpha} = \ell_\kappa(\sigma_n, \hat{\sigma}_\alpha)$ , a **non-singular matrix**.

- Selecting  $\alpha = 1, \dots, g$  we have an **explicit formula** for a basis of the **first de Rham cohomology group**.