Numerical analysis of problems in electromagnetism

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Finite elements

A finite element method is an approximation method for variational problems of the form

$$\text{find } u \in V \ : \ a(u,v) = \mathcal{F}(v) \quad \forall\, v \in V\,, \tag{1}$$

where the real/complex vector space V, the bilinear/sesquilinear form $a(\cdot,\cdot)$ and the linear/antilinear functional $\mathcal{F}(\cdot)$ are data of the problem.

Its basic ingredients are:

- a triangulation of the computational domain Ω (mesh)
- a (finite dimensional) vector space V_h constituted by piecewise-polynomial functions.

Finite elements (cont'd)

The finite element method thus reads

$$\text{find } u_h \in V_h \,:\, a_h(u_h, v_h) = \mathcal{F}_h(v_h) \quad \forall\, v_h \in V_h\,. \qquad (2)$$

Here:

- $a_h(\cdot,\cdot)$ and $\mathcal{F}_h(\cdot)$ are suitable approximations of $a(\cdot,\cdot)$ and $\mathcal{F}(\cdot)$ (often, they coincide with them).

Remark. A first natural requirement is that V_h must be a "good" approximation of V in the sense that

$$\operatorname{dist}(v, V_h) \to 0 \quad \forall\, v \in V\,. \qquad (3)$$

- It is not necessary that $V_h \subset V$, but very often this is the case.

Degrees of freedom and basis functions

- In order to operate with V_h, it is necessary to find a basis of it (easy to construct and suitable for computations...).

Denoting by N_h the dimension of V_h, it is enough to find N_h linear functionals $\mathcal{G}_i$ such that

$$v_h \in V_h \, , \; \mathcal{G}_i(v_h) = 0 \;\; \forall \, i = 1, \ldots, N_h \;\Longrightarrow\; v_h = 0 \, . \qquad (4)$$

[The $\mathcal{G}_i$ are called degrees of freedom.]

The basis is then given by the functions $\varphi_j \in V_h$ such that

$$\mathcal{G}_i(\varphi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \, . \end{cases} \qquad (5)$$

[Hint: check directly that φ_j are linearly independent...]

Nodal degrees of freedom

A natural choice (not the only possible one... we will see another example later on) of the degrees of freedom is the following: having selected N_h nodes $\mathbf{x}_i$ in the computational domain Ω, define

$$\mathcal{G}_i(\varphi) = \varphi(\mathbf{x}_i)\,. \tag{6}$$

[This definition requires that the point values of φ are well-defined scalar quantities; this is surely true if φ is a continuous scalar function, not necessarily if $\varphi \in V$...]

Clearly, the choice of the nodes must be co-ordinated with the choice of V_h, in order to satisfy (4).

Nodal finite elements

Let us make precise the context in a specific case.

- Assume that $\Omega \subset \mathbb{R}^3$ and that the elements K of the triangulation are tetrahedra.

A natural choice of the finite elements is the following:

$$V_h = L_h^r := \{ v_h \in C^0(\Omega) \,|\, v_{h|K} \in \mathbb{P}_r \ \forall \ K \} \,, \tag{7}$$

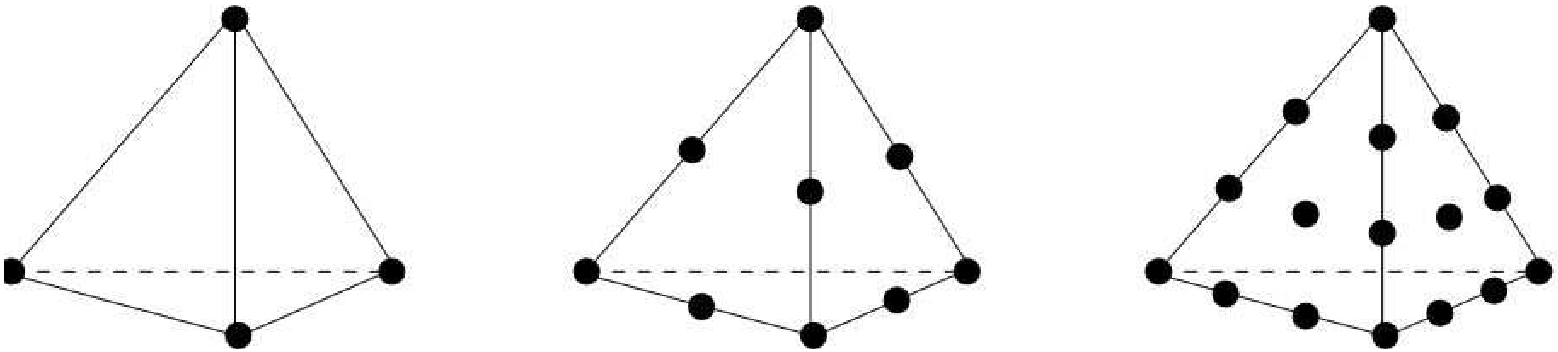
having denoted by $\mathbb{P}_r$ the set of polynomials of degree less than or equal to r, $r \geq 1$.

Nodal finite elements (cont'd)

It is not difficult to determine how to choose the nodes in this situation: for instance,

- $r = 1$: the vertices of all the tetrahedra
- $r = 2$: the vertices of all the tetrahedra and the middle points of all the edges
- $r = 3$: the vertices of all the tetrahedra, all the points dividing an edge in three equal parts and the barycenters of all the faces.

Nodal finite elements (cont'd)



The degrees of freedom for tetrahedra ($r = 1$, $r = 2$, $r = 3$).
Only the visible nodes are indicated.

Exercise. Condition (4) is satisfied. [Hint: show that an element of $\mathbb{P}_r$ vanishing at the nodes of a face must vanish on that face...]

Nodal finite elements (cont'd)

Remark. In the proof of the exercise one verifies that it is possible to construct element-by-element a polynomial $q \in \mathbb{P}_r$ by assigning the value of its nodal degrees of freedom, and that on the interelements it is uniquely determined (if it vanishes on the nodes of a face, then it vanishes on the whole face...).
Hence putting the pieces together one finds a continuous function, namely, an element of the finite element space V_h defined in (7).
This element is uniquely determined by the values of the assigned degrees of freedom: in other words, the total number of the nodal degrees of freedom is equal to the dimension of V_h.

Nodal finite elements (cont'd)

Remark. Indeed, for the finite elements introduced in (7), with nodal degrees of freedom, a more restrictive condition than (4) is satisfied. In fact, denoting by N_K the number of nodes belonging to the element K, one has

$$q \in \mathbb{P}_r \ , \ \mathcal{G}_i(q) = 0 \ \ \forall\, i = 1, \ldots, N_K \implies q = 0 \, ,$$

and consequently

$$v_h \in V_h \ , \ \mathcal{G}_i(v_{h|K}) = 0 \ \ \forall\, i = 1, \ldots, N_K \implies v_{h|K} = 0 \, . \qquad (8)$$

Therefore, it is easily seen that the basis functions have a "small" support: φ_i is non-vanishing only in the elements K of the triangulation that contain the node $\mathbf{x}_i$.

Approximation error

Question. Having done the choice

$$V_h = L_h^r := \{v_h \in C^0(\Omega) \,|\, v_{h|K} \in \mathbb{P}_r \ \forall\ K\}$$

with nodal degrees of freedom, is condition (3) satisfied?

To find an answer, let us begin with this remark. Denote by $\mathcal{V}$ the space of "smooth" functions and suppose that each function in V can be approximated by an element of $\mathcal{V}$ [this is very often the case for partial differential equations expressed in variational form: but there are exceptions...].

Then, given $v \in V$, a proof of (3) can start observing that

$$\text{dist}(v, V_h) \leq \text{dist}(v, w) + \text{dist}(w, V_h) \,,$$

where $w \in \mathcal{V}$, and $\text{dist}(v, w)$ can be taken arbitrarily small.

Finite element interpolant

On the other hand,

$$\text{dist}(w, V_h) \leq \text{dist}(w, w_h) \quad \forall\, w_h \in V_h\,,$$

therefore the problem is to select a “good” approximation w_h of a smooth function w.

To this end, it is useful to consider the finite element interpolant of a function. It is defined as follows: given a function φ (say, continuous), the interpolant $\pi_h\varphi$ of φ is the unique function belonging to V_h such that

$$(\pi_h\varphi)(\mathbf{x}_i) = \varphi(\mathbf{x}_i) \quad \forall\, i = 1, \ldots, N_h\,. \tag{9}$$

[Existence and uniqueness of $\pi_h\varphi$ are a consequence of (4)...]

Interpolation operator

The interpolation operator $\pi_h : C^0(\overline{\Omega}) \to V_h$ is then trivially defined as the operator which associates to a function its interpolant:

$$\pi_h : \varphi \longrightarrow \pi_h \varphi \,. \tag{10}$$

It is readily seen that

$$\pi_h \varphi = \sum_{j=1}^{N_h} \varphi(\mathbf{x}_j) \varphi_j \,. \tag{11}$$

[Hint: just check that $\sum_{j=1}^{N_h} \varphi(\mathbf{x}_j)\varphi_j(\mathbf{x}_i) = \varphi(\mathbf{x}_i)$...]

Interpolation error

Let us focus now on the estimate of the interpolation error for a “smooth” function.

An estimate of the interpolation error depends on the characteristics of the space V, namely, depends on the distance defined in V. [Clearly, there are many distances defined in a vector space V: the right one is that making V a Hilbert space...]

Typically, for second order partial differential equations we have that V is a closed subspace of $H^1(\Omega)$, the Sobolev space of first order. (This is not always the case... we will see a different situation later on.)

Therefore one can think that

$$\text{dist}(w, \pi_h w) = \|w - \pi_h w\|_{1,\Omega} .$$

Interpolation error (cont'd)

It can be proved that for a "regular" family of triangulations and for the choice (7) with nodal degrees of freedom one has

$$\|w - \pi_h w\|_{1,\Omega} \leq C(w) h^r \qquad (12)$$

for each "smooth" function w, hence condition (3) is satisfied.

[A family of triangulations $\mathcal{T}_h$, $h > 0$, is said "regular" if

$$\frac{\text{diam } K}{\text{diam } B_K} \leq \text{const} \quad \forall\, K \in \mathcal{T}_h \;\forall\, h > 0\,,$$

where B_K denotes the largest ball contained in K: namely, the elements are not becoming more and more distorted as the mesh is refined.]

Interpolation error (cont'd)

It can be useful to look deeper at the interpolation error estimate (12), in order to make explicit the regularity of w that is sufficient for obtaining the result.

In this respect, it can be proved that (12) holds provided that w belongs to $L^2(\Omega)$ together with all its derivatives up to order $r+1$: in other words, the interpolation error is of order r (with respect to the natural $H^1(\Omega)$-norm) if the (Sobolev) regularity of the solution is equal to $r+1$.

This result will be useful for checking that the order of convergence of the finite element method is related to the (Sobolev) regularity of the exact solution.

Discretization error

What is missing now is an estimate of the discretization error, namely, the distance between the exact solution $u \in V$ of problem (1) and the approximate solution $u_h \in V_h$ of problem (2).

[Clearly, we expect that the approximation condition (3), dist $(v, V_h) \to 0$ for each $v \in V$, is a crucial one; but the discretization error cannot avoid reading also the type of differential problem we have at hand...]

The procedure we present is quite general (for linear problems). However, let us assume for the sake of simplicity that

$$a_h(\cdot,\cdot) = a(\cdot,\cdot) \ \ , \ \ \mathcal{F}_h(\cdot) = \mathcal{F}(\cdot) \ \ , \ \ V_h \subset V \, . \qquad (13)$$

Discretization error (cont'd)

[Note that the condition $V_h \subset V$ is clearly satisfied for the choice (7)...]

The argument of the so-called Céa lemma is the following.

By subtracting (2) from (1) (for $v = v_h \in V$) we have

$$a(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h\,. \tag{14}$$

[This property is often called consistency of the finite element scheme.]

Hence

$$\begin{aligned} a(u - u_h, u - u_h) &= a(u - u_h, u) \\ &= a(u - u_h, u - v_h) \quad \forall\, v_h \in V_h\,. \end{aligned} \tag{15}$$

Discretization error (cont'd)

Suppose now that

- V is a Hilbert space
- the (bilinear/sesquilinear) form $a(\cdot,\cdot)$ is
  - continuous, namely

$$|a(w,v)| \leq \gamma \, \|w\|_V \|v\|_V \quad \forall\, w,v \in V \tag{16}$$

  - coercive, namely

$$|a(v,v)| \geq \alpha \, \|v\|_V^2 \quad \forall\, v \in V\,. \tag{17}$$

[In particular, by Lax–Milgram lemma these conditions guarantee that there exists a unique solution u to (1) and a unique solution u_h to (2), for any linear/antilinear and continuous functional $\mathcal{F}$.]

Discretization error (cont'd)

From (15) one has

$$\begin{aligned} \alpha\|u-u_h\|_V^2 &\le a(u-u_h, u-u_h) \\ &= a(u-u_h, u-v_h) \\ &\le \gamma\,\|u-u_h\|_V\|u-v_h\|_V \quad \forall v_h \in V_h\,, \end{aligned}$$

hence

$$\|u-u_h\|_V \le \frac{\gamma}{\alpha}\,\text{dist}\,(u, V_h)\,, \tag{18}$$

and convergence is proved, provided that (3) holds.

Order of convergence

Suppose now that V is a closed subspace of $H^1(\Omega)$ and that (16) and (17) are satisfied.

If one is working with the finite elements (7) with nodal degrees of freedom, it is possible to estimate the order of convergence of the finite element method.

In fact, we start from (18) and we find

$$\begin{aligned} \|u - u_h\|_{1,\Omega} &\leq \frac{\gamma}{\alpha} \, \text{dist}\,(u, V_h) \\ &\leq \frac{\gamma}{\alpha} \, \|u - \pi_h u\|_{1,\Omega} \leq C(u)\, h^r \, , \end{aligned} \tag{19}$$

provided that $\mathcal{T}_h$ is a "regular" family of triangulations and the (Sobolev) regularity of u is equal to $r+1$.

Maxwell equations in electromagnetism

The complete Maxwell system of electromagnetism reads

$$\begin{cases} \dfrac{\partial \mathcal{D}}{\partial t} + \mathcal{J} = \mathrm{curl}\, \mathcal{H} & \text{Maxwell–Ampère equation} \\ \dfrac{\partial \mathcal{B}}{\partial t} + \mathrm{curl}\, \mathcal{E} = 0 & \text{Faraday equation} \\ \mathrm{div}\, \mathcal{D} = \rho & \text{Gauss electrical equation} \\ \mathrm{div}\, \mathcal{B} = 0 & \text{Gauss magnetic equation .} \end{cases}$$

- $\mathcal{H}$ and $\mathcal{E}$ are the magnetic field and electric field, respectively
- $\mathcal{B}$ and $\mathcal{D}$ are the magnetic induction and electric induction, respectively
- $\mathcal{J}$ and ρ are the (surface) electric current density and (volume) electric charge density, respectively.

Maxwell equations in electromagnetism (cont'd)

These fields are related through some constitutive equations: it is usually assumed a linear dependence like

$$\mathcal{D} = \boldsymbol{\varepsilon}\mathcal{E} \ , \ \ \mathcal{B} = \boldsymbol{\mu}\mathcal{H} \ , \ \ \mathcal{J} = \boldsymbol{\sigma}\mathcal{E} + \mathcal{J}_e \ ,$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are the electric permittivity and magnetic permeability, respectively, and $\boldsymbol{\sigma}$ is the electric conductivity.

[In general, $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are not constant, but are symmetric and uniformly positive definite matrices (with entries that are bounded functions of the space variable $\mathbf{x}$). Clearly, the conductivity $\boldsymbol{\sigma}$ is only present in conductors, and is identically vanishing in any insulator.]

- $\mathcal{J}_e$ is the applied electric current density.

Eddy currents

As observed in experiments and stated by the Faraday law, a time-variation of the magnetic field generates an electric field. Therefore, in each conductor a current density $\mathbf{J}_{\text{eddy}} = \boldsymbol{\sigma}\mathbf{E}$ arises; this term expresses the presence in conducting media of the so-called eddy currents.

This phenomenon, and the related heating of the conductor, was observed and studied in the mid of the nineteenth century by the French physicist L. Foucault, and in fact the generated eddy currents are also known as Foucault currents.

Slowly varying fields

In many real-life applications, the time of propagation of the electromagnetic waves is very small with respect to some characteristic time scale, or, equivalently, their wave length is much larger than the diameter of the physical domain.

Therefore one can think that the speed of propagation is infinite, and take into account only the diffusion of the electromagnetic fields, neglecting electromagnetic waves.

Rephrasing this concept, one can also say that, when considering time-dependent problems in electromagnetism, one can distinguish between "fast" varying fields and "slowly" varying fields. In the latter case, one is led to simplify the set of equations, neglecting time derivatives, or, depending on the specific situation at hand, one time derivative, either $\frac{\partial \mathcal{D}}{\partial t}$ or $\frac{\partial \mathcal{B}}{\partial t}$.

Eddy current approximation

Typically, problems of this type are peculiar of electrical engineering, where low frequencies are involved, but not of electronic engineering, where the frequency ranges in much larger bands.

Let us focus on the case in which the displacement current term $\frac{\partial \mathcal{D}}{\partial t}$ can be disregarded, while the time-variation of the magnetic induction is still important, as well as the related presence of eddy currents in the conductors.

- The resulting equations are called eddy current equations.

Eddy current approximation (cont'd)

A thumb rule for deciding wheter $\frac{\partial \mathcal{D}}{\partial t}$ can be dropped is the following: if L is a typical length in Ω (say, its diameter) and we choose the inverse of the angular frequency ω^{-1} as a typical time, it is possibile to disregard the displacement current term provided that

$$|\mathcal{D}||\omega| \ll |\mathcal{H}|L^{-1} \ , \ \ |\mathcal{D}||\omega| \ll |\boldsymbol{\sigma}\mathcal{E}| \, .$$

Using the Faraday equation, we can write $\mathcal{E}$ is terms of $\mathcal{H}$, finding

$$|\mathcal{E}|L^{-1} \approx |\omega||\boldsymbol{\mu}\mathcal{H}| \, .$$

Eddy current approximation (cont'd)

Hence, recalling that $\mathcal{D} = \boldsymbol{\varepsilon}\mathcal{E}$ and putting everything together, one should have

$$\mu_{\max}\,\varepsilon_{\max}\omega^2 L^2 \ll 1 \;\;,\;\; \sigma_{\min}^{-1}\varepsilon_{\max}|\omega| \ll 1\;,$$

where $\mu_{\max}$ and $\varepsilon_{\max}$ are uniform upper bounds in Ω for the maximum eigenvalues of $\boldsymbol{\mu}(\mathbf{x})$ and $\boldsymbol{\varepsilon}(\mathbf{x})$, respectively, and $\sigma_{\min}$ denotes a uniform lower bound in Ω_C for the minimum eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$.

Since the magnitude of the velocity of the electromagnetic wave can be estimated by $(\mu_{\max}\,\varepsilon_{\max})^{-1/2}$, the first relation is requiring that the wave length is large compared to L.

Eddy current approximation (cont'd)

Let us also note that for electrical industry applications some typical values of the parameters involved are $\mu_0 = 4\pi \times 10^{-7}$ H/m, $\varepsilon_0 = 8.9 \times 10^{-12}$ F/m, $\sigma_{\mathrm{copper}} = 5.7 \times 10^7$ S/m, $\omega = 2\pi \times 50$ rad/s (power frequency of 50 Hz), hence in that case

$$\frac{1}{\sqrt{\mu_0 \varepsilon_0}|\omega|} \approx 10^6 \,\mathrm{m} \ \ , \ \ \sigma_{\mathrm{copper}}^{-1} \varepsilon_0 |\omega| \approx 4.9 \times 10^{-17} \, ,$$

and dropping the displacement current term looks appropriate.

Though less apparent, the same is true for a typical conductivity in physiological problem, say, $\sigma_{\mathrm{tissue}} \approx 10^{-1}$ S/m, for which $\sigma_{\mathrm{tissue}}^{-1} \varepsilon_0 |\omega| \approx 2.8 \times 10^{-8}$.

Time-harmonic Maxwell and eddy current equations

When interested in time-periodic phenomena, it is assumed that

$$\begin{aligned} \mathcal{J}_e(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{J}_e(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{E}(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{E}(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{H}(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{H}(\mathbf{x}) \exp(i\omega t)] \, . \end{aligned} \tag{20}$$

- $\omega \neq 0$ is the (angular) frequency.

Inserting these relations in the Maxwell equations one obtains the so-called time-harmonic Maxwell equations

$$\begin{cases} \operatorname{curl} \mathbf{H} - i\omega\boldsymbol{\varepsilon}\mathbf{E} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \operatorname{curl} \mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \, . \end{cases} \tag{21}$$

Time-harmonic Maxwell and eddy current equations (cont'd)

As a consequence one has div$(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω, and the electric charge in conductors is defined by $\rho =$ div$(\boldsymbol{\varepsilon}\mathbf{E})$.

It can be proved that the time-harmonic Maxwell equations have a unique solution (provided that suitable boundary conditions are added, and that the conductor is not empty; we will come back later on to the case in which the conductor is empty).

On the other hand, dropping the displacement current term the time-harmonic eddy current equations are

$$\begin{cases} \text{curl}\, \mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \text{curl}\, \mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega\,. \end{cases} \tag{22}$$

Gauge conditions for the electric field

Let us spend some more words about eddy current equations.
Since in an insulator one has $\boldsymbol{\sigma} = \mathbf{0}$, it follows that $\mathbf{E}$ is not uniquely determined in that region ($\mathbf{E} + \nabla\psi$ is still a solution).
Some additional conditions ("gauge" conditions) are thus necessary: the most natural idea is to impose the conditions satisfied by the solution $\mathbf{E}$ of the Maxwell equations.
As in the insulator Ω_I we have no charges, the first additional condition is

$$\text{div}(\varepsilon_I \mathbf{E}_I) = 0 \qquad \text{in } \Omega_I \tag{23}$$

($\mathbf{E}_I$ means $\mathbf{E}_{|\Omega_I}$, and similarly for other quantities).

Topological gauge conditions for the electric field

Other gauge conditions are related to the topology of the insulator Ω_I. Denoting by Ω_C the conductor (strictly contained in the physical domain Ω, and surrounded by the insulator Ω_I) and by $\Gamma := \overline{\Omega_C} \cap \overline{\Omega_I}$, let us define

$$\mathcal{H}_I := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \text{curl}\,\mathbf{G}_I = \mathbf{0}, \text{div}(\boldsymbol{\varepsilon}_I \mathbf{G}_I) = 0$$
$$\mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \mathit{BC}_E(\mathbf{G}_I) = 0 \text{ on } \partial\Omega\}\,,$$

where BC_E denotes the boundary condition imposed on $\mathbf{E}_I$ (see later on for a precise description).
The topological gauge conditions can be written as

$$\boldsymbol{\varepsilon}_I \mathbf{E}_I \perp \mathcal{H}_I\,. \tag{24}$$

Topological gauge conditions for the electric field (cont'd)

Thus these conditions are ensuring that, if in addition one has curl $\mathbf{E}_I = \mathbf{0}$ in Ω_I, div$(\boldsymbol{\varepsilon}_I \mathbf{E}_I) = 0$ in Ω_I, $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on Γ and *BC*$_E(\mathbf{E}_I) = 0$ on $\partial\Omega$, then it follows $\mathbf{E}_I = \mathbf{0}$ in Ω_I.

- It can be shown that the orthogonality condition $\boldsymbol{\varepsilon}_I \mathbf{E}_I \perp \mathcal{H}_I$ is equivalent to impose that the flux of $\boldsymbol{\varepsilon}_I \mathbf{E}_I$ is vanishing on a suitable set of surfaces.
  [These surfaces depend on the choice of the boundary condition for $\mathbf{E}_I$; for instance, for $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ they are the connected components of $\partial\Omega \cup \Gamma$.]

Boundary conditions

We will distinguish between two types of boundary conditions.

- Electric. One imposes $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. [As a consequence, one also has $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$.]
- Magnetic (Maxwell). One imposes $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. [As a consequence, one also has $\boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = -(i\omega)^{-1}\mathbf{J}_e \cdot \mathbf{n}$ on $\partial\Omega$.]
- Magnetic (eddy currents). One imposes $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$. [Note that $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ implies $\mathbf{J}_e \cdot \mathbf{n} = 0$ on $\partial\Omega$.]

For eddy current equations, the notation $\mathit{BC}_E(\mathbf{E}_I)$ on $\partial\Omega$ therefore refers to $\mathbf{E}_I \times \mathbf{n}$ for the electric boundary condition, and to $\boldsymbol{\varepsilon}_I\mathbf{E}_I \cdot \mathbf{n}$ for the magnetic boundary conditions.

The spaces of harmonic fields

Let us consider a couple of questions.

- If a vector field satisfies curl $\mathbf{v} = \mathbf{0}$ and div $\mathbf{v} = 0$ in a domain, together with the boundary conditions $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on a part of the boundary and $\mathbf{v} \cdot \mathbf{n} = 0$ on the other part, is it non-trivial, namely, not vanishing everywhere in the domain? [A field like that is called harmonic field.]
- If that is the case, do harmonic fields appear in electromagnetism?

Both questions have an affermative answer.

The spaces of harmonic fields (cont'd)

Let us start from the first question.

If the domain $\mathcal{O}$ is homeomorphic to a three-dimensional ball, a curl-free vector field $\mathbf{v}$ must be a gradient of a scalar function ψ, that must be harmonic due to the constraint on the divergence.

If the boundary condition is $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{O}$, which in this case is a connected surface, then it follows $\psi = \text{const.}$ on $\partial\mathcal{O}$, and therefore $\psi = \text{const.}$ in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

The spaces of harmonic fields (cont'd)

If the boundary condition is $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\mathcal{O}$, then ψ satisfies a homogeneous Neumann boundary condition and thus $\psi = \text{const.}$ in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

The same result follows if the boundary conditions are $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_N, and Γ_D is a connected surface: in fact, we still have $\psi = \text{const.}$ on Γ_D and $\text{grad}\,\psi \cdot \mathbf{n} = 0$ on Γ_N, hence ψ satisfies a mixed boundary value problem and we obtain $\psi = \text{const.}$ in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

The spaces of harmonic fields (cont'd)

However, the problem is different in a more general geometry.

In fact, take the magnetic field generated in the vacuum by a current of constant intensity I^0 passing along the x_3-axis: as it is well-known, for $x_1^2 + x_2^2 > 0$ it is given by

$$\mathbf{H}(x_1, x_2, x_3) = \frac{I^0}{2\pi} \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0 \right) \ .$$

The spaces of harmonic fields (cont'd)

It is easily checked that, as Maxwell equations require, $\text{curl}\,\mathbf{H} = \mathbf{0}$ and $\text{div}\,\mathbf{H} = 0$.

Let us consider now the torus $\mathcal{T}$ obtained by rotating around the x_3-axis the disk of centre $(a, 0, 0)$ and radius b, with $0 < b < a$. One sees at once that $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}$; hence we have found a non-trivial harmonic field $\mathbf{H}$ in $\mathcal{T}$ satisfying $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}$.

The spaces of harmonic fields (cont'd)

On the other hand, consider now the electric field generated in the vacuum by a pointwise charge ρ_0 placed at the origin. For $\mathbf{x} \neq \mathbf{0}$ it is given by

$$\mathbf{E}(x_1, x_2, x_3) = \frac{\rho_0}{4\pi\varepsilon_0} \frac{\mathbf{x}}{|\mathbf{x}|^3} \ ,$$

where ε_0 is the electric permittivity of the vacuum.

It satisfies $\operatorname{div} \mathbf{E} = 0$ and $\operatorname{curl} \mathbf{E} = \mathbf{0}$, and moreover $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on the boundary of $\mathcal{C} := B_{R_2} \setminus \overline{B_{R_1}}$, where $0 < R_1 < R_2$ and $B_R := \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x}| < R\}$ is the ball of centre $\mathbf{0}$ and radius R. We have thus found a non-trivial harmonic field $\mathbf{E}$ in $\mathcal{C}$ satisfying $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{C}$.

The spaces of harmonic fields (cont'd)

These two examples show that the geometry of the domain and the type of boundary conditions play an essential role when considering harmonic fields.

What are the relevant differences between the set $\mathcal{O}$, homeomorphic to a ball, and the sets $\mathcal{T}$ and $\mathcal{C}$?

For the former, the point is that in $\mathcal{T}$ we have a non-bounding cycle, namely, a cycle that is not the boundary of a surface contained in $\mathcal{T}$ (take for instance the circle of centre $\mathbf{0}$ and radius a in the (x_1, x_2)-plane).

In the latter case, the boundary of $\mathcal{C}$ is not connected.

The spaces of harmonic fields (cont'd)

Four types of spaces of harmonic fields are coming into play.

- For the electric field

$$\mathcal{H}_I^{(A)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \text{curl}\, \mathbf{G}_I = \mathbf{0}, \text{div}(\boldsymbol{\varepsilon}_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\},$$

$$\mathcal{H}_I^{(B)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \text{curl}\, \mathbf{G}_I = \mathbf{0}, \text{div}(\boldsymbol{\varepsilon}_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \boldsymbol{\varepsilon}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

The spaces of harmonic fields (cont'd)

- For the magnetic field

$$\mathcal{H}_I^{(C)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \text{curl}\,\mathbf{G}_I = \mathbf{0}, \text{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0$$
$$\boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma, \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,$$

$$\mathcal{H}_I^{(D)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \text{curl}\,\mathbf{G}_I = \mathbf{0}, \text{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0$$
$$\boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma, \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$$

All are finite dimensional! Their dimension is a topological invariant (precisely,... see below!).

The basis functions of the spaces of harmonic fields

Let us make precise which are the basis functions of $\mathcal{H}_I^{(D)}$ and $\mathcal{H}_I^{(C)}$.

For $\mathcal{H}_I^{(D)}$ one has first to introduce the "cutting" surfaces $\Xi_\alpha^* \subset \Omega_I$, $\alpha = 1, \ldots, n_{\Omega_I}$, with $\partial\Xi_\alpha^* \subset \partial\Omega \cup \Gamma$, such that every curl-free vector field in Ω_I has a global potential in $\Omega_I \setminus \cup_\alpha \Xi_\alpha^*$.

The number n_{Ω_I} is the number of (independent) non-bounding cycles in Ω_I, namely, the first Betti number of Ω_I, or, equivalently, the dimension of the first homology space of Ω_I.

These surfaces "cut" the non-bounding cycles in Ω_I.

The basis functions of the spaces of harmonic fields (cont'd)

The basis functions $\boldsymbol{\rho}^*_{\alpha,I}$ are the $(L^2(\Omega_I))^3$-extensions of $\text{grad}\, p^*_{\alpha,I}$, where $p^*_{\alpha,I}$ is the solution to

$$\begin{cases} \text{div}(\boldsymbol{\mu}_I \,\text{grad}\, p^*_{\alpha,I}) = 0 & \text{in } \Omega_I \setminus \Xi^*_\alpha \\ \boldsymbol{\mu}_I \,\text{grad}\, p^*_{\alpha,I} \cdot \mathbf{n}_I = 0 & \text{on } (\partial\Omega \cup \Gamma) \setminus \partial\Xi^*_\alpha \\ \left[\boldsymbol{\mu}_I \,\text{grad}\, p^*_{\alpha,I} \cdot \mathbf{n}_{\Xi^*}\right]_{\Xi^*_\alpha} = 0 & \\ \left[p^*_{\alpha,I}\right]_{\Xi^*_\alpha} = 1\,, & \end{cases} \qquad (25)$$

having denoted by $[\,\cdot\,]_{\Xi^*_\alpha}$ the jump across the surface Ξ^*_α and by $\mathbf{n}_{\Xi^*}$ the unit normal vector on Ξ^*_α.

The basis functions of the spaces of harmonic fields (cont'd)

The basis functions for $\mathcal{H}_I^{(C)}$ can be defined as follows.

First of all we have $\operatorname{grad} z_{l,I}$, the solutions to

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} z_{l,I}) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \operatorname{grad} z_{l,I} \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\ z_{l,I} = 0 & \text{on } \partial\Omega \setminus (\partial\Omega)_l \\ z_{l,I} = 1 & \text{on } (\partial\Omega)_l \ , \end{cases} \tag{26}$$

where $l = 1, \ldots, p_{\partial\Omega}$, and $p_{\partial\Omega} + 1$ is the number of connected components of $\partial\Omega$.

The basis functions of the spaces of harmonic fields (cont'd)

To complete the construction of the basis functions we have to proceed further.
For that, as in the preceding case, let us recall that in Ω_I there exists a set of "cutting" surfaces Ξ_q, with $\partial\Xi_q \subset \Gamma$, such that every curl-free vector field in Ω_I with vanishing tangential component on $\partial\Omega$ has a global potential in $\Omega_I \setminus \cup_q \Xi_q$.

These surfaces "cut" the $\partial\Omega$-independent non-bounding cycles in Ω_I (whose number is denoted by n_Γ).

The basis functions of the spaces of harmonic fields (cont'd)

Then introduce the functions $p_{q,I}$, $q = 1, \dots, n_\Gamma$, defined in $\Omega_I \setminus \Xi_q$ and solutions to

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} p_{q,I}) = 0 & \text{in } \Omega_I \setminus \Xi_q \\ \boldsymbol{\mu}_I \operatorname{grad} p_{q,I} \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \setminus \partial\Xi_q \\ p_{q,I} = 0 & \text{on } \partial\Omega \\ \left[\boldsymbol{\mu}_I \operatorname{grad} p_{q,I} \cdot \mathbf{n}_\Xi\right]_{\Xi_q} = 0 & \\ \left[p_{q,I}\right]_{\Xi_q} = 1 \ , & \end{cases} \tag{27}$$

having denoted by $[\,\cdot\,]_{\Xi_q}$ the jump across the surface Ξ_q and by $\mathbf{n}_\Xi$ the unit normal vector on Ξ_q.

The other basis functions $\boldsymbol{\rho}_{q,I}$ are the $(L^2(\Omega_I))^3$-extensions of $\operatorname{grad} p_{q,I}$ (computed in $\Omega_I \setminus \Xi_q$).

Vector potential formulation

Motivated by the fact that the magnetic induction $\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$ is divergence-free in Ω, a classical approach to the Maxwell equations and to eddy current problems is that based on the introduction of a vector magnetic potential $\mathbf{A}$ such that curl $\mathbf{A} = \boldsymbol{\mu}\mathbf{H}$. Often, this is also accompanied by the use of a scalar electric potential V_C in the conductor Ω_C, satisfying $-i\omega\mathbf{A}_C - \text{grad}\, V_C = \mathbf{E}_C$.

Summing up, one looks for $\mathbf{A}$ and V_C such that

$$\mathbf{E}_C = -i\omega\mathbf{A}_C - \text{grad}\, V_C \quad , \quad \boldsymbol{\mu}\mathbf{H} = \text{curl}\, \mathbf{A} \; . \qquad (28)$$

[Note that $\mathbf{A}$ and V_C are not uniquely defined...]

For the time being, let us focus on the eddy current equations. For the sake of definiteness we consider the electric boundary condition.

Vector potential formulation (cont'd)

Imposing the Ampère equation one has:

$$\text{curl}(\boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{A}) + i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\text{grad}\,V_C = \mathbf{J}_e \quad \text{in } \Omega\,.$$

On the other hand, from (28) we see at once that

$$\text{curl}\,\mathbf{E}_C = -i\omega\,\text{curl}\,\mathbf{A}_C = -i\omega\boldsymbol{\mu}_C\mathbf{H}_C\,,$$

thus the Faraday equation in Ω_C is satisfied. Moreover, $\boldsymbol{\mu}\mathbf{H}$ is equal to $\text{curl}\,\mathbf{A}$ in Ω, therefore it is a solenoidal vector field in Ω.

If we require $\mathbf{A}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, the boundary condition $\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$ is satisfied: in fact,

$$\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n} = \text{curl}\,\mathbf{A}_I \cdot \mathbf{n} = \text{div}_\tau(\mathbf{A}_I \times \mathbf{n}) = 0\,.$$

Vector potential formulation (cont'd)

[A remark on the relation

$$\text{curl}\,\mathbf{A}\cdot\mathbf{n} = \text{div}_\tau(\mathbf{A}\times\mathbf{n}) \quad \text{on } \partial\Omega\,,$$

which is very often used in electromagnetism.
Given a function η defined in $\overline{\Omega}$, we have

$$\begin{aligned}\int_{\partial\Omega}\text{curl}\,\mathbf{A}\cdot\mathbf{n}\,\eta &= \int_\Omega \text{div}(\eta\ \text{curl}\,\mathbf{A}) = \int_\Omega \text{grad}\,\eta\cdot\text{curl}\,\mathbf{A}\\ &= \int_{\partial\Omega}\text{grad}\,\eta\cdot(\mathbf{n}\times\mathbf{A}) = -\int_{\partial\Omega}\eta\ \text{div}_\tau(\mathbf{n}\times\mathbf{A})\,,\end{aligned}$$

and, since η is arbitrary, the conclusion follows.]

Don’t forget the Faraday equation!

A little bit surprisingly, what we have presented is not the complete formulation in terms of $\mathbf{H}$ and $\mathbf{E}_C$: something is still missing.

- In fact, the Faraday equation is not completely solved.

More precisely, in Ω_C we have solved the Faraday equation in differential form, but we are not imposing the Faraday equation in integral form for all the surfaces contained in Ω.

Let us see in more detail: the Faraday equation relates the flux of the magnetic induction through a surface with the line integral of the electric field on the boundary of that surface.

Don't forget the Faraday equation! (cont'd)

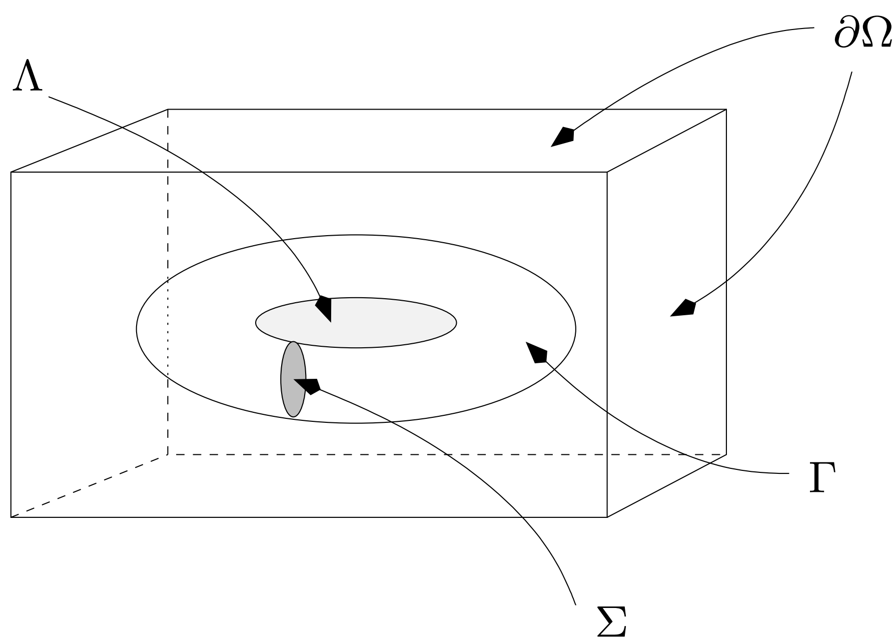
Since we know the magnetic field in the whole Ω, surfaces can stay everywhere in Ω; but we know the electric field only in Ω_C, therefore the boundary of the surface must stay in $\overline{\Omega_C}$.

On the other hand, since the Faraday equation (in differential form) is satisfied in Ω_C, for a surface contained in Ω_C everything is all right.

Thus we must verify if there are surfaces in Ω_I with boundary on Γ, and moreover such that this boundary is not the boundary of a surface in Ω_C [if this is not the case, the Divergence Theorem says that again everything is all right, as the magnetic induction is divergence free in Ω...].

Don't forget the Faraday equation! (cont'd)

- Conclusion: the Faraday equation has not been imposed on the "cutting" surface Λ! [The non-bounding cycle is the boundary of the surface Σ.]



Back to the vector potential formulation

It can be seen that the integral form of the Faraday equation on these surfaces is satisfied if

$$\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I^* = -\int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I^* \,,$$

where $\boldsymbol{\rho}_I^*$ is curl-free in Ω_I.
Let us verify if this condition holds when the $(\mathbf{A}, V_C)$ formulation is used: we have

$$\begin{aligned} \textstyle\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I^* &= \textstyle\int_{\Omega_I} i\omega \,\mathrm{curl}\, \mathbf{A}_I \cdot \boldsymbol{\rho}_I^* \\ &= i\omega \textstyle\int_\Gamma (\mathbf{n}_I \times \mathbf{A}_I) \cdot \boldsymbol{\rho}_I^* = i\omega \int_\Gamma (\mathbf{A}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I^* \\ &= -\textstyle\int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I^* - \int_\Gamma (\mathrm{grad}\, V_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I^* \,. \end{aligned}$$

Back to the vector potential formulation (cont'd)

On the other hand

$$
\begin{aligned}
\int_\Gamma \quad & (\text{grad}\, V_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I^* \\
&= \int_\Gamma (\boldsymbol{\rho}_I^* \times \mathbf{n}_I) \cdot \text{grad}\, V_C \\
&= -\int_\Gamma \text{div}_\tau (\boldsymbol{\rho}_I^* \times \mathbf{n}_I)\, V_C \\
&= -\int_\Gamma \text{curl}\, \boldsymbol{\rho}_I^* \cdot \mathbf{n}_I\, V_C = 0 \ .
\end{aligned}
$$

In conclusion, using of the $(\mathbf{A}, V_C)$ formulation guarantees that the Faraday equation is completely solved.

- This approach opens the problem of determining correct gauge conditions ensuring the uniqueness of $\mathbf{A}$ and V_C (these conditions can be necessary when considering numerical approximation, in order to avoid that the discrete problem becomes singular).

Gauge conditions

The most frequently used is the Coulomb gauge

$$\text{div}\,\mathbf{A} = 0 \qquad \text{in } \Omega\,. \tag{29}$$

In a general geometrical situation, this can be not enough for determining a unique vector potential $\mathbf{A}$ in Ω. In fact, there exist non-trivial irrotational, solenoidal vector fields with vanishing tangential component, namely, the elements of the space of harmonic fields

$$\mathcal{H}(e;\Omega) := \{\mathbf{w} \in (L^2(\Omega))^3 \mid \text{curl}\,\mathbf{w} = \mathbf{0}, \text{div}\,\mathbf{w} = 0, \\ \mathbf{w} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\ ,$$

Gauge conditions (cont'd)

whose dimension is given by the number of connected components of $\partial\Omega$ minus 1 (say, as stated before, $p_{\partial\Omega}$). Imposing orthogonality, namely, $\mathbf{A} \perp \mathcal{H}(e;\Omega)$, turns out to be equivalent to require

$$\int_{(\partial\Omega)_l} \mathbf{A} \cdot \mathbf{n} = 0 \qquad \forall\, l = 1, \ldots, p_{\partial\Omega}\,. \tag{30}$$

In conclusion, we are left with the problem

$$\begin{cases} \mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}) + i\omega\boldsymbol{\sigma}\mathbf{A} & \\ \qquad\qquad + \boldsymbol{\sigma}\,\mathrm{grad}\,V_C = \mathbf{J}_e & \text{in } \Omega \\ \mathrm{div}\,\mathbf{A} = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_l} \mathbf{A} \cdot \mathbf{n} = 0 & \forall\, l = 1, \ldots, p_{\partial\Omega} \\ \mathbf{A} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases} \tag{31}$$

Penalization

[Clearly, V_C is determined up to an additive constant in each connected component $\Omega_{C,j}$ of Ω_C, $j = 1, \ldots, p_\Gamma + 1$.]

The solenoidal constraint can be imposed by adding a penalization term. Introducing the constant $\mu_* > 0$, representing a suitable average in Ω of the entries of the matrix $\boldsymbol{\mu}$, the Coulomb gauge condition $\operatorname{div} \mathbf{A} = 0$ in Ω can be incorporated in the Ampère equation, which becomes

$$\begin{aligned} &\operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}) - \mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{A} + i\omega \boldsymbol{\sigma} \mathbf{A} + \boldsymbol{\sigma} \operatorname{grad} V_C \\ &\quad = \mathbf{J}_e \qquad \text{in } \Omega \,. \end{aligned}$$

A boundary condition for $\operatorname{div} \mathbf{A}$ is now necessary, and we impose

$$\operatorname{div} \mathbf{A} = 0 \qquad \text{on } \partial\Omega \,.$$

Penalization (cont'd)

Moreover one adds the two equations

$$\operatorname{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\operatorname{grad} V_C) = \operatorname{div}\mathbf{J}_{e,C} \quad \text{in } \Omega_C$$

$$(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\operatorname{grad} V_C)\cdot\mathbf{n}_C = \mathbf{J}_{e,C}\cdot\mathbf{n}_C + \mathbf{J}_{e,I}\cdot\mathbf{n}_I \quad \text{on } \Gamma\,,$$

that are necessary as, due to the modification in the Ampère equation, it is no more ensured that the electric field $\mathbf{E}_C = -i\omega\mathbf{A}_C - \operatorname{grad} V_C$ satisfies the necessary conditions

$$\operatorname{div}(\boldsymbol{\sigma}\mathbf{E}_C) = -\operatorname{div}\mathbf{J}_{e,C} \quad \text{in } \Omega_C$$

$$\boldsymbol{\sigma}\mathbf{E}_C\cdot\mathbf{n}_C = -\mathbf{J}_{e,C}\cdot\mathbf{n}_C - \mathbf{J}_{e,I}\cdot\mathbf{n}_I \quad \text{on } \Gamma\,.$$

Vector potential strong formulation

The complete $(\mathbf{A}, V_C)$ formulation is therefore

$$\left\{\begin{array}{ll}
\text{curl}(\boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{A}) - \mu_*^{-1}\,\text{grad}\,\text{div}\,\mathbf{A} & \\
\qquad\qquad +i\omega\boldsymbol{\sigma}\mathbf{A} + \boldsymbol{\sigma}\,\text{grad}\,V_C = \mathbf{J}_e & \text{in } \Omega \\
\text{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\text{grad}\,V_C) = \text{div}\,\mathbf{J}_{e,C} & \text{in } \Omega_C \\
(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\text{grad}\,V_C)\cdot\mathbf{n}_C & \\
\qquad\qquad = \mathbf{J}_{e,C}\cdot\mathbf{n}_C + \mathbf{J}_{e,I}\cdot\mathbf{n}_I & \text{on } \Gamma \\
\int_{(\partial\Omega)_l}\mathbf{A}\cdot\mathbf{n} = 0 & \forall\, l = 1,\dots,p_{\partial\Omega} \\
\text{div}\,\mathbf{A} = 0 & \text{on } \partial\Omega \\
\mathbf{A}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,.
\end{array}\right. \tag{32}$$

[For the magnetic boundary conditions see Bíró and V. (2007).]

Vector potential strong formulation (cont'd)

This formulation deals directly with curl $\mathbf{A}$ and div $\mathbf{A}$, hence nor $\mathbf{A} \times \mathbf{n}$ neither $\mathbf{A} \cdot \mathbf{n}$ are admitted to jump on a surface: in other words, the vector $\mathbf{A}$ cannot jump on a surface internal to Ω.

Therefore at the finite element level one is led to approximate each component of $\mathbf{A}$ by continuous nodal finite elements (say, the elements belonging to the space V_h introduced in (7)).

[If the constraint div $\mathbf{A} = 0$ is imposed by requiring that $\mathbf{A}$ is orthogonal to a suitable space of gradients, it is no longer mandatory that $\mathbf{A} \cdot \mathbf{n}$ has no jumps: therefore one could also use vector finite elements for which some components are not continuous. We will see a different example of this type later on...]

Vector potential strong formulation (cont'd)

It is important to show that any solution to (32) satisfies $\operatorname{div}\mathbf{A} = 0$ in Ω. In fact, taking the divergence of $(32)_1$ and using $(32)_2$ we have $-\Delta \operatorname{div}\mathbf{A}_C = 0$ in Ω_C. Moreover, since $\operatorname{div}\mathbf{J}_{e,I} = 0$ in Ω_I, one also obtains $-\Delta \operatorname{div}\mathbf{A}_I = 0$ in Ω_I. On the other hand, using $(32)_3$, on the interface Γ we have

$$
\begin{aligned}
-\mu_*^{-1} \operatorname{grad}\operatorname{div}\mathbf{A}_C \cdot \mathbf{n}_C \\
&= -\mathbf{J}_{e,I} \cdot \mathbf{n}_I - \operatorname{curl}\big(\boldsymbol{\mu}_C^{-1} \operatorname{curl}\mathbf{A}_C\big) \cdot \mathbf{n}_C \\
&= -\mathbf{J}_{e,I} \cdot \mathbf{n}_I - \operatorname{div}_\tau\big[\big(\boldsymbol{\mu}_C^{-1} \operatorname{curl}\mathbf{A}_C\big) \times \mathbf{n}_C\big] \ ,
\end{aligned}
$$

and also

$$
\begin{aligned}
-\mu_*^{-1} \operatorname{grad}\operatorname{div}\mathbf{A}_I \cdot \mathbf{n}_I \\
&= \mathbf{J}_{e,I} \cdot \mathbf{n}_I - \operatorname{curl}\big(\boldsymbol{\mu}_I^{-1} \operatorname{curl}\mathbf{A}_I\big) \cdot \mathbf{n}_I \\
&= \mathbf{J}_{e,I} \cdot \mathbf{n}_I - \operatorname{div}_\tau\big[\big(\boldsymbol{\mu}_I^{-1} \operatorname{curl}\mathbf{A}_I\big) \times \mathbf{n}_I\big] \ .
\end{aligned}
$$

Vector potential strong formulation (cont'd)

Moreover, a solution to $(32)_1$ satisfies on the interface Γ

$$\begin{aligned}\mathbf{n}_C \times (\boldsymbol{\mu}_C^{-1}\,\text{curl}\,\mathbf{A}_C) - \mu_*^{-1}\,\text{div}\,\mathbf{A}_C\,\mathbf{n}_C \\ + \mathbf{n}_I \times (\boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{A}_I) - \mu_*^{-1}\,\text{div}\,\mathbf{A}_I\,\mathbf{n}_I = \mathbf{0}\ ,\end{aligned}$$

therefore, due to orthogonality,

$$\mathbf{n}_C \times (\boldsymbol{\mu}_C^{-1}\,\text{curl}\,\mathbf{A}_C) + \mathbf{n}_I \times (\boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{A}_I) = \mathbf{0}\ ,\quad \text{div}\,\mathbf{A}_C = \text{div}\,\mathbf{A}_I\ .$$

Hence we have obtained

$$\text{grad}\,\text{div}\,\mathbf{A}_C \cdot \mathbf{n}_C + \text{grad}\,\text{div}\,\mathbf{A}_I \cdot \mathbf{n}_I = 0 \quad \text{on } \Gamma\ ,$$

and this last condition, together with the matching of $\text{div}\,\mathbf{A}$ on Γ, furnishes that $\text{div}\,\mathbf{A}$ is a harmonic function in the whole Ω. Since it vanishes on $\partial\Omega$, it vanishes in Ω.

Vector potential weak formulation

We are now interested in finding a weak formulation of (32).

First of all, multiplying $(32)_1$ by $\overline{\mathbf{w}}$ with $\mathbf{w} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and integrating in Ω, we obtain by integration by parts

$$\begin{aligned} \int_\Omega (\boldsymbol{\mu}^{-1} \,\text{curl}\, \mathbf{A} \cdot \text{curl}\, \overline{\mathbf{w}} + \mu_*^{-1} \,\text{div}\, \mathbf{A} \ \text{div}\, \overline{\mathbf{w}}) \\ + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma} \,\text{grad}\, V_C \cdot \overline{\mathbf{w}_C}) \\ = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \ , \end{aligned}$$

having used $(32)_5$.

Let us now multiply $(32)_2$ by $i\omega^{-1}\overline{Q_C}$ and integrate in Ω_C: by integration by parts and using $(32)_3$ we find

$$\begin{aligned} \int_{\Omega_C} (-\boldsymbol{\sigma} \mathbf{A}_C \cdot \text{grad}\, \overline{Q_C} + i\omega^{-1} \boldsymbol{\sigma} \,\text{grad}\, V_C \cdot \text{grad}\, \overline{Q_C}) \\ = i\omega^{-1} \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\, \overline{Q_C} + i\omega^{-1} \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{Q_C} \ . \end{aligned}$$

Vector potential weak formulation (cont'd)

Introducing the sesquilinear form

$$\begin{aligned}
&\mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w}, Q_C)] \\
&\quad := \int_\Omega (\boldsymbol{\mu}^{-1} \,\text{curl}\, \mathbf{A} \cdot \text{curl}\, \overline{\mathbf{w}} + \mu_*^{-1} \,\text{div}\, \mathbf{A} \;\text{div}\, \overline{\mathbf{w}}) \\
&\qquad + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma} \,\text{grad}\, V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad - \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{A}_C \cdot \text{grad}\, \overline{Q_C} \\
&\qquad + i\omega^{-1} \int_{\Omega_C} \boldsymbol{\sigma} \,\text{grad}\, V_C \cdot \text{grad}\, \overline{Q_C} \;,
\end{aligned} \tag{33}$$

we have finally rewritten (32) as

Find $(\mathbf{A}, V_C) \in W_\sharp \times H^1_\sharp(\Omega_C)$ such that

$$\begin{aligned}
\mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w}, Q_C)] &= \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \\
&\quad + i\omega^{-1} \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\, \overline{Q_C} + i\omega^{-1} \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{Q_C}
\end{aligned} \tag{34}$$

for all $(\mathbf{w}, Q_C) \in W_\sharp \times H^1_\sharp(\Omega_C)$,

Vector potential weak formulation (cont'd)

where

$$W_\sharp := \{\mathbf{w} \in H_0(\mathsf{curl};\Omega) \cap H(\mathsf{div};\Omega) \mid \int_{(\partial\Omega)_l} \mathbf{w}\cdot\mathbf{n} = 0 \;\; \forall\, l = 1,\dots,p_{\partial\Omega}\} \,,$$

and

$$H^1_\sharp(\Omega_C) := \prod_{j=1}^{p_\Gamma+1} H^1(\Omega_{C,j})/\mathbb{C} \,.$$

- The sesquilinear form $\mathcal{A}[\cdot,\cdot]$ is continuous and coercive [we will see this result later on...], therefore existence and uniqueness of the solution is ensured by the Lax–Milgram lemma.

Vector potential: from the weak to the strong formulation

To complete the argument, it is necessary to show that a solution of the weak problem is in fact a solution of the eddy current problem.

- This is not a trivial fact, as the functional spaces $W_\sharp$ and $H^1_\sharp(\Omega_C)$ contain some constraints.

The first step is to show that (34) is satisfied for any $\mathbf{w} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$, $Q_C \in H^1(\Omega_C)$.
First note that (34) does not change if we add to Q_C a (different) constant in $\Omega_{C,j}$. In fact, the necessary conditions on $\mathbf{J}_{e,I}$ are $\text{div}\,\mathbf{J}_{e,I} = 0$ in Ω_I and $\mathbf{J}_{e,I} \perp \mathcal{H}_I$, and the latter can be rewritten as $\int_{\Gamma_j} \mathbf{J}_{e,I} \cdot \mathbf{n}_I = 0$ for each $j = 1, \ldots, p_\Gamma + 1$ and $\int_{(\partial\Omega)_l} \mathbf{J}_{e,I} \cdot \mathbf{n} = 0$ for each $l = 1, \ldots, p_{\partial\Omega}$. Hence a solution $(\mathbf{A}, V_C)$ of (34) satisfies it also for each $Q_C \in H^1(\Omega_C)$.

Vector potential: from the weak to the strong formulation (cont'd)

Taking $\mathbf{w} = \mathbf{0}$, a first general result is that any solution to (34) satisfies

$$\begin{cases} \operatorname{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\operatorname{grad} V_C) = \operatorname{div}\mathbf{J}_{e,C} & \text{in } \Omega_C \\ (i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\operatorname{grad} V_C)\cdot\mathbf{n}_C = \mathbf{J}_{e,C}\cdot\mathbf{n}_C + \mathbf{J}_{e,I}\cdot\mathbf{n}_I & \text{on } \Gamma\ . \end{cases}$$

Therefore, setting

$$\mathbf{J} := \begin{cases} -i\omega\boldsymbol{\sigma}\mathbf{A}_C - \boldsymbol{\sigma}\operatorname{grad} V_C + \mathbf{J}_{e,C} & \text{in } \Omega_C \\ \mathbf{J}_{e,I} & \text{in } \Omega_I\ , \end{cases}$$

we have proved that $\operatorname{div}\mathbf{J} = 0$ in Ω.

Vector potential: from the weak to the strong formulation (cont'd)

For any $\mathbf{w} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ we can define by $\mathbf{w}_e$ the harmonic field in $\mathcal{H}(e; \Omega)$ satisfying $\int_{(\partial\Omega)_l} \mathbf{w}_e \cdot \mathbf{n} = \int_{(\partial\Omega)_l} \mathbf{w} \cdot \mathbf{n}$ for all $l = 1, \ldots, p_{\partial\Omega}$. Clearly, the difference $\mathbf{w} - \mathbf{w}_e$ belongs to $W_\sharp$. Hence

$$
\begin{aligned}
&\mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w}, Q_C)] \\
&\quad = \mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w} - \mathbf{w}_e, Q_C)] + \mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w}_e, 0)] \\
&\quad = \int_\Omega \mathbf{J}_e \cdot (\overline{\mathbf{w}} - \overline{\mathbf{w}_e}) + i\omega^{-1} \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\, \overline{Q_C} \\
&\qquad\quad + i\omega^{-1} \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{Q_C} \\
&\qquad\quad + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C + \boldsymbol{\sigma}\, \text{grad}\, V_C) \cdot \overline{\mathbf{w}_{e,C}} \\
&\quad = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} + i\omega^{-1} \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\, \overline{Q_C} \\
&\qquad\quad + i\omega^{-1} \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{Q_C} - \int_\Omega \mathbf{J} \cdot \overline{\mathbf{w}_e} \; .
\end{aligned}
$$

Vector potential: from the weak to the strong formulation (cont'd)

Therefore, the only result that remains to be proved is

$$\int_\Omega \mathbf{J} \cdot \overline{\mathbf{w}_e} = 0 \; .$$

The basis functions of $\mathcal{H}(e;\Omega)$ are given by grad w_l^*, $l = 1, \ldots, p_{\partial\Omega}$, where w_l^* is the (real-valued) solution to

$$\begin{cases} \Delta w_l^* = 0 & \text{in } \Omega \\ w_l^* = 0 & \text{on } (\partial\Omega) \setminus (\partial\Omega)_l \\ w_l^* = 1 & \text{on } (\partial\Omega)_l \; , \end{cases}$$

and we have

$$\begin{aligned} \int_\Omega \mathbf{J} \cdot \text{grad}\, w_l^* &= -\int_\Omega \text{div}\, \mathbf{J}\, w_l^* + \int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n}\, w_l^* \\ &= \int_{(\partial\Omega)_l} \mathbf{J} \cdot \mathbf{n} = \int_{(\partial\Omega)_l} \mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \, . \end{aligned}$$

Vector potential: from the weak to the strong formulation (cont'd)

Taking now in (34) a test function $\mathbf{w} \in (C_0^\infty(\Omega))^3$, by integration by parts we find at once that

$$\begin{aligned}\mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}) - \mu_*^{-1}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{A}&\\ +i\omega\boldsymbol{\sigma}\mathbf{A} + \boldsymbol{\sigma}\,\mathrm{grad}\,V_C = \mathbf{J}_e \quad \text{in } \Omega\,.&\end{aligned}$$

Repeating the same argument for $\mathbf{w} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ gives $\mathrm{div}\,\mathbf{A} = 0$ on $\partial\Omega$, and therefore a weak solution $(\mathbf{A}, V_C)$ to (34) is a solution to the strong problem (32).

Vector potential formulation: existence and uniqueness

The proof of existence and uniqueness derives from the Lax–Milgram lemma.
We have only to check that the sesquilinear form $\mathcal{A}[\cdot,\cdot]$ is coercive in $W_\sharp \times H^1_\sharp(\Omega_C)$, namely, that there exists a constant $\kappa_0 > 0$ such that for each $(\mathbf{w}, Q_C) \in W_\sharp \times H^1(\Omega_C)$ with $\int_{\Omega_{C,j}} Q_{C|\Omega_j} = 0$, $j = 1, \ldots, p_\Gamma + 1$, it holds

$$\begin{aligned}
&|\mathcal{A}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)]| \\
&\qquad \geq \kappa_0 \Big(\int_\Omega (|\mathbf{w}|^2 + |\operatorname{curl} \mathbf{w}|^2 + |\operatorname{div} \mathbf{w}|^2) \\
&\qquad\qquad + \int_{\Omega_C} (|Q_C|^2 + |\operatorname{grad} Q_C|^2) \Big) \,.
\end{aligned} \tag{35}$$

Vector potential formulation: existence and uniqueness (cont'd)

First of all, we can easily obtain

$$\begin{aligned}
&\mathcal{A}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)] \\
&\quad = \int_\Omega (\boldsymbol{\mu}^{-1} \,\text{curl}\, \mathbf{w} \cdot \text{curl}\, \overline{\mathbf{w}} + \mu_*^{-1} |\,\text{div}\, \mathbf{w}|^2) \\
&\qquad + i\omega^{-1} \int_{\Omega_C} \boldsymbol{\sigma} (i\omega \mathbf{w}_C + \text{grad}\, Q_C) \cdot (-i\omega \overline{\mathbf{w}_C} + \text{grad}\, \overline{Q_C}) \ .
\end{aligned}$$

Then, observe that, given a couple of real numbers a and b, for each $0 < \delta < 1$ it holds

$$|2ab| \leq \delta a^2 + \delta^{-1} b^2 \, .$$

Vector potential formulation: existence and uniqueness (cont'd)

Hence one has

$$
\begin{aligned}
|\omega|^{-1} \int_{\Omega_C} \boldsymbol{\sigma}(i\omega \mathbf{w}_C + \mathrm{grad}\, Q_C) \cdot (-i\omega \overline{\mathbf{w}_C} + \mathrm{grad}\, \overline{Q_C}) \\
\geq |\omega|^{-1}\sigma_{\min} \int_{\Omega_C} [|\,\mathrm{grad}\, Q_C|^2 + \omega^2 |\mathbf{w}_C|^2 \\
+2\,\mathrm{Re}(i\omega \mathbf{w}_C \cdot \mathrm{grad}\, \overline{Q_C})] \\
\geq |\omega|^{-1}\sigma_{\min}(1-\delta) \int_{\Omega_C} |\,\mathrm{grad}\, Q_C|^2 \\
-|\omega|\sigma_{\min}(1-\delta)\delta^{-1} \int_{\Omega_C} |\mathbf{w}_C|^2 \,,
\end{aligned}
$$

where $\sigma_{\min}$ is an uniform lower bound in Ω_C of the minimum eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$.

Vector potential formulation: existence and uniqueness (cont'd)

The Poincaré inequality gives that

$$\begin{aligned}\int_{\Omega_C} |\operatorname{grad} Q_C|^2 &= \sum_{j=1}^{p_\Gamma+1} \int_{\Omega_{C,j}} |\operatorname{grad} Q_{C|\Omega_{C,j}}|^2 \\ &\geq K_1 \sum_{j=1}^{p_\Gamma+1} \int_{\Omega_{C,j}} (|\operatorname{grad} Q_{C|\Omega_{C,j}}|^2 + |Q_{C|\Omega_{C,j}}|^2) \\ &= K_1 \int_{\Omega_C} (|\operatorname{grad} Q_C|^2 + |Q_C|^2)\end{aligned}$$

[recall that $\int_{\Omega_{C,j}} Q_{C|\Omega_{C,j}} = 0$, $j = 1, \ldots, p_\Gamma + 1$].
Moreover, the Poincaré-like inequality yields

$$\begin{aligned}\int_\Omega (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{w} \cdot \operatorname{curl} \overline{\mathbf{w}} + \mu_*^{-1} |\operatorname{div} \mathbf{w}|^2) &\geq \int_\Omega (\mu_{\max}^{-1} |\operatorname{curl} \mathbf{w}|^2 + \mu_*^{-1} |\operatorname{div} \mathbf{w}|^2) \\ &\geq K_2 \int_\Omega (|\operatorname{curl} \mathbf{w}|^2 + |\operatorname{div} \mathbf{w}|^2 + |\mathbf{w}|^2) \ ,\end{aligned}$$

Vector potential formulation: existence and uniqueness (cont'd)

where $\mu_{\max}$ is a uniform upper bound in Ω of the maximum eigenvalues of $\boldsymbol{\mu}(\mathbf{x})$ [recall that, for a divergence-free vector field, the conditions $\int_{(\partial\Omega)_l} \mathbf{w} \cdot \mathbf{n} = 0$ for all $l = 1, \ldots, p_{\partial\Omega}$ are equivalent to the orthogonality to $\mathcal{H}(e;\Omega)$].
Choosing $(1-\delta)$ so small that $\sigma_{\min}|\omega|(1-\delta) < K_2\delta$, we find at once (35).

Vector potential formulation: numerical approximation

- Numerical approximation is performed by means of nodal finite elements, for all the components of $\mathbf{A}$ and for V_C [all the components of $\mathbf{A}_h$ and $V_{C,h}$ are elements of the space V_h introduced in (7)].

Via Céa lemma we have

$$\begin{aligned}\Big(\int_\Omega (|\mathbf{A}-\mathbf{A}_h|^2 + |\,\mathrm{curl}(\mathbf{A}-\mathbf{A}_h)|^2 + |\,\mathrm{div}(\mathbf{A}-\mathbf{A}_h)|^2) \\ + \int_{\Omega_C} |\,\mathrm{grad}(V_C - V_{C,h})|^2 \Big)^{1/2} \\ \leq C_0 \Big(\int_\Omega (|\mathbf{A}-\mathbf{w}_h|^2 + |\,\mathrm{curl}(\mathbf{A}-\mathbf{w}_h)|^2 + |\,\mathrm{div}(\mathbf{A}-\mathbf{w}_h)|^2) \\ + \int_{\Omega_C} |\,\mathrm{grad}(V_C - Q_{C,h})|^2 \Big)^{1/2} ,\end{aligned}$$

for each choice of $\mathbf{w}_h$ and $Q_{C,h}$ (the former satisfying the constraints $\int_{(\partial\Omega)_l} \mathbf{w}_h \cdot \mathbf{n} = 0$ for all $l = 1, \ldots, p_{\partial\Omega}$).

Vector potential formulation: numerical approximation (cont'd)

Denote by $\mathbf{I}_h\mathbf{w}$ the nodal interpolant of $\mathbf{w}$ [this means that $\mathbf{I}_h\mathbf{w} = (\pi_h w_1, \pi_h w_2, \pi_h w_3)$, with $\mathbf{w} = (w_1, w_2, w_3)$].

- It is not possible to choose $\mathbf{w}_h = \mathbf{I}_h\mathbf{A}$, as the constraints $\int_{(\partial\Omega)_l} \mathbf{w}_h \cdot \mathbf{n} = 0$ have to be satisfied for all $l = 1, \ldots, p_{\partial\Omega}$. However, for each unconstrained discrete function $\mathbf{v}_h$ it is possible to find a constrained discrete function $\mathbf{w}_h$ such that

$$\|\mathbf{A} - \mathbf{w}_h\|_W \quad \leq C\|\mathbf{A} - \mathbf{v}_h\|_W \,.$$

  [Here notation is $W := H(\text{curl};\Omega) \cap H(\text{div};\Omega)$.]
  In particular, this can be done for $\mathbf{v}_h = \mathbf{I}_h\mathbf{A}$. Therefore, convergence is ensured provided that $\mathbf{A}$ is smooth enough [precisely, the convergence is of order r provided $\mathbf{A}$ is in the Sobolev space of order $r+1$].

Vector potential formulation: numerical approximation (cont'd)

- The regularity of $\mathbf{A}$ is a delicate point! In fact, it must be noted that it is not guaranteed if Ω has re-entrant corners or edges, namely, if it is a non-convex polyhedron (see Costabel and Dauge (2000), Costabel, Dauge and Nicaise (2003)).
  More important, in that case the space $H^1_n(\Omega) := (H^1(\Omega))^3 \cap H_0(\text{curl};\Omega)$ turns out to be a proper closed subspace of $H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ ($H^1_n(\Omega)$ and $H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ coincide if and only if Ω is convex).
  Hence the nodal finite element approximate solution $\mathbf{A}_h \in H^1_n(\Omega)$ cannot approach an exact solution $\mathbf{A} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ with $\mathbf{A} \notin H^1_n(\Omega)$, and convergence in $W = H(\text{curl};\Omega) \cap H(\text{div};\Omega)$ is lost: this is a general problem for the nodal finite element approximation of Maxwell equations.

Vector potential formulation: numerical approximation (cont'd)

Remark. This is a case in which "smooth" functions are not approximating the functions belonging to the variational space $H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$, but only the functions belonging the closed proper subspace $H^1_n(\Omega)$: Céa lemma and interpolation estimates are not enough to conclude the convergence proof...

Vector potential formulation: numerical approximation (cont'd)

- Summing up: the nodal finite element approximation is convergent either if the solution is regular (and this information could be available even for a non-convex polyhedron Ω) or else if the domain Ω is a convex polyhedron, as in this case the space of smooth normal vector fields is dense in $H^1_n(\Omega) = H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$, and one can apply Céa lemma and interpolation estimates in the standard way.
- Let us also note that the assumption that Ω is convex is not a severe restriction, as in most real-life applications $\partial\Omega$ arises from a somehow arbitrary truncation of the whole space. Hence, re-entrant corners and edges of Ω can be easily avoided.

Vector potential formulation: numerical approximation (cont'd)

- It is worth noting that a cure for the lack of convergence of nodal finite element approximations in the presence of re-entrant corners and edges has been proposed by Costabel and Dauge (2002). They introduce a special weight in the grad div penalization term, thus permitting to use standard nodal finite elements in a numerically efficient way.
- In numerical implementation, imposing the boundary condition $\mathbf{A}_h \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ is clearly straightforward if the boundary of the computational domain Ω is formed by planar surfaces, parallel to the reference planes.

Vector potential formulation: numerical approximation (cont'd)

- If that is not the case, for each node $\mathbf{p}$ on $\partial\Omega$ introduce a local system of coordinates with one axis aligned with $\mathbf{n}_a$, a suitable average of the normals to the surface elements containing $\mathbf{p}$, and express, through a rotation, the vector $\mathbf{A}_h$ with respect to that system: the condition $\mathbf{A}_h \times \mathbf{n}_a = \mathbf{0}$ is then trivially imposed (see Rodger and Eastham (1985)).
- Another possible approach, which avoids the arbitrariness inherent in the averaging process of the normals at corner points, is described by Bossavit (1999). It is based on imposing $\mathbf{A}_h \times \mathbf{n} = \mathbf{0}$ at the center of the element faces on $\partial\Omega$: the drawback is that it results in a constrained problem, requiring the introduction of as many Lagrange multipliers as the (double of the) number of surface elements on $\partial\Omega$.

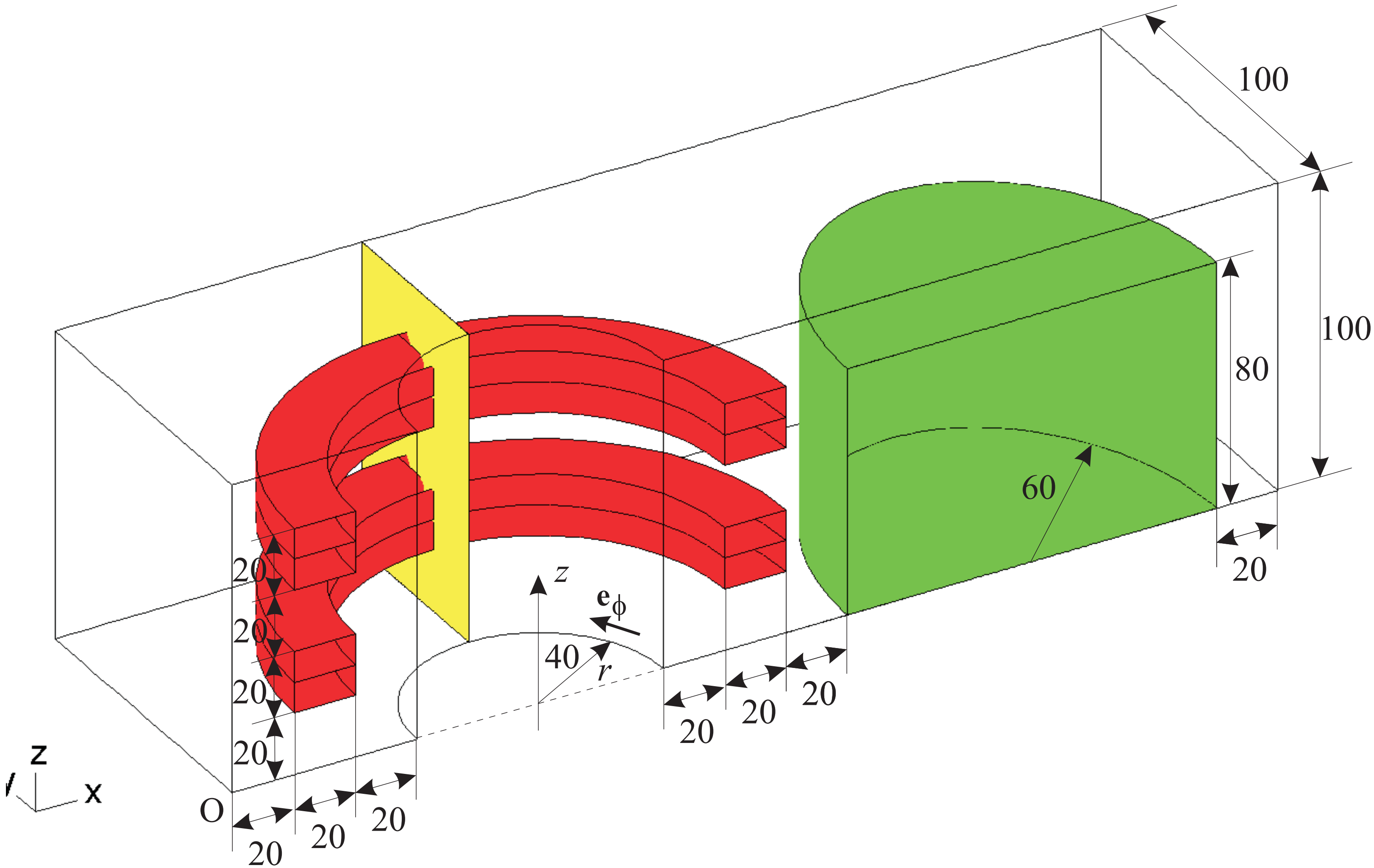
Vector potential formulation: numerical approximation (cont'd)

- Ungauged formulation have been also proposed (see Ren (1996), Kameari and Koganezawa (1997), Bíró (1999)): edge elements are employed for the approximation of the potential $\mathbf{A}$, without requiring that the gauge condition div $\mathbf{A} = 0$ in Ω is satisfied.
  Clearly, in this way the resulting linear system is singular: however, in many cases the right-hand sides turn out to be compatible, so that suitable iterative algebraic solvers can still be convergent.
  [Warning: lack of a complete theory...]

Numerical results

The numerical results we present here have been obtained in Bíró and V. (2007), for the magnetic boundary conditions (Ω is a torus and Ω_C is a ball-like set).
The employed finite elements are second order hexahedral “serendipity" elements, with 20 nodes (8 at the vertices and 12 at the midpoints of each edge), for all the components of $\mathbf{A}_h$ and for V_h.
The values of the physical coefficients have been assumed as follows: $\mu = \mu_* = 4\pi \times 10^{-7}$ H/m, $\sigma = 5.7 \times 10^7$ S/m, $\omega = 2\pi \times f = 100\pi$ rad/s, i.e., $f = 50$ Hz.
The half of the domain is described here below. The coils (the support of $\mathbf{J}_{e,I}$, therefore modeled as insulators) are red, while the conductor Ω_C is green; the yellow “cutting" surface Σ_1 is also drawn.

Numerical results (cont'd)



The computational domain [one half].

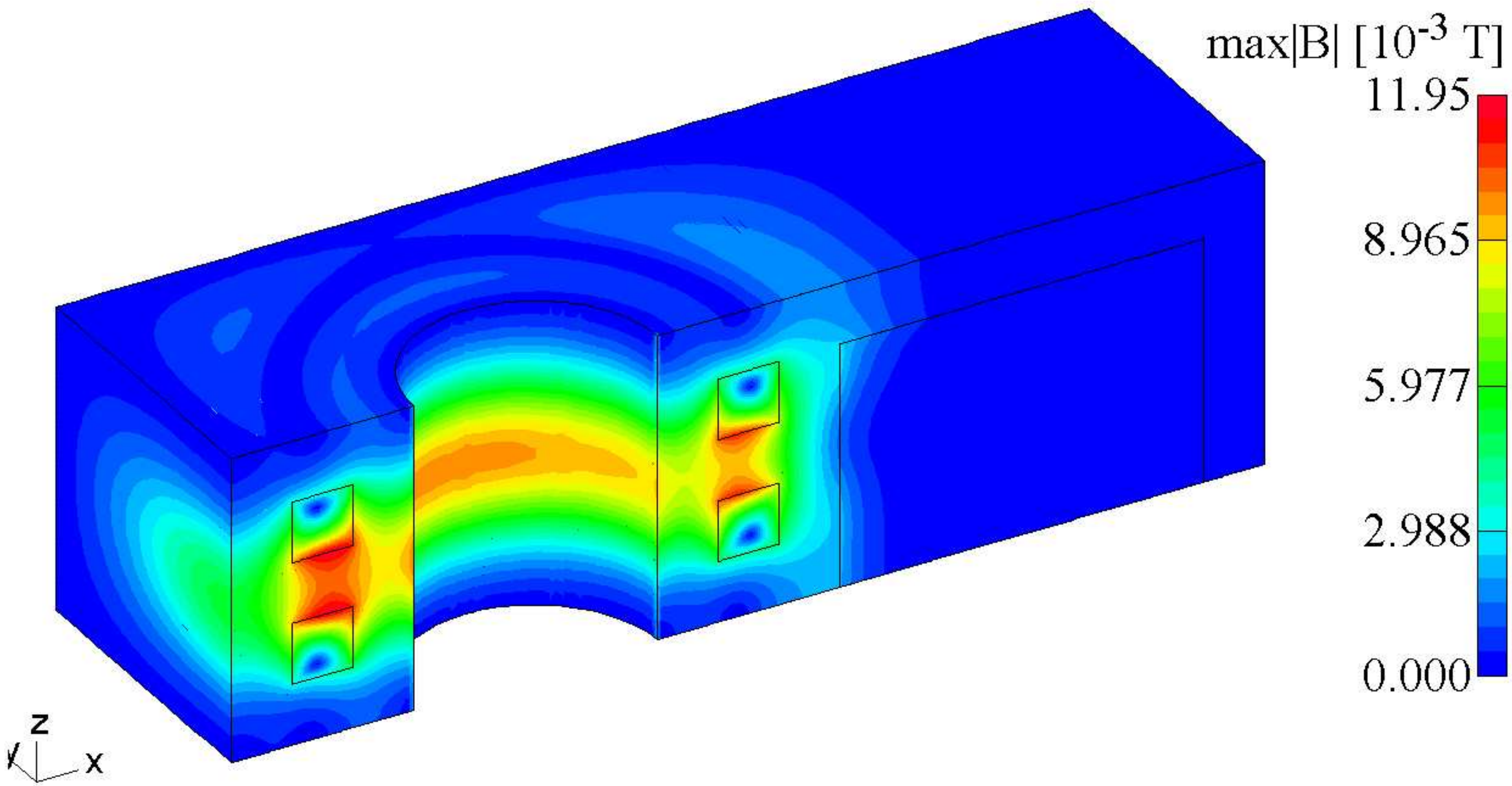
Numerical results (cont'd)

The current density is given by $\mathbf{J}_{e,C} = \mathbf{0}$ and $\mathbf{J}_{e,I} = J_{e,I}\mathbf{e}_\phi$, where $\mathbf{e}_\phi$ is the azymuthal unit vector in the cylindrical system centered at the point (100,0,0), oriented counterclockwise, and

$$J_{e,I} = \begin{cases} 10^6 \text{ A/m}^2 & \text{if } 60 < r < 80\ ,\ 60 < z < 80 \\ -10^6 \text{ A/m}^2 & \text{if } 60 < r < 80\ ,\ 20 < z < 40 \\ 0 & \text{otherwise}\ . \end{cases}$$

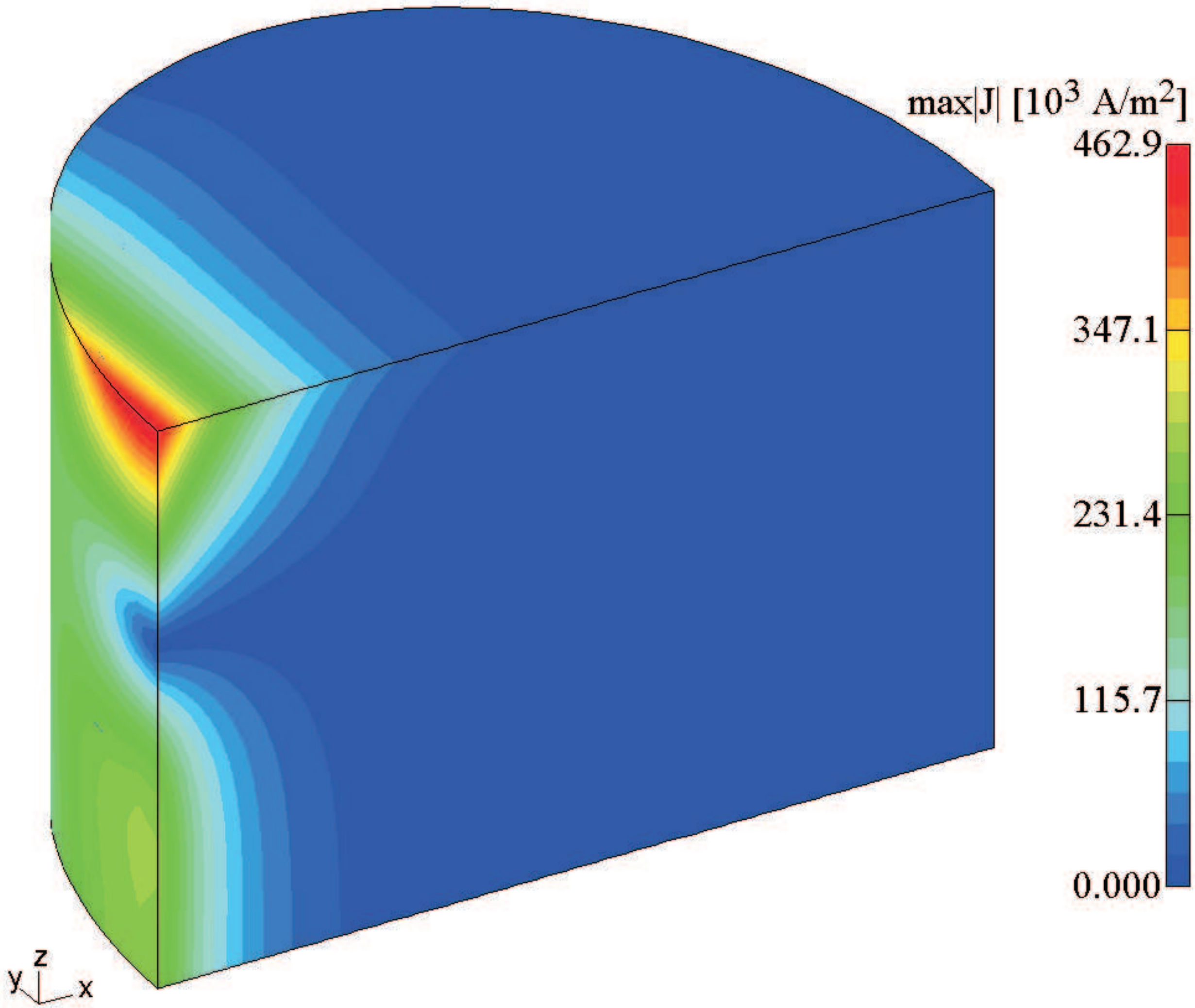
In the two figures below some details of the computed solution are presented: the magnitude of the computed flux density $\mathbf{B}$ in the first figure, the magnitude of the computed current density $\mathbf{J}_C := -i\omega\sigma\mathbf{A}_C - \sigma\,\text{grad}\,V_C$ in the second figure.

Numerical results (cont'd)



The magnitude of the flux density B.

Numerical results (cont'd)



The magnitude of the current density $\mathbf{J}_C := -i\omega\sigma\mathbf{A}_C - \sigma\,\text{grad}\,V_C$.

Pros and cons

- *Pros*
  - standard nodal finite elements for all the unknowns;
  - no difficulty with the topology of the conducting domain;
  - "positive definite" algebraic problem.
- *Cons*
  - many degrees of freedom;
  - lack of convergence for re-entrant corners of the computational domain.

Edge finite elements

Electromagnetic problems can be approximated by means of a different type of vector finite elements, for which the continuity of all the components is not required.

In fact, looking at Maxwell or eddy current equations it is apparent that what is really needed is that the curl operator is well-defined: not necessarily the gradient operator or the divergence operator (see (21) and (22)).

Therefore, in order that a discrete function $\mathbf{w}_h$ is also an element of the variational space [still to be defined... but only involving the curl operator!], what is needed is the continuity of $\mathbf{w}_h \times \mathbf{n}$ on all the interelements.

Edge finite elements (cont'd)

- These elements are called edge elements, and have been proposed by Nédélec (1980).

Let us assume that the triangulation is composed by tetrahedra.
For $r \geq 1$ denote by $\widetilde{\mathbb{P}}_r$ the space of homogeneous polynomials of degree r and define

$$S_r := \{\mathbf{q} \in (\widetilde{\mathbb{P}}_r)^3 \,|\, \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0\}$$

$$R_r := (\mathbb{P}_{r-1})^3 \oplus S_r \,.$$

The first family of Nédélec finite elements is

$$N_h^r := \{\mathbf{w}_h \in H(\mathrm{curl};\Omega) \,|\, \mathbf{w}_{h|K} \in R_r \;\forall\, K \in \mathcal{T}_h\} \,. \qquad (36)$$

Edge finite elements (cont'd)

The degrees of freedom are not nodal values, but:

- edge degrees of freedom $m_e(\mathbf{w})$

$$\left\{ \int_e \mathbf{w} \cdot \boldsymbol{\tau}_e \, q \, ds \;\forall q \in \mathbb{P}_{r-1}(e) \right\} \tag{37}$$

- face degrees of freedom $m_f(\mathbf{w})$ (for $r \geq 2$)

$$\left\{ \int_f \mathbf{w} \times \mathbf{n}_f \cdot \mathbf{q} \, dS \;\forall \mathbf{q} \in (\mathbb{P}_{r-2}(f))^2 \right\} \tag{38}$$

- volume degrees of freedom $m_K(\mathbf{w})$ (for $r \geq 3$)

$$\left\{ \int_K \mathbf{w} \cdot \mathbf{q} \, dV \;\forall \mathbf{q} \in (\mathbb{P}_{r-3})^3 \right\} . \tag{39}$$

Edge finite elements (cont'd)

Here $\boldsymbol{\tau}_e$ denotes a unit vector with the direction of e, while $\mathbf{n}_f$ is the unit normal vector on f.

The total number of degrees of freedom on a tetrahedron K is equal to the dimension of R_r, and it can be shown that, if all the degrees of freedom vanish, then a polynomial $\mathbf{w} \in R_r$ is identically vanishing in K, hence conditions (8) and (4) are satisfied.

It can also be proved that, if a vector function $\mathbf{w} \in R_r$ has all its degrees of freedom vanishing on a face f of K and on the three edges contained in f, then the tangential component of $\mathbf{w}$ vanishes on f. This means that, using these degrees of freedom for identifying a piecewise-polynomial function that locally belongs to R_r, we obtain an element of $H(\text{curl}; \Omega)$, hence an element of N_h^r.

Lowest order edge finite elements

- Let us specify the form of Nédélec edge elements and their degrees of freedom for $r = 1$.

The condition $\mathbf{q} \cdot \mathbf{x} = 0$ for $\mathbf{q} \in (\widetilde{\mathbb{P}}_1)^3$ says that $\mathbf{q} = \mathbf{a} \times \mathbf{x}$ with $\mathbf{a} \in \mathbb{R}^3$. Hence the space R_1 is given by the polynomials of the form

$$\mathbf{q}(\mathbf{x}) = \mathbf{b} + \mathbf{a} \times \mathbf{x} \quad , \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \,. \tag{40}$$

For $r = 1$ only edge degrees of freedom are active, and are given by

$$\int_e (\mathbf{b} + \mathbf{a} \times \mathbf{x}) \cdot \boldsymbol{\tau}_e \, ds \tag{41}$$

for the six edges e of the tetrahedron K.

Lowest order edge finite elements (cont'd)

- Let us show that if all the degrees of freedom of $\mathbf{q} = \mathbf{b} + \mathbf{a} \times \mathbf{x}$ on K are equal to 0, then $\mathbf{q} = \mathbf{0}$: in other words, (8) and (4) are satisfied.

A direct computation shows that $\text{curl}\, \mathbf{q} = 2\, \mathbf{a}$. Moreover, from Stokes theorem for each face f we have

$$
\begin{aligned}
0 \quad &= \sum_e \int_e \mathbf{q} \cdot \boldsymbol{\tau}_e \, ds = \int_{\partial f} \mathbf{q} \cdot \boldsymbol{\tau} \, ds \\
&= \int_f \text{curl}\, \mathbf{q} \cdot \mathbf{n}_f \, dS = 2\, \mathbf{a} \cdot \mathbf{n}_f \, \text{meas}(f) \,,
\end{aligned}
$$

hence $\mathbf{a} \cdot \mathbf{n}_f = 0$ on f. Since three of the vectors $\mathbf{n}_f$ are linearly independent, it follows $\mathbf{a} = \mathbf{0}$.

Lowest order edge finite elements (cont'd)

Then for each edge e

$$\begin{aligned} 0 &= \int_e \mathbf{q} \cdot \boldsymbol{\tau}_e \, ds = \int_e \mathbf{b} \cdot \boldsymbol{\tau}_e \, ds \\ &= \mathbf{b} \cdot \boldsymbol{\tau}_e \, \text{length}(e) \,, \end{aligned}$$

and three of the vectors $\boldsymbol{\tau}_e$ are linearly independent, so that $\mathbf{b} = \mathbf{0}$ and in conclusion $\mathbf{q} = \mathbf{0}$.

Lowest order edge finite elements (cont'd)

- Another point is to prove that if the three edge degrees of freedom of $\mathbf{q} = \mathbf{b} + \mathbf{a} \times \mathbf{x}$ on a face f are equal to 0 then $\mathbf{q} \times \mathbf{n}_f = \mathbf{0}$ on f.

We have already seen that $\mathbf{a} \cdot \mathbf{n}_f = 0$ on f. On the other hand,

$$\begin{aligned} \mathbf{q} \times \mathbf{n}_f &= \mathbf{b} \times \mathbf{n}_f + (\mathbf{a} \times \mathbf{x}) \times \mathbf{n}_f \\ &= \mathbf{b} \times \mathbf{n}_f + (\mathbf{a} \cdot \mathbf{n}_f)\, \mathbf{x} - (\mathbf{x} \cdot \mathbf{n}_f)\, \mathbf{a}\,. \end{aligned}$$

Since on a face one has $\mathbf{x} \cdot \mathbf{n}_f = \text{const}$, it follows that $\mathbf{q} \times \mathbf{n}_f$ is equal on f to a constant vector $\mathbf{c}_f$, with $\mathbf{c}_f \cdot \mathbf{n}_f = 0$.

Lowest order edge finite elements (cont'd)

Finally,

$$\begin{aligned} 0 &= \int_e \mathbf{q} \cdot \boldsymbol{\tau}_e \, ds = \int_e (\mathbf{n}_f \times \mathbf{q} \times \mathbf{n}_f) \cdot \boldsymbol{\tau}_e \, ds \\ &= (\mathbf{n}_f \times \mathbf{c}_f) \cdot \boldsymbol{\tau}_e \, \text{length}(e) \, . \end{aligned}$$

Since two of the vectors $\boldsymbol{\tau}_e$ are generating the plane containing f (and the vector $\mathbf{n}_f \times \mathbf{c}_f$), it follows $\mathbf{c}_f = \mathbf{0}$ and consequently $\mathbf{q} \times \mathbf{n}_f = \mathbf{0}$ on f.

Lowest order edge finite elements (cont'd)

- In particular, we have shown that the dimension of N_h^1 is equal to the total number of the edge degrees of freedom (i.e., the total number of edges).

The basis functions are defined as in (5), namely, for each edge e_m we construct the function $\boldsymbol{\varphi}_m$ such that

$$\int_{e_l} \boldsymbol{\varphi}_m \cdot \boldsymbol{\tau}\, ds = \begin{cases} 1 & \text{if } m = l \\ 0 & \text{if } m \neq l\,. \end{cases} \qquad (42)$$

Since (8) is satisfied, the basis functions have a "small" support: $\boldsymbol{\varphi}_m$ is non-vanishing only in the elements K of the triangulation that contain the edge e_m.

Lowest order edge finite elements (cont'd)

- The explicit construction of a basis for the edge element space N_h^1 is easily done.

In fact, it can be proved that the basis function $\boldsymbol{\varphi}_{i,j}$ associated to the edge $e_{i,j}$ joining the nodes $\mathbf{x}_i$ and $\mathbf{x}_j$ and satisfying $\int_{e_{i,j}} \boldsymbol{\varphi}_{i,j} \cdot \boldsymbol{\tau}\, ds = 1$ is given by

$$\boldsymbol{\varphi}_{i,j} = \varphi_i \,\mathrm{grad}\, \varphi_j - \varphi_j \,\mathrm{grad}\, \varphi_i \,, \tag{43}$$

where φ_i is the piecewise-linear nodal basis function associated to the node $\mathbf{x}_i$.

Interpolation operator

As usual, the interpolant $\mathbf{r}_h\mathbf{w}$ of a (smooth enough) vector function $\mathbf{w}$ is the unique vector function belonging to N_h^r such that

$$
\begin{aligned}
m_e(\mathbf{r}_h\mathbf{w}) &= m_e(\mathbf{w}) \\
m_f(\mathbf{r}_h\mathbf{w}) &= m_f(\mathbf{w}) \\
m_K(\mathbf{r}_h\mathbf{w}) &= m_K(\mathbf{w})
\end{aligned}
\tag{44}
$$

for each edge e, face f and element K.

The interpolation operator $\mathbf{r}_h : S \to N_h^r$ is defined as

$$
\mathbf{r}_h : \mathbf{w} \to \mathbf{r}_h\mathbf{w} \tag{45}
$$

(having denoted by S the space of “smooth enough” vector functions: we will come back to this here below...).

Interpolation operator (cont'd)

The interpolant $\mathbf{r}_h\mathbf{w}$ can be written as

$$\mathbf{r}_h\mathbf{w} = \sum_e m_e(\mathbf{w})\boldsymbol{\varphi}_e + \sum_f m_f(\mathbf{w})\boldsymbol{\varphi}_f + \sum_K m_K(\mathbf{w})\boldsymbol{\varphi}_K \qquad (46)$$

(having denoted by $\boldsymbol{\varphi}_e$ the set of basis functions associated to the edge e and similarly for the other cases).

- Question: what about the space S, where the interpolation operator is defined?

It is necessary to give a meaning to line integrals and surface integrals, which is not possible for functions belonging to the space $H(\text{curl};\Omega)$.

Interpolation operator (cont'd)

Up today, the best result is due to Amrouche, Bernardi, Dauge and Girault (1998): if we know that for some $p > 2$ the function $\mathbf{w}$ satisfies $\mathbf{w} \in (L^p(\Omega))^3$ with $\mathrm{curl}\, \mathbf{w} \in (L^p(\Omega))^3$ and $\mathbf{w}_{|K} \times \boldsymbol{\nu} \in ((L^p(\partial K))^3$ for each $K \in \mathcal{T}_h$, then the interpolant $\mathbf{r}_h \mathbf{w}$ is well-defined.

For instance, this is true if $\mathbf{w}$ has a sufficiently large Sobolev regularity, namely, if $\mathbf{w} \in H^s(\mathrm{curl}; \Omega)$ for $s > 1/2$, where

$$H^s(\mathrm{curl}; \Omega) := \{\mathbf{w} \in (H^s(\Omega))^3 \,|\, \mathrm{curl}\, \mathbf{w} \in (H^s(\Omega))^3\} \,. \qquad (47)$$

[Since the exponent s can be non-integer, this space looks a little bit “exotic”... However, it is necessary to take it into consideration, as in general the solutions of Maxwell and eddy current equations are not very regular in the scale of Sobolev spaces: it happens that $s < 1$.]

Interpolation error

If the family of triangulations $\mathcal{T}_h$ is regular and $\mathbf{w} \in H^s(\text{curl}; \Omega)$, $1/2 < s \leq r$, it is possible to prove the following interpolation error estimate

$$\begin{gathered} \|\mathbf{w} - \mathbf{r}_h \mathbf{w}\|_{0,\Omega} + \| \operatorname{curl} \mathbf{w} - \operatorname{curl}(\mathbf{r}_h \mathbf{w})\|_{0,\Omega} \\ \leq C h^s (\|\mathbf{w}\|_{s,\Omega} + \| \operatorname{curl} \mathbf{w}\|_{s,\Omega}) \end{gathered} \tag{48}$$

(see Alonso and V. (1999)).

Since each vector function belonging to $H(\text{curl}; \Omega)$ can be approximated by smooth vector functions, we can conclude that approximation property (3), namely,

$$\operatorname{dist}(v, V_h) \to 0 \quad \forall\, v \in V$$

is satisfied for $V = H(\text{curl}; \Omega)$ and $V_h = N_h^r$.

The cavity problem

- Edge elements are therefore a suitable tool for numerical approximation of Maxwell and eddy current equations.

In order to give an example, let us consider the cavity problem for the time-harmonic Maxwell equations (21), with electric boundary condition. This means that the computational domain Ω is an empty cavity surrounded by a perfectly conducting medium.

In this situation, it is also reasonable to assume that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are scalar constants, say, $\boldsymbol{\varepsilon} = \varepsilon_0$ and $\boldsymbol{\mu} = \mu_0$, the electric permittivity and the magnetic permeability of the vacuum.

The cavity problem (cont'd)

Therefore the problem reads

$$\begin{cases} \text{curl}\,\mathbf{H} - i\omega\varepsilon_0\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \text{curl}\,\mathbf{E} + i\omega\mu_0\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases} \tag{49}$$

Using the Faraday equation to write $\mathbf{H}$ in terms of $\mathbf{E}$ and substituting the result $\mathbf{H} = -(i\omega\mu_0)^{-1}\,\text{curl}\,\mathbf{E}$ in the Ampère equation, one is left with

$$\begin{cases} \text{curl}\,\text{curl}\,\mathbf{E} - \omega^2\mu_0\varepsilon_0\mathbf{E} = -i\omega\mu_0\mathbf{J}_e & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases}$$

The cavity problem (cont'd)

Introducing the wave number

$$k := |\omega|\sqrt{\mu_0\varepsilon_0}\,, \tag{50}$$

we can finally write

$$\begin{cases} \textsf{curl}\,\textsf{curl}\,\mathbf{E} - k^2\mathbf{E} = -i\omega\mu_0\mathbf{J}_e & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases}$$

Splitting $\mathbf{J}_e$ into its real and imaginary parts, we can solve two problems of the same form for the real and imaginary parts of $\mathbf{E}$.

The cavity problem (cont'd)

Hence, we can focus on the problem

$$\begin{cases} \text{curl}\,\text{curl}\,\mathbf{E} - k^2\mathbf{E} = \mathbf{F} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,, \end{cases} \tag{51}$$

where all the functions are real valued.

- Problem (51) is associated to a bilinear form that is not coercive in $H(\text{curl};\Omega)$ [$-k^2$ has the "wrong" sign...]. What we can say about existence and uniqueness of a solution?

Maxwell eigenvalue problem

Consider the Maxwell eigenvalue problem

$$\begin{cases} \text{curl}\,\text{curl}\,\mathbf{E} = \lambda\mathbf{E} & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases} \tag{52}$$

The classical Hilbert–Schmidt theory can be applied to obtain

- Besides $\lambda_0 = 0$, there exists a sequence of positive, increasing and diverging to ∞ eigenvalues λ_m of problem (52) [see, e.g., Leis (1986)].

The cavity problem: existence and uniqueness

Fredholm alternative theory can be used to prove

- When $k \neq \sqrt{\lambda_m}$, $m = 0, 1, 2, \ldots,$ there exists a unique solution of problem (51).

Numerical approximation of (51) is important in order to simulate the real physical situation and obtain informations for shape optimization (for instance, an electromagnetic cavity is a model for microwave ovens).

[Clearly, to this aim another issue is the numerical simulation of (52); however, here we do not consider this problem, referring to Boffi, Fernandes, Gastaldi and Perugia (1999), Caorsi, Fernandes and Raffetto (2000) and Monk (2003a).]

The cavity problem: variational formulations

The variational formulation of (51) is

$$\begin{aligned}
&\text{find } \mathbf{E} \in H_0(\text{curl};\Omega) \ : \\
&\quad \int_\Omega \text{curl}\,\mathbf{E} \cdot \text{curl}\,\mathbf{w} - k^2 \int_\Omega \mathbf{E} \cdot \mathbf{w} = \int_\Omega \mathbf{F} \cdot \mathbf{w} \\
&\forall\ \mathbf{w} \in H_0(\text{curl};\Omega)\,.
\end{aligned} \tag{53}$$

The finite element approximation problem with edge elements reads

$$\begin{aligned}
&\text{find } \mathbf{E}_h \in W_h \ : \\
&\quad \int_\Omega \text{curl}\,\mathbf{E}_h \cdot \text{curl}\,\mathbf{w}_h - k^2 \int_\Omega \mathbf{E}_h \cdot \mathbf{w}_h = \int_\Omega \mathbf{F} \cdot \mathbf{w}_h \\
&\forall\ \mathbf{w}_h \in W_h\,,
\end{aligned} \tag{54}$$

where

$$W_h := N_h^r \cap H_0(\text{curl};\Omega)\,.$$

The cavity problem: numerical approximation

The existence and uniqueness of the solution to the discrete problem (54) has to be proved. We will do that later on, and for the time being we assume that the solution $\mathbf{E}_h$ does exist.

Let us focus on the convergence of the numerical scheme and on the error estimate, following Monk (2003b) [for different approaches, see Monk and Demkowicz (2001), Boffi and Gastaldi (2002)]. Setting $\mathbf{e}_h := \mathbf{E} - \mathbf{E}_h$, by subtracting (54) from (53) we find

$$\int_\Omega \text{curl}\, \mathbf{e}_h \cdot \text{curl}\, \mathbf{w}_h - k^2 \int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h = 0 \qquad \forall\, \mathbf{w}_h \in W_h\,. \quad (55)$$

The cavity problem: numerical approximation (cont'd)

A first trivial remark is that $\text{grad}\, L_h^r \subset N_h^r$ (L_h^r defined in (7)), therefore using in (55) $\mathbf{w}_h = \text{grad}\, v_h$ with $v_h \in L_h^r \cap H_0^1(\Omega)$ we have

$$\int_\Omega \mathbf{e}_h \cdot \text{grad}\, v_h = 0\,. \qquad (56)$$

In other words, $\mathbf{e}_h$ is discrete divergence free.

Denote by P_h the orthogonal projection from $H(\text{curl};\Omega)$ onto W_h, by $m(\cdot,\cdot)$ the bilinear form at the left hand side of (55), and by $\|\cdot\|_{\mathrm{curl},\Omega}$ (respectively, $(\cdot,\cdot)_{\mathrm{curl},\Omega}$) the norm (respectively, the scalar product) in $H(\text{curl};\Omega)$. One obtains

$$\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \le \|\mathbf{E} - P_h\mathbf{E}\|_{\mathrm{curl},\Omega} + (1+k^2) \sup_{\mathbf{w}_h \in W_h} \frac{\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega}}\,. \qquad (57)$$

The cavity problem: numerical approximation (cont'd)

Let us prove (57). We have

$$\begin{aligned}\|\mathbf{e}_h\|^2_{\mathrm{curl},\Omega} &= (\mathbf{e}_h, \mathbf{E} - P_h\mathbf{E})_{\mathrm{curl},\Omega} + (\mathbf{e}_h, P_h\mathbf{E} - \mathbf{E}_h)_{\mathrm{curl},\Omega} \\ &= (\mathbf{e}_h, \mathbf{E} - P_h\mathbf{E})_{\mathrm{curl},\Omega} + m(\mathbf{e}_h, P_h\mathbf{E} - \mathbf{E}_h) \\ &\qquad\qquad +(1+k^2)\int_\Omega \mathbf{e}_h \cdot (P_h\mathbf{E} - \mathbf{E}_h) \\ &= (\mathbf{e}_h, \mathbf{E} - P_h\mathbf{E})_{\mathrm{curl},\Omega} + (1+k^2)\int_\Omega \mathbf{e}_h \cdot (P_h\mathbf{E} - \mathbf{E}_h)\,,\end{aligned}$$

having used (55).
On the other hand,

$$\int_\Omega \mathbf{e}_h \cdot (P_h\mathbf{E} - \mathbf{E}_h) \leq \sup_{\mathbf{w}_h \in W_h} \frac{\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega}} \|P_h\mathbf{E} - \mathbf{E}_h\|_{\mathrm{curl},\Omega}\,.$$

Since $\mathbf{E}_h = P_h\mathbf{E}_h$ and $\|P_h\mathbf{e}_h\|_{\mathrm{curl},\Omega} \leq \|\mathbf{e}_h\|_{\mathrm{curl},\Omega}$, (57) follows at once.

The cavity problem: numerical approximation (cont'd)

Let us estimate

$$\sup_{\mathbf{w}_h \in W_h} \frac{\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega}} .$$

A Helmholtz orthogonal decomposition result ensures that we can write $\mathbf{e}_h = \mathrm{curl}\, \mathbf{q}_0 + \mathbf{k}_0 + \mathrm{grad}\, p_0$, where $\mathrm{grad}\, p_0$ is the $(L^2(\Omega))^3$-orthogonal projection of $\mathbf{e}_h$ on $\mathrm{grad}\, H^1_0(\Omega)$ (in particular, $p_0 \in H^1_0(\Omega)$), and $\mathbf{k}_0$ is a harmonic field belonging to $\mathcal{H}(e;\Omega)$ (namely, $\mathrm{curl}\, \mathbf{k}_0 = \mathbf{0}$, $\mathrm{div}\, \mathbf{k}_0 = 0$ and $\mathbf{k}_0 \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$). We set $\mathbf{e}_0 := \mathrm{curl}\, \mathbf{q}_0 + \mathbf{k}_0$, and thus $\mathrm{div}\, \mathbf{e}_0 = 0$, $\mathrm{curl}\, \mathbf{e}_0 = \mathrm{curl}\, \mathbf{e}_h$, $\mathbf{e}_0 \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.

Since $\mathbf{e}_h$ is discrete divergence free, it follows that $\mathrm{grad}\, p_0$ is discrete divergence free, too.

The cavity problem: numerical approximation (cont'd)

Due to the properties of orthogonal projections, we also have $\|\operatorname{grad} p_0\|_{0,\Omega} \leq \|\mathbf{e}_h\|_{0,\Omega}$.

Similarly, the discrete orthogonal decomposition $\mathbf{w}_h = \mathbf{w}_{0,h} + \operatorname{grad} \xi_h$ holds, with $\xi_h \in L_h^r \cap H_0^1(\Omega)$ and $\mathbf{w}_{0,h} \in W_h$. The function $\mathbf{w}_{0,h}$ is discrete divergence free and clearly satisfies $\operatorname{curl} \mathbf{w}_{0,h} = \operatorname{curl} \mathbf{w}_h$ and $\|\mathbf{w}_{0,h}\|_{0,\Omega} \leq \|\mathbf{w}_h\|_{0,\Omega}$.

Having obtained these preliminaries results, we find

$$\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h = \int_\Omega (\mathbf{e}_0 + \operatorname{grad} p_0) \cdot \mathbf{w}_h = \int_\Omega \mathbf{e}_0 \cdot \mathbf{w}_h + \int_\Omega \operatorname{grad} p_0 \cdot \mathbf{w}_{0,h}\,.$$

We will see later on how to estimate $\int_\Omega \operatorname{grad} p_0 \cdot \mathbf{w}_{0,h}$.

The cavity problem: numerical approximation (cont'd)

Concerning the term $\int_\Omega \mathbf{e}_0 \cdot \mathbf{w}_h$ we find

$$\int_\Omega \mathbf{e}_0 \cdot \mathbf{w}_h \leq \|\mathbf{e}_0\|_{0,\Omega} \, \|\mathbf{w}_h\|_{0,\Omega} \, , \tag{58}$$

and we need to estimate $\|\mathbf{e}_0\|_{0,\Omega}$.

The required estimate can be obtained by means of a duality argument (see Nitsche (1970), Schatz (1974)). Let $\mathbf{z} \in H(\text{curl};\Omega)$ be the solution to

$$\begin{cases} \text{curl}\,\text{curl}\,\mathbf{z} - k^2\mathbf{z} = \mathbf{e}_0 & \text{in } \Omega \\ \mathbf{z} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \, , \end{cases} \tag{59}$$

which satisfies the estimate $\|\mathbf{z}\|_{\text{curl},\Omega} \leq C\|\mathbf{e}_0\|_{0,\Omega}$. Since $\text{div}\,\mathbf{e}_0 = 0$, we also have $\text{div}\,\mathbf{z} = 0$.

The cavity problem: numerical approximation (cont'd)

Moreover, $\operatorname{curl} \mathbf{z}$ satisfies

$$\begin{cases} \operatorname{curl}(\operatorname{curl} \mathbf{z}) = k^2 \mathbf{z} + \mathbf{e}_0 & \text{in } \Omega \\ \operatorname{div}(\operatorname{curl} \mathbf{z}) = 0 & \text{in } \Omega \\ \operatorname{curl} \mathbf{z} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \,. \end{cases}$$

A couple of regularity results due to Amrouche, Bernardi, Dauge and Girault (1998) say that $\mathbf{z} \in H^s(\Omega)$ with $\operatorname{curl} \mathbf{z} \in H^s(\Omega)$ for $s > 1/2$, and the following estimates hold

$$\|\mathbf{z}\|_{s,\Omega} \le C \|\mathbf{z}\|_{\mathrm{curl},\Omega} \le C \|\mathbf{e}_0\|_{0,\Omega}$$

$$\begin{aligned} \|\operatorname{curl} \mathbf{z}\|_{s,\Omega} &\le C(\|\operatorname{curl} \operatorname{curl} \mathbf{z}\|_{0,\Omega} + \|\operatorname{curl} \mathbf{z}\|_{0,\Omega}) \\ &\le C(\|\mathbf{z}\|_{\mathrm{curl},\Omega} + \|\mathbf{e}_0\|_{0,\Omega}) \le C \|\mathbf{e}_0\|_{0,\Omega} \,. \end{aligned}$$

The cavity problem: numerical approximation (cont'd)

Hence the interpolant $\mathbf{r}_h\mathbf{z}$ is defined and we have

$$\|\mathbf{z}-\mathbf{r}_h\mathbf{z}\|_{\mathrm{curl},\Omega} \leq Ch^s(\|\mathbf{z}\|_{s,\Omega} + \|\,\mathsf{curl}\,\mathbf{z}\|_{s,\Omega}) \leq Ch^s\|\mathbf{e}_0\|_{0,\Omega}\,.$$

Using (59) we find

$$\|\mathbf{e}_0\|^2_{0,\Omega} = m(\mathbf{z},\mathbf{e}_0) = m(\mathbf{z},\mathbf{e}_h - \mathsf{grad}\,p_0) = m(\mathbf{z},\mathbf{e}_h)\,,$$

since $\mathbf{z}$ is divergence free and $p_{0|\partial\Omega} = 0$.
Moreover, taking into account (55)

$$\begin{aligned} m(\mathbf{z},\mathbf{e}_h) &= m(\mathbf{z}-\mathbf{r}_h\mathbf{z},\mathbf{e}_h) \leq C\|\mathbf{z}-\mathbf{r}_h\mathbf{z}\|_{\mathrm{curl},\Omega}\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \\ &\leq Ch^s\|\mathbf{e}_0\|_{0,\Omega}\|\mathbf{e}_h\|_{\mathrm{curl},\Omega}\,. \end{aligned}$$

The cavity problem: numerical approximation (cont'd)

In conclusion,

$$\|\mathbf{e}_0\|_{0,\Omega} \leq Ch^s \|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \,. \tag{60}$$

Let us come to the estimate of $\int_\Omega \mathsf{grad}\, p_0 \cdot \mathbf{w}_{0,h}$.

Since $\mathbf{w}_{0,h}$ is discrete divergence free, it is possible to find a divergence free vector function $\mathbf{U}_0$ such that

$$\begin{aligned} \|\mathbf{w}_{0,h} - \mathbf{U}_0\|_{0,\Omega} &\leq Ch^s(\|\mathbf{w}_{0,h}\|_{0,\Omega} + \|\,\mathsf{curl}\,\mathbf{w}_{0,h}\|_{0,\Omega}) \\ &\leq Ch^s(\|\mathbf{w}_h\|_{0,\Omega} + \|\,\mathsf{curl}\,\mathbf{w}_h\|_{0,\Omega}) \,. \end{aligned}$$

The cavity problem: numerical approximation (cont'd)

[This can be done by taking the solution $\mathbf{U}_0$ of the problem

$$\begin{cases} \text{curl}\,\mathbf{U}_0 = \text{curl}\,\mathbf{w}_{0,h} & \text{in } \Omega \\ \text{div}\,\mathbf{U}_0 = 0 & \text{in } \Omega \\ \mathbf{U}_0 \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \int_\Omega \mathbf{U}_0 \cdot \text{grad}\,\psi_l = \int_\Omega \mathbf{w}_{0,h} \cdot \text{grad}\,\psi_l & \forall\, l = 1, \ldots, p_{\partial\Omega}\,, \end{cases}$$

where ψ_l is the discrete function, defined on a fixed coarse mesh, taking value 1 on $(\partial\Omega)_l$ and value 0 on all the other nodes in $\overline{\Omega}$. It can be shown that

$$\begin{aligned} \|\mathbf{U}_0\|_{\text{curl},\Omega} &\leq C(\|\,\text{curl}\,\mathbf{w}_{0,h}\|_{0,\Omega} + \textstyle\sum_l |\int_\Omega \mathbf{w}_{0,h} \cdot \text{grad}\,\psi_l|) \\ &\leq C\|\mathbf{w}_{0,h}\|_{\text{curl},\Omega}\,, \end{aligned}$$

and that $\mathbf{w}_{0,h} = \mathbf{r}_h \mathbf{U}_0 + \text{grad}\,\phi_h$, with $\phi_h \in L^r_h$ and constant on each $(\partial\Omega)_l$; hence $\mathbf{w}_{0,h} = \mathbf{r}_h \mathbf{U}_0 + \text{grad}\,v_h + \sum_l c_l\,\text{grad}\,\psi_l$ with $v_h \in L^r_h \cap H^1_0(\Omega)$.

The cavity problem: numerical approximation (cont'd)

Therefore

$$
\begin{aligned}
\|\mathbf{w}_{0,h} - \mathbf{U}_0\|^2_{0,\Omega} &= \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{w}_{0,h} - \mathbf{U}_0) \\
&= \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{w}_{0,h} - \mathbf{r}_h \mathbf{U}_0) \\
&\quad + \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0) \\
&= \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\operatorname{grad} v_h + \textstyle\sum_l c_l \operatorname{grad} \psi_l) \\
&\quad + \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0) \\
&= \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0) \\
&\leq \|\mathbf{w}_{0,h} - \mathbf{U}_0\|_{0,\Omega} \|\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0\|_{0,\Omega} \, .
\end{aligned}
$$

On the other hand, if $\operatorname{curl} \mathbf{U}_0 \in \operatorname{curl} W_h$ it can be proved that

$$
\begin{aligned}
\|\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0\|_{0,\Omega} &\leq C h^s (\|\mathbf{U}_0\|_{s,\Omega} + \|\operatorname{curl} \mathbf{U}_0\|_{0,\Omega}) \\
&\leq C h^s \|\mathbf{w}_{0,h}\|_{\mathrm{curl},\Omega} \, . \,]
\end{aligned}
$$

The cavity problem: numerical approximation (cont'd)

Then

$$\begin{aligned}\int_\Omega \operatorname{grad} p_0 \cdot \mathbf{w}_{0,h} &= \int_\Omega \operatorname{grad} p_0 \cdot (\mathbf{w}_{0,h} - \mathbf{U}_0) \\ &\leq \|\operatorname{grad} p_0\|_{0,\Omega} \|\mathbf{w}_{0,h} - \mathbf{U}_0\|_{0,\Omega} \\ &\leq Ch^s \|\mathbf{w}_h\|_{\mathrm{curl},\Omega} \|\operatorname{grad} p_0\|_{0,\Omega} \\ &\leq Ch^s \|\mathbf{w}_h\|_{\mathrm{curl},\Omega} \|\mathbf{e}_h\|_{0,\Omega} \,.\end{aligned} \tag{61}$$

In conclusion

$$\sup_{\mathbf{w}_h \in W_h} \frac{\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega}} \leq Ch^s \|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \quad , \tag{62}$$

and from (57) for h small enough we have

$$\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \leq C\|\mathbf{E} - P_h\mathbf{E}\|_{\mathrm{curl},\Omega} \,. \tag{63}$$

The cavity problem: numerical approximation (cont'd)

This estimate ensures that for h small enough problem (54) is well-posed. Since it is enough to prove uniqueness, suppose that $\mathbf{E}_h$ is a solution corresponding to $\mathbf{F} = \mathbf{0}$. We know that for this right hand side the exact solution $\mathbf{E}$ of (53) is vanishing, therefore $\mathbf{e}_h = -\mathbf{E}_h$. Using (63) it follows $\mathbf{e}_h = \mathbf{0}$, hence the uniqueness of the solution to (54).

Moreover, since

$$\|\mathbf{E} - P_h\mathbf{E}\|_{\mathrm{curl},\Omega} = \inf_{\mathbf{w}_h \in W_h} \|\mathbf{E} - \mathbf{w}_h\|_{\mathrm{curl},\Omega} \,,$$

we have also obtained the quasi-optimal error estimate

$$\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \leq C \inf_{\mathbf{w}_h \in W_h} \|\mathbf{E} - \mathbf{w}_h\|_{\mathrm{curl},\Omega} \,, \tag{64}$$

valid for h small enough.

E and H formulations

We want now to present some coupled approaches. In order to understand more clearly the situation, we start going back to the formulations of the Maxwell and eddy current problems.

In terms of the electric field, the time-harmonic Maxwell equations read

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E}) - \omega^2 \boldsymbol{\eta} \mathbf{E} = -i\omega \mathbf{J}_e & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \,, \end{cases} \tag{65}$$

having set $\boldsymbol{\eta} := \boldsymbol{\varepsilon} - i\omega^{-1}\boldsymbol{\sigma}$.

[The condition $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ has to be substituted by $\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ when considering the magnetic boundary condition.]

E and H formulations (cont'd)

Similarly, in terms of the magnetic field they are written as

$$\begin{cases} \text{curl}(\boldsymbol{\eta}^{-1}\,\text{curl}\,\mathbf{H}) - \omega^2 \boldsymbol{\mu}\mathbf{H} = \text{curl}(\boldsymbol{\eta}^{-1}\mathbf{J}_e) & \text{in } \Omega \\ \boldsymbol{\eta}^{-1}\,\text{curl}\,\mathbf{H} \times \mathbf{n} = \boldsymbol{\eta}^{-1}\mathbf{J}_e \times \mathbf{n} & \text{on } \partial\Omega\,. \end{cases} \tag{66}$$

Once the electric field $\mathbf{E}$ is available, one sets

$$\mathbf{H} = i\omega^{-1}\boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{E} \qquad \text{in } \Omega\,. \tag{67}$$

On the other hand, from $\mathbf{H}$ one determines

$$\mathbf{E} = -i\omega^{-1}\boldsymbol{\eta}^{-1}(\text{curl}\,\mathbf{H} - \mathbf{J}_e) \qquad \text{in } \Omega\,. \tag{68}$$

E and H formulations (cont'd)

The structure of (65) and (66) is quite similar, since Re $\boldsymbol{\eta} = \boldsymbol{\varepsilon}$ is positive definite.

A Fredholm alternative approach can be used for proving well-posedness, and, similarly to the case of the cavity problem, numerical approximation by means of edge elements is the standard option.

In this framework, coupled formulations are not particularly appealing.

$\mathbf{E}$ and $\mathbf{H}$ formulations (cont'd)

For the eddy current equations, this symmetry is broken, and a degeneration occurs where $\boldsymbol{\sigma}$ is vanishing. We have

- $\mathbf{E}$ formulation

$$\begin{cases} \text{curl}(\boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{E}) + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{J}_e & \text{in } \Omega \\ \text{div}(\boldsymbol{\varepsilon}_I\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ BC_E(\mathbf{E}_I) = 0 & \text{on } \partial\Omega \\ \boldsymbol{\varepsilon}_I\mathbf{E}_I \perp \mathcal{H}_I \end{cases} \qquad (69)$$

[where the condition $\boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{E}\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$ has to be dropped if considering the electric boundary condition].

The magnetic field $\mathbf{H}$ is computed as in (67).

E and H formulations (cont'd)

H formulation

$$\begin{cases} \text{curl}(\boldsymbol{\sigma}^{-1}\,\text{curl}\,\mathbf{H}_C) + i\omega\boldsymbol{\mu}_C\mathbf{H}_C & \\ \qquad\qquad = \text{curl}(\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}) & \text{in } \Omega_C \\ \text{curl}\,\mathbf{H}_I = \mathbf{J}_{e,I} & \text{in } \Omega_I \\ \text{div}(\boldsymbol{\mu}\mathbf{H}) = 0 & \text{in } \Omega \\ \mathit{BC}_H(\mathbf{H}_I) = 0 & \text{on } \partial\Omega \\ \mathbf{H}_I \times \mathbf{n}_I + \mathbf{H}_C \times \mathbf{n}_C = \mathbf{0} & \text{on } \Gamma \\ \mathit{TOP}(\mathbf{H}) = 0\ , & \end{cases} \tag{70}$$

where $\mathit{BC}_H(\mathbf{H}_I)$ means $\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n}$ for the electric boundary condition, and $\mathbf{H}_I \times \mathbf{n}$ for the magnetic boundary conditions, and $\mathit{TOP}(\mathbf{H}) = 0$ is a set of topological conditions that have to be satisfied by the magnetic field $\mathbf{H}$.

E and H formulations (cont'd)

Having determined $\mathbf{H}$, the electric field is obtained by setting

$$\mathbf{E}_C = \boldsymbol{\sigma}^{-1}(\text{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C}) \qquad \text{in } \Omega_C\ , \tag{71}$$

and solving the problem

$$\begin{cases} \text{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}_I\mathbf{H}_I & \text{in } \Omega_I \\ \text{div}(\boldsymbol{\varepsilon}_I\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ BC_E(\mathbf{E}_I) = 0 & \text{on } \partial\Omega \\ \mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C & \text{on } \Gamma \\ \boldsymbol{\varepsilon}_I\mathbf{E}_I \perp \mathcal{H}_I\ . & \end{cases} \tag{72}$$

This last problem is not always solvable, but needs that some compatibility conditions on the data are satisfied.

Topological conditions on the magnetic field

Besides the conditions $\mathrm{div}(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω and $\boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$ (if $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$), that are clearly satisfied, it is important to underline that the other needed compatibility conditions are the topological conditions $\mathit{TOP}(\mathbf{H}) = 0$.

Let us make clear their structure. For the sake of definiteness, let us focus on the electric boundary condition. We need to consider again the (finite dimensional) space

$$\mathcal{H}_I^{(D)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathrm{curl}\, \mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0 \\ \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \cup \Gamma\} \,,$$

and its basis functions $\boldsymbol{\rho}^*_{\alpha,I}$, $\alpha = 1, \ldots, n_{\Omega_I}$ [let us recall that n_{Ω_I} is the first Betti number of Ω_I, or, equivalently, the number of (independent) non-bounding cycles in Ω_I].

Topological conditions on the magnetic field (cont'd)

The topological conditions *TOP*$(\mathbf{H}) = 0$ mean that

$$\begin{aligned} &\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}^*_{\alpha,I} \\ &\quad + \int_\Gamma [\boldsymbol{\sigma}^{-1}(\text{curl}\, \mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot \boldsymbol{\rho}^*_{\alpha,I} = 0 \end{aligned} \tag{73}$$

for each $\alpha = 1, \dots, n_{\Omega_I}$.

Note that one has $n_{\Omega_I} \geq 1$ if the conductor Ω_C is not simply-connected, and therefore in that case these conditions have to be taken into account.

- It can be proved that the topological conditions *TOP*$(\mathbf{H}) = 0$ are equivalent to the integral form of the Faraday equation on each surface that "cuts" a non-bounding cycle [Seifert surface].

A FEM–BEM approach

We are now ready to present an approach based on a coupled formulation: variational in Ω_C, by means of potential theory in Ω_I.

In this case, it is reasonable to consider $\Omega_I := \mathbb{R}^3 \setminus \overline{\Omega_C}$. Moreover, for the sake of simplicity let us require that Ω_C is a simply-connected open set with a connected boundary, so that we have not to take into account the topological conditions on $\mathbf{H}$.

Finally, it is assumed that the applied current density $\mathbf{J}_e$ is vanishing in Ω_I, and that the magnetic permeability $\boldsymbol{\mu}_I$ and the electric permittivity $\boldsymbol{\varepsilon}_I$ are positive constants in Ω_I, say $\mu_0 > 0$ and $\varepsilon_0 > 0$, the permeability and the permittivity of the vacuum.

A FEM–BEM approach (cont'd)

As we have seen before, in terms of the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}_C$ the eddy current problem reads

$$\begin{cases} \text{curl}\,\mathbf{E}_C + i\omega\boldsymbol{\mu}_C\mathbf{H}_C = \mathbf{0} & \text{in } \Omega_C \\ \text{curl}\,\mathbf{H}_C - \boldsymbol{\sigma}\mathbf{E}_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\ \text{curl}\,\mathbf{H}_I = \mathbf{0} & \text{in } \Omega_I \\ \text{div}(\mu_0\mathbf{H}_I) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}_C\mathbf{H}_C \cdot \mathbf{n}_C + \mu_0\mathbf{H}_I \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\ \mathbf{H}_C \times \mathbf{n}_C + \mathbf{H}_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\ \mathbf{H}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\ . \end{cases} \tag{74}$$

A FEM–BEM approach (cont'd)

[If needed, the electric field $\mathbf{E}_I$ can be computed after having determined $\mathbf{H}_I$ and $\mathbf{E}_C$ in (74), by solving

$$\begin{cases} \text{curl}\, \mathbf{E}_I = -i\omega\mu_0 \mathbf{H}_I & \text{in } \Omega_I \\ \text{div}(\varepsilon_0 \mathbf{E}_I) = 0 & \text{in } \Omega_I \\ \mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C & \text{on } \Gamma \\ \int_\Gamma \varepsilon_0 \mathbf{E}_I \cdot \mathbf{n}_I = 0 & \\ \mathbf{E}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \text{ .]} \end{cases}$$

A FEM–BEM approach (cont'd)

For obtaining a formulation which is stable with respect to the frequency ω, it is better to look for a vector magnetic potential $\mathbf{A}_C$, a scalar electric potential V_C and a scalar magnetic potential ψ_I such that

$$\boldsymbol{\mu}_C \mathbf{H}_C = \text{curl}\, \mathbf{A}_C \quad , \quad \mathbf{E}_C = -i\omega \mathbf{A}_C - \text{grad}\, V_C \quad , \quad \mathbf{H}_I = \text{grad}\, \psi_I \; .$$

[See Pillsbury (1983), Rodger and Eastham (1983), Emson and Simkin (1983).]

Gauging is necessary only in Ω_C: we require the Coulomb gauge $\text{div}\, \mathbf{A}_C = 0$ in Ω_C, with $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ. Moreover, we also impose that

$$|\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty \; .$$

A FEM–BEM approach (cont'd)

We have thus obtained the problem

$$
\begin{cases}
\mathrm{curl}(\boldsymbol{\mu}_C^{-1}\,\mathrm{curl}\,\mathbf{A}_C) & \\
\qquad\qquad +i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathrm{grad}\,V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\
\Delta\psi_I = 0 & \text{in } \Omega_I \\
\mathrm{div}\,\mathbf{A}_C = 0 & \text{in } \Omega_C \\
\mathbf{A}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \\
\mathrm{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C + \mu_0\,\mathrm{grad}\,\psi_I \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\
(\boldsymbol{\mu}_C^{-1}\,\mathrm{curl}\,\mathbf{A}_C) \times \mathbf{n}_C + \mathrm{grad}\,\psi_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\
|\psi_I(\mathbf{x})| + |\,\mathrm{grad}\,\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\ ,
\end{cases}
$$

where V_C is determined up to an additive constant.

A FEM–BEM approach (cont'd)

Inserting the Coulomb gauge condition in the Ampère equation as a penalization term, one has

$$
\left\{
\begin{array}{ll}
\mathrm{curl}(\boldsymbol{\mu}_C^{-1}\,\mathrm{curl}\,\mathbf{A}_C) - \mu_*^{-1}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{A}_C & \\
\qquad + i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathrm{grad}\,V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\
\Delta\psi_I = 0 & \text{in } \Omega_I \\
\mathrm{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathrm{grad}\,V_C) = \mathrm{div}\,\mathbf{J}_{e,C} & \text{in } \Omega_C \\
(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathrm{grad}\,V_C)\cdot\mathbf{n}_C & \\
\qquad = \mathbf{J}_{e,C}\cdot\mathbf{n}_C & \text{on } \Gamma \\
\mathbf{A}_C\cdot\mathbf{n}_C = 0 & \text{on } \Gamma \\
\mathrm{curl}\,\mathbf{A}_C\cdot\mathbf{n}_C + \mu_0\,\mathrm{grad}\,\psi_I\cdot\mathbf{n}_I = 0 & \text{on } \Gamma \\
(\boldsymbol{\mu}_C^{-1}\,\mathrm{curl}\,\mathbf{A}_C)\times\mathbf{n}_C & \\
\qquad + \mathrm{grad}\,\psi_I\times\mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\
|\psi_I(\mathbf{x})| + |\,\mathrm{grad}\,\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}|\to\infty\ .
\end{array}
\right.
\tag{75}
$$

A FEM–BEM approach (cont'd)

Since in Ω_I we have to solve the Laplace equation, using potential theory it is possible to transform the problem for ψ_I into a problem on the interface Γ, thus reducing in a significative way the number of unknowns in numerical computations.

We introduce on Γ (using suitable functional spaces...) the single layer and double layer potentials

$$\mathcal{S}(\xi)(\mathbf{x}) := \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\, \xi(\mathbf{y}) dS_y$$

$$\mathcal{D}(\eta)(\mathbf{x}) := \int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \cdot \eta(\mathbf{y}) \mathbf{n}_C(\mathbf{y}) dS_y$$

A FEM–BEM approach (cont'd)

and the hypersingular integral operator

$$\mathcal{H}(\eta)(\mathbf{x}) := -\,\text{grad}\left(\int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3}\cdot\eta(\mathbf{y})\mathbf{n}_C(\mathbf{y})dS_y\right)\cdot\mathbf{n}_C(\mathbf{x})\ .$$

We also recall that the adjoint operator $\mathcal{D}'$ reads

$$\mathcal{D}'(\xi)(\mathbf{x}) = \left(\int_\Gamma \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3}\,\xi(\mathbf{y})dS_y\right)\cdot\mathbf{n}_C(\mathbf{x})\ .$$

A FEM–BEM approach (cont'd)

We have $\Delta\psi_I = 0$ in Ω_I and $\operatorname{grad}\psi_I \cdot \mathbf{n}_I = -\frac{1}{\mu_0}\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C$ on Γ, therefore from potential theory the trace $\psi_\Gamma := \psi_{I|\Gamma}$ satisfies the bounday integral equations

$$\frac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) + \frac{1}{\mu_0}\mathcal{S}(\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C) = 0 \text{ on } \Gamma \tag{76}$$

$$\frac{1}{2\mu_0}\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C + \frac{1}{\mu_0}\mathcal{D}'(\operatorname{curl}\mathbf{A}_C \cdot \mathbf{n}_C) + \mathcal{H}(\psi_\Gamma) = 0 \text{ on } \Gamma\,, \tag{77}$$

and the unknown ψ_I can be replaced by its trace ψ_Γ.

We can now devise a weak form of this $(\mathbf{A}_C, V_C) - \psi_\Gamma$ formulation. From the matching condition

$$\mathbf{n}_C \times \boldsymbol{\mu}_C^{-1}\operatorname{curl}\mathbf{A}_C + \mathbf{n}_I \times \operatorname{grad}\psi_I = \mathbf{0} \text{ on } \Gamma$$

A FEM–BEM approach (cont'd)

we find

$$\int_\Gamma \mathbf{n}_C \times \boldsymbol{\mu}_C^{-1} \,\text{curl}\, \mathbf{A}_C \cdot \overline{\mathbf{w}_C} = - \int_\Gamma \mathbf{n}_I \times \text{grad}\, \psi_I \cdot \overline{\mathbf{w}_C}$$
$$= - \int_\Gamma \psi_\Gamma \,\text{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \,,$$

the last equality coming from standard integration by parts on Γ.
Hence, multiplying by suitable test functions $(\mathbf{w}_C, Q_C, \eta)$ with $\mathbf{w}_C \cdot \mathbf{n}_C = 0$ on Γ, integrating in Ω_C and Γ, and integrating by parts we end up with the following weak problem

A FEM–BEM approach (cont'd)

$$\begin{aligned}
&\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1}\,\text{curl}\,\mathbf{A}_C \cdot \text{curl}\,\overline{\mathbf{w}_C} + \mu_*^{-1}\,\text{div}\,\mathbf{A}_C\,\text{div}\,\overline{\mathbf{w}_C}) \\
&\qquad + \int_{\Omega_C} (i\omega\boldsymbol{\sigma}\mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma}\,\text{grad}\,V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad + \int_\Gamma [-\tfrac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) \\
&\qquad\qquad\qquad + \frac{1}{\mu_0}\mathcal{S}(\text{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C)]\,\text{curl}\,\overline{\mathbf{w}_C} \cdot \mathbf{n}_C \\
&\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C} \\
&\int_{\Omega_C} (i\omega\boldsymbol{\sigma}\mathbf{A}_C \cdot \text{grad}\,\overline{Q_C} + \boldsymbol{\sigma}\,\text{grad}\,V_C \cdot \text{grad}\,\overline{Q_C}) \\
&\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\,\overline{Q_C} \\
&\int_\Gamma [\tfrac{1}{2}\,\text{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C + \mathcal{D}'(\text{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C) + \mu_0\mathcal{H}(\psi_\Gamma)]\overline{\eta} = 0\ ,
\end{aligned} \tag{78}$$

having used (76) for obtaining the first equation.

[See Alonso Rodríguez and V. (2009).]

A FEM–BEM approach (cont'd)

- The sesquilinear form at the left hand side is coercive in $[H(\text{curl}; \Omega_C) \cap H_0(\text{div}; \Omega_C)] \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma)/\mathbb{C}$, uniformly with respect to ω (the case $\omega = 0$ is admitted!). [The crucial point is that $\mathcal{S}$ and $\mathcal{H}$ are coercive; the rest of the proof is similar to that employed for the $(\mathbf{A}, V_C)$-formulation.]
- Existence and uniqueness follow by the Lax–Milgram lemma.
- Having determined $\mathbf{A}_C$ and ψ_Γ (up to an additive constant), then $\psi_I := \mathcal{D}(\psi_\Gamma) - \frac{1}{\mu_0}\mathcal{S}(\text{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C)$.
- Numerical approximation is performed with nodal finite elements in Ω_C and on Γ [boundary elements on Γ].

A FEM–BEM approach (cont'd)

- Convergence is ensured provided that Ω_C is a convex polyhedron. If this is not true, one can modify the approach, using the vector potential $\mathbf{A}$ on a convex set Ω_A larger than Ω_C, keeping V_C in Ω_C and looking for ψ_{Γ_A} on $\Gamma_A := \partial\Omega_A$.

Other FEM–BEM couplings

- Bossavit and Vérité(1982, 1983) (for the magnetic field, and using the Steklov–Poincaré operator) [numerical code TRIFOU].
- Mayergoyz, Chari and Konrad (1983) (for the electric field, and using special basis functions near Γ).
- Hiptmair (2002) (unknowns: $\mathbf{E}_C$ in Ω_C, $\mathbf{H} \times \mathbf{n}$ on Γ).
- Meddahi and Selgas (2003) (unknowns: $\mathbf{H}_C$ in Ω_C, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n}$ on Γ).
- Bermúdez, Gómez, Muñiz and Salgado (2007) (for axisymmetric problems associated to the modeling of induction furnaces).

Weak formulations for $\mathbf{H}$ and $\mathbf{E}$

Other coupled formulations stem from a deeper analysis of the weak formulations for the magnetic and electric fields.

First of all, under the necessary conditions

$$\text{div}\,\mathbf{J}_{e,I} = 0 \text{ in } \Omega_I \ , \ \mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \ , \ \mathbf{J}_{e,I} \perp \mathcal{H}_I \ ,$$

it can be shown that there exists a field $\mathbf{H}_e \in H(\text{curl}; \Omega)$ satisfying

$$\begin{cases} \text{curl}\,\mathbf{H}_{e,I} = \mathbf{J}_{e,I} & \text{in } \Omega_I \\ BC_H(\mathbf{H}_{e,I}) = 0 & \text{on } \partial\Omega \end{cases}$$

[the boundary conditions for $\mathbf{J}_{e,I}$ and $\mathbf{H}_{e,I}$ have to be dropped if considering the electric boundary condition].

Weak H-formulation (cont'd)

Setting

$$V := \{\mathbf{v} \in H(\text{curl};\Omega) \mid \text{curl}\,\mathbf{v}_I = \mathbf{0} \text{ in } \Omega_I, \mathbf{v}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$$

[the boundary condition has to be dropped if considering the electric boundary condition], multiplying the Faraday equation by $\overline{\mathbf{v}}$, with $\mathbf{v} \in V$, integrating in Ω and integrating by parts one finds

$$\int_{\Omega_C} \mathbf{E}_C \cdot \text{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega_I} \mathbf{E}_I \cdot \text{curl}\,\overline{\mathbf{v}_I} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{v}} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} = 0 \ ,$$

thus

$$\int_{\Omega_C} \mathbf{E}_C \cdot \text{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} = 0 \ ,$$

as $\text{curl}\,\mathbf{v}_I = \mathbf{0}$ in Ω_I.

Weak H-formulation (cont'd)

Using the Ampère equation in Ω_C for expressing $\mathbf{E}_C$, we end up with the following problem

Find $(\mathbf{H} - \mathbf{H}_e) \in V$:

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \operatorname{curl} \mathbf{H}_C \cdot \operatorname{curl} \overline{\mathbf{v}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{v}} = \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \operatorname{curl} \overline{\mathbf{v}_C} \qquad (79)$$

for each $\mathbf{v} \in V$.

This formulation is well-posed via the Lax–Milgram lemma, as the sesquilinear form

$$a(\mathbf{u}, \mathbf{v}) := \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \operatorname{curl} \mathbf{u}_C \cdot \operatorname{curl} \overline{\mathbf{v}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{u} \cdot \overline{\mathbf{v}}$$

is clearly continuous and coercive in V.

Weak E-formulation

For deriving the weak $\mathbf{E}$-formulation one starts from the Ampère equation: multiplying by $\overline{\mathbf{z}}$, integrating in Ω and integrating by parts one easily sees that

$$\int_\Omega \mathbf{H} \cdot \mathrm{curl}\, \overline{\mathbf{z}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{H} \cdot \overline{\mathbf{z}} - \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}}$$

for all $\mathbf{z} \in H(\mathrm{curl}; \Omega)$.
The boundary term disappears if $\mathbf{H}$ satisfies the magnetic boundary condition, or if $\mathbf{z}$ satisfies the electric boundary condition.
Set

$$\begin{aligned} Z \quad &:= \{\mathbf{z} \in H(\mathrm{curl}; \Omega) \,|\, \mathrm{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I) = 0 \text{ in } \Omega_I, \\ &\qquad\qquad\qquad\qquad BC_E(\mathbf{z}_I) = 0,\ \boldsymbol{\varepsilon}_I \mathbf{z}_I \perp \mathcal{H}_I\} \,. \end{aligned}$$

Weak E-formulation (cont'd)

Expressing $\mathbf{H}$ through the Faraday equation, the weak $\mathbf{E}$-formulation finally reads

Find $\mathbf{E} \in Z$:

$$\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} = -i\omega \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}} \tag{80}$$

for each $\mathbf{z} \in Z$.

Though less straightforward, it can be proved that the sesquilinear form

$$a_e(\mathbf{w}, \mathbf{z}) := \int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{w} \cdot \operatorname{curl} \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{z}_C}$$

is continuous and coercive in Z, and well-posedness of the weak $\mathbf{E}$-formulation follows from Lax–Milgram lemma.

Numerical approximation

Both problems (79) and (80) contain a differential constraint: the former on the curl, the latter on the divergence.

- Numerical approximation needs some care!

Possible ways of attack:

- saddle-point formulations [Lagrange multipliers]
- a scalar potential for $\mathbf{H}_I - \mathbf{H}_{e,I}$
- a vector potential for $\varepsilon_I \mathbf{E}_I$.

Numerical approximation (cont'd)

The first choice has been considered by Alonso Rodríguez, Hiptmair and V. (2004a) (for the magnetic field) and by Alonso Rodríguez and V. (2004) (for the electric field); hybrid (coupled) formulations in terms of $(\mathbf{H}_C, \mathbf{E}_I)$ or $(\mathbf{E}_C, \mathbf{H}_I)$ have been also proposed and analyzed (Alonso Rodríguez, Hiptmair and V. (2004b, 2005)).

The second possibility, also leading to coupled formulations, will be described here below.

To our knowledge, the third choice has not been completely exploited. [However, in a different though related situation we have before presented a similar procedure: the (classical) approach based on a vector potential for the divergence free vector field $\mu\mathbf{H}$.]

Scalar potential formulation

For the sake of definiteness let us consider the electric boundary condition.

The starting point is to consider $\mathbf{H}_e \in H(\text{curl}; \Omega)$ satisfying

$$\text{curl}\, \mathbf{H}_{e,I} = \mathbf{J}_{e,I} \qquad \text{in } \Omega_I \,.$$

Then the main step is to use the Helmholtz orthogonal decomposition

$$\mathbf{H}_I - \mathbf{H}_{e,I} = \text{grad}\, \psi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \;, \tag{81}$$

where $\psi_I^* \in H^1(\Omega_I)/\mathbb{C}$ and $\eta_{I,\alpha}^* \in \mathbb{C}$ (the two terms of the decomposition are orthogonal, with respect to the scalar product $(\mathbf{u}_I, \mathbf{v}_I)_{\mu_I,\Omega_I} := \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{u}_I \cdot \mathbf{v}_I$).

Orthogonal decompositions

There are infinitely many of these decomposition results...

Let us recall the two that are interesting for the magnetic field:

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{Q}_I^* + \text{grad}\,\chi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \theta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^*$$

and

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{Q}_I + \text{grad}\,\chi_I + \sum_{l=1}^{p_{\partial\Omega}} a_{I,l}\,\text{grad}\,z_{l,I} + \sum_{m=1}^{n_\Gamma} b_{I,m}\boldsymbol{\rho}_{m,I}\ .$$

Orthogonal decompositions (cont'd)

Let us explain the first decomposition.

The vector function $\mathbf{Q}_I^*$ is the solution to

$$\begin{cases} \text{curl}(\boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{Q}_I^*) = \text{curl}\,\mathbf{v}_I & \text{in } \Omega_I \\ \text{div}\,\mathbf{Q}_I^* = 0 & \text{in } \Omega_I \\ \mathbf{Q}_I^* \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \cup \partial\Omega \\ \mathbf{Q}_I^* \perp \mathcal{H}_{I,\varepsilon_0}^{(A)} & \end{cases}$$

[$\mathcal{H}_{I,\varepsilon_0}^{(A)}$ denotes $\mathcal{H}_I^{(A)}$ for $\boldsymbol{\varepsilon}_I = \varepsilon_0$, a positive constant].
The scalar function χ_I^* is the solution to the elliptic Neumann boundary value problem

$$\begin{cases} \text{div}(\boldsymbol{\mu}_I\,\text{grad}\,\chi_I^*) = \text{div}(\boldsymbol{\mu}_I \mathbf{v}_I) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I\,\text{grad}\,\chi_I^* \cdot \mathbf{n}_I = \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I & \text{on } \Gamma \cup \partial\Omega\,. \end{cases}$$

Orthogonal decompositions (cont'd)

Finally the vector $\theta^*_{I,\alpha}$ is the solution of the linear system

$$\sum_{\alpha=1}^{n_{\Omega_I}} A^*_{\beta\alpha}\theta^*_{I,\alpha} = \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \boldsymbol{\rho}^*_{\beta,I} \ ,$$

where

$$A^*_{\beta\alpha} := \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}^*_{\alpha,I} \cdot \boldsymbol{\rho}^*_{\beta,I} \ ,$$

and the harmonic vector fields $\boldsymbol{\rho}^*_{\alpha,I}$ are the basis functions of the space $\mathcal{H}_I^{(D)}$.

Orthogonal decompositions (cont'd)

Let us explain the second decomposition.

The vector function $\mathbf{Q}_I$ is the solution to

$$\begin{cases} \text{curl}(\boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{Q}_I) = \text{curl}\,\mathbf{v}_I & \text{in } \Omega_I \\ \text{div}\,\mathbf{Q}_I = 0 & \text{in } \Omega_I \\ \mathbf{Q}_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\ \mathbf{Q}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ (\boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{Q}_I) \times \mathbf{n} = \mathbf{v}_I \times \mathbf{n} & \text{on } \partial\Omega \\ \mathbf{Q}_I \perp \mathcal{H}_{I,\varepsilon_0}^{(B)} & \end{cases}$$

[$\mathcal{H}_{I,\varepsilon_0}^{(B)}$ denotes $\mathcal{H}_I^{(B)}$ for $\boldsymbol{\varepsilon}_I = \varepsilon_0$, a positive constant].

Orthogonal decompositions (cont'd)

The scalar function χ_I is the solution to the elliptic mixed boundary value problem

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I \,\mathrm{grad}\, \chi_I) = \mathrm{div}(\boldsymbol{\mu}_I \mathbf{v}_I) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \,\mathrm{grad}\, \chi_I \cdot \mathbf{n}_I = \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I & \text{on } \Gamma \\ \chi_I = 0 & \text{on } \partial\Omega \,. \end{cases}$$

Finally the vector $(a_{I,l}, b_{I,m})$ is the solution of the linear system

$$A \begin{pmatrix} a_{I,l} \\ b_{I,m} \end{pmatrix} = \begin{pmatrix} \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathrm{grad}\, z_{s,I} \\ \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \boldsymbol{\rho}_{n,I} \end{pmatrix} ,$$

Orthogonal decompositions (cont'd)

where $A := \begin{pmatrix} D & B \\ B^T & C \end{pmatrix}$ with

$$
\begin{aligned}
D_{sl} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \, \mathrm{grad}\, z_{l,I} \cdot \mathrm{grad}\, z_{s,I} \\
B_{sm} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_{m,I} \cdot \mathrm{grad}\, z_{s,I} \\
C_{mn} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_{n,I} \cdot \boldsymbol{\rho}_{m,I} \ ,
\end{aligned}
$$

and the harmonic vector fields $\mathrm{grad}\, z_{s,I}$ and $\boldsymbol{\rho}_{n,I}$ are the basis functions of the space $\mathcal{H}_I^{(C)}$.

Scalar potential formulation (cont'd)

Coming back to the scalar potential formulation, in (79) each test function $\mathbf{v} \in V$ can be thus written as

$$\mathbf{v}_I = \text{grad}\, \chi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \theta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \,. \tag{82}$$

Inserting (81) and (82) in (79) and using orthogonality one easily finds, for the unknowns $\mathbf{Z}_C := \mathbf{H}_C - \mathbf{H}_{e,C}$, ψ_I^*, $\eta_{I,\alpha}^*$,

$$\begin{aligned}
&\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \text{curl}\, \mathbf{Z}_C \cdot \text{curl}\, \overline{\mathbf{v}_C} + \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{Z}_C \cdot \overline{\mathbf{v}_C} \\
&\qquad + \int_{\Omega_I} i\omega \boldsymbol{\mu}_I\, \text{grad}\, \psi_I^* \cdot \text{grad}\, \overline{\chi_I^*} + i\omega [A^* \boldsymbol{\eta}_I^*, \boldsymbol{\theta}_I^*] \\
&\quad = - \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \text{curl}\, \mathbf{H}_{e,C} \cdot \text{curl}\, \overline{\mathbf{v}_C} - \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_{e,C} \cdot \overline{\mathbf{v}_C} \\
&\qquad - \int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot (\text{grad}\, \overline{\chi_I^*} + \textstyle\sum_{\alpha=1}^{n_{\Omega_I}} \overline{\theta_{I,\alpha}^*} \boldsymbol{\rho}_{\alpha,I}^*) \\
&\qquad + \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \text{curl}\, \overline{\mathbf{v}_C} \,,
\end{aligned} \tag{83}$$

Scalar potential formulation (cont'd)

where we recall that the matrix A^* is defined by

$$A^*_{\beta\alpha} := \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}^*_{\alpha,I} \cdot \boldsymbol{\rho}^*_{\beta,I} \,,$$

and is symmetric and positive definite (the fields $\boldsymbol{\rho}^*_{\alpha,I}$ form a basis for the space $\mathcal{H}^{(D)}_I$).
Clearly, the solutions $\mathbf{Z}_C$, ψ^*_I and $\boldsymbol{\eta}^*_I$ have to satisfy on Γ the matching condition

$$\mathbf{Z}_C \times \mathbf{n}_C + \mathrm{grad}\, \psi^*_I \times \mathbf{n}_I + \sum_{\alpha=1}^{n_{\Omega_I}} \eta^*_{I,\alpha} \boldsymbol{\rho}^*_{\alpha,I} \times \mathbf{n}_I = \mathbf{0}\,.$$

The same holds for the test functions $\mathbf{v}_C$, χ^*_I and $\boldsymbol{\theta}^*_I$.

Scalar potential formulation (cont'd)

The left hand side in (83) is a continuous and coercive sesquilinear form, therefore the problem is well-posed.

The numerical approximation is standard:

- (vector) edge finite elements in Ω_C
- (scalar) nodal finite elements in Ω_I.

In addition, one looks for

- other n_{Ω_I} degrees of freedom (expressing the line integrals of $\mathbf{H}_I - \mathbf{H}_{e,I}$ along the non-bounding cycles contained in $\overline{\Omega_I}$).

Convergence is ensured by Céa lemma.

[Bermúdez, Rodríguez and Salgado (2002), Alonso Rodríguez, Fernandes and V. (2003).]

Scalar potential formulation (cont'd)

Some remarks about implementation issues:

- The matching condition on the interface Γ is easily imposed by eliminating the degrees of freedom of $\mathbf{v}_{C,h}$ associated to the edges and faces on Γ in terms of those of grad $\chi^*_{I,h} + \sum_{\alpha=1}^{n_{\Omega_I}} \theta^*_{I,\alpha} \boldsymbol{\rho}^*_{\alpha,I}$.
- The construction of the fields $\boldsymbol{\rho}^*_{\alpha,I}$ (or of a suitable approximation of them) is not needed.
  It is enough to construct n_{Ω_I} interpolants $\boldsymbol{\lambda}^*_\alpha$, each one jumping by 1 on a "cutting" surface (and continuous across all the others).
  One looses (in part) orthogonality properties, but everything works well.

Scalar potential formulation (cont'd)

- For the electric boundary condition, the construction of the vector $\mathbf{H}_{e,I}$ can be done through the Biot–Savart formula

$$\begin{aligned}\mathbf{H}_{e,I}(\mathbf{x}) \quad &:= \text{curl}\left(\int_{\Omega_I} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\, \mathbf{J}_{e,I}(\mathbf{y})\, d\mathbf{y}\right) \\ &= \int_{\Omega_I} \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \times \mathbf{J}_{e,I}(\mathbf{y})\, d\mathbf{y}\end{aligned}$$

[at least for $\mathbf{J}_{e,I} \cdot \mathbf{n} = 0$ on $\partial\Omega \cup \Gamma$; if this is not satisfied, one has to extend $\mathbf{J}_{e,I}$ on a set larger than Ω_I, in such a way that $\mathbf{J}_{e,I}$ is tangential on the boundary of this set].

Scalar potential formulation (cont'd)

- When considering the magnetic boundary condition, it must be noted that the Biot–Savart formula gives a vector field $\mathbf{H}_{e,I}$ that does not satisfy the boundary condition $\mathbf{H}_{e,I} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.
  Then, a couple of procedures can be adopted:
  - construct $\mathbf{H}_{e,I}$ (or a suitable approximation of it) by means of a different approach, in such a way that $\mathbf{H}_{e,I} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, and decompose $\mathbf{H}_I - \mathbf{H}_{e,I}$ as a sum of orthogonal terms, each one with vanishing tangential value on $\partial\Omega$;
  - use again the Biot–Savart formula, and decompose $\mathbf{H}_I - \mathbf{H}_{e,I}$ as in the case of the electric boundary condition.

Scalar potential formulation (cont'd)

Let us illustrate this second approach: we again write

$$\mathbf{Z}_I = \mathbf{H}_I - \mathbf{H}_{e,I} = \text{grad}\,\psi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \ ,$$

but now we have to consider a non-homogeneous boundary value problem (on $\partial\Omega$ we have $\mathbf{Z}_I \times \mathbf{n} \neq \mathbf{0}$).

The problem reads as follows: one looks for $\mathbf{Z}_C$, ψ_I^*, $\boldsymbol{\eta}_I^*$ such that

Scalar potential formulation (cont'd)

$$\text{grad}\,\psi_I^* \times \mathbf{n} + \textstyle\sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^*\,\boldsymbol{\rho}_{\alpha,I}^* \times \mathbf{n} = -\mathbf{H}_{e,I} \times \mathbf{n} \text{ on } \partial\Omega$$

$$\begin{aligned}
&\textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\text{curl}\,\mathbf{Z}_C \cdot \text{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega_C} i\omega\boldsymbol{\mu}_C \mathbf{Z}_C \cdot \overline{\mathbf{v}_C} \\
&\qquad + \textstyle\int_{\Omega_I} i\omega\boldsymbol{\mu}_I\,\text{grad}\,\psi_I^* \cdot \text{grad}\,\overline{\chi_I^*} + i\omega[A^*\boldsymbol{\eta}_I^*, \boldsymbol{\theta}_I^*] \\
&\quad = -\textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\text{curl}\,\mathbf{H}_{e,C} \cdot \text{curl}\,\overline{\mathbf{v}_C} - \int_{\Omega_C} i\omega\boldsymbol{\mu}_C \mathbf{H}_{e,C} \cdot \overline{\mathbf{v}_C} \\
&\qquad - \textstyle\int_{\Omega_I} i\omega\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot (\text{grad}\,\overline{\chi_I^*} + \sum_{\alpha=1}^{n_{\Omega_I}} \overline{\theta_{I,\alpha}^*}\boldsymbol{\rho}_{\alpha,I}^*) \\
&\qquad + \textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \text{curl}\,\overline{\mathbf{v}_C}\,,
\end{aligned} \tag{84}$$

where the test functions have to satisfy

$$\text{grad}\,\chi_I^* \times \mathbf{n} + \sum_{\alpha=1}^{n_{\Omega_I}} \theta_{I,\alpha}^*\,\boldsymbol{\rho}_{\alpha,I}^* \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\,,$$

Scalar potential formulation (cont'd)

and moreover the matching condition on Γ

$$\mathbf{Z}_C \times \mathbf{n}_C + \text{grad}\, \psi_I^* \times \mathbf{n}_I + \sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \times \mathbf{n}_I = \mathbf{0}$$

is still imposed (also for $\mathbf{v}_C$, χ_I^*, $\boldsymbol{\theta}_I^*$).

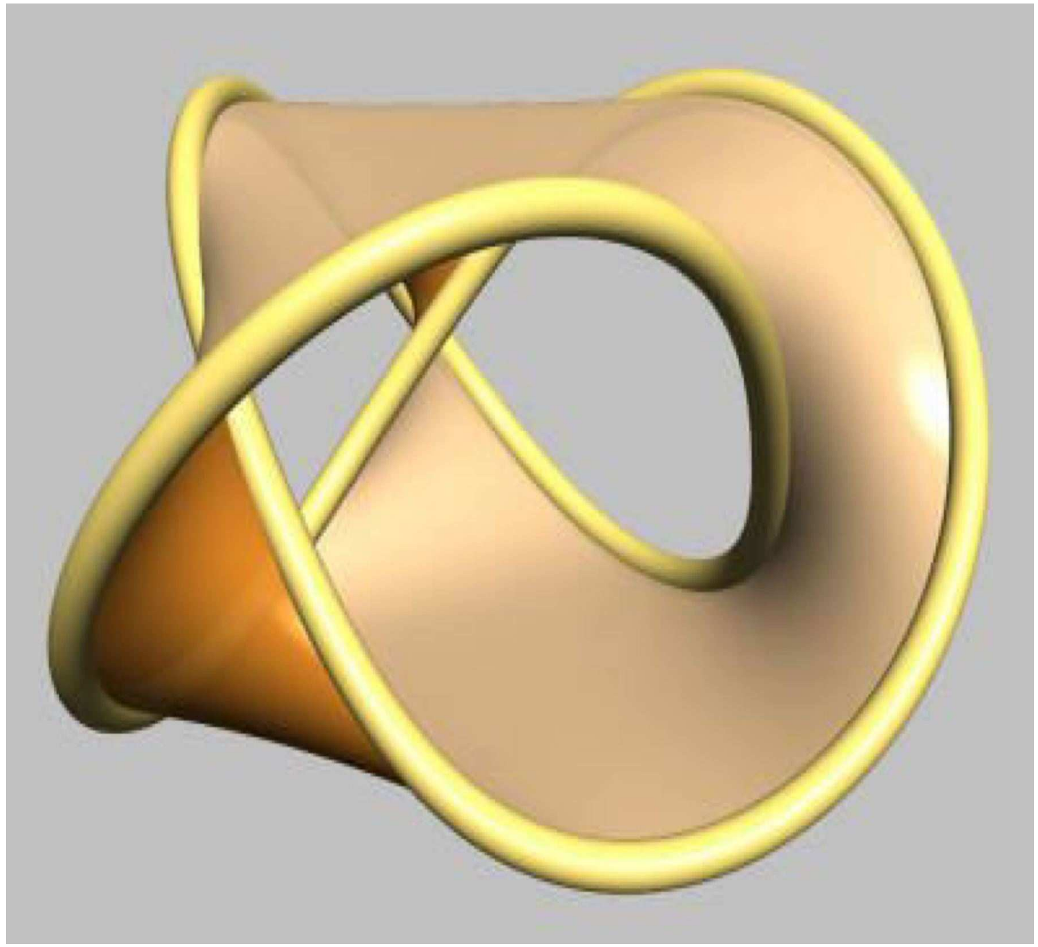
At the finite dimensional level the constraint on $\partial\Omega$ can be imposed by means of a Lagrange multiplier [Bermúdez, Rodríguez and Salgado (2002)].

Scalar potential formulation (cont'd)

- For implementation it is necessary to determine the "cutting" surfaces of the non-bounding cycles (their knowledge is necessary for constructing the basis functions $\boldsymbol{\rho}^*_{\alpha,I}$ or the interpolants $\boldsymbol{\lambda}^*_\alpha$). This can be easy in many situations, but for a general topological domain it can be computationally expensive.

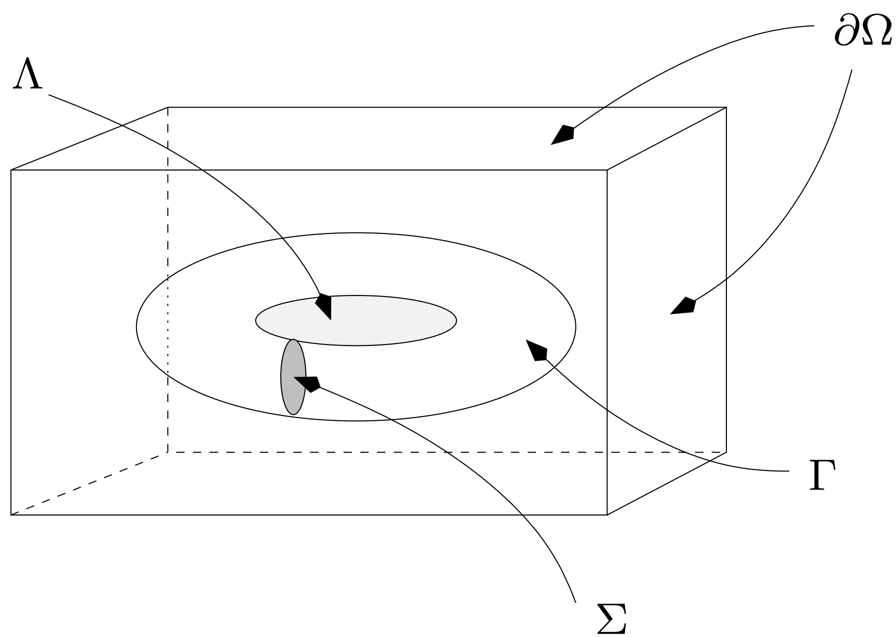
Let us see a picture of the "cutting" surface when Ω_C is the trefoil knot (thanks to J.J. van Wijk).

Scalar potential formulation (cont'd)



Scalar potential formulation (cont'd)

Instead, if Ω_C is a torus, we have the "cutting" surface Λ:



Some algorithms have been proposed to the aim of constructing "cutting" surfaces: see Kotiuga (1987, 1988, 1989), Leonard and Rodger (1989) and the book by Gross and Kotiuga (2004).

Scalar potential formulation (cont'd)

- A coupled formulation in terms of $\mathbf{E}_C$, ψ_I^* and $\boldsymbol{\eta}_I^*$ is also possible.

  From the Ampère equation in Ω_C, multiplying by $\overline{\mathbf{z}_C}$, integrating in Ω_C and integrating by parts one finds

$$\int_{\Omega_C} \mathbf{H}_C \cdot \operatorname{curl} \overline{\mathbf{z}_C} + \int_\Gamma \mathbf{n}_C \times \mathbf{H}_C \cdot \overline{\mathbf{z}_C} - \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \, .$$

  Using the Faraday equation for expressing $\mathbf{H}_C$ and recalling that $\mathbf{n}_C \times \mathbf{H}_C = \mathbf{n}_C \times \mathbf{H}_I$ on Γ, it holds

$$\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{E}_C \cdot \operatorname{curl} \overline{\mathbf{z}_C} + i\omega \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C}) + i\omega \int_\Gamma \mathbf{H}_I \times \mathbf{n}_C \cdot \overline{\mathbf{z}_C} = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \, .$$

Scalar potential formulation (cont'd)

On the other hand, multiplying the Faraday equation in Ω_I by a test function $\overline{\mathbf{v}_I}$ such that $\text{curl}\,\mathbf{v}_I = \mathbf{0}$ in Ω_I and recalling that $\mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C$ on Γ, by integration by parts one has

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \overline{\mathbf{v}_I} = -\int_{\Omega_I} \text{curl}\,\mathbf{E}_I \cdot \overline{\mathbf{v}_I} = -\int_{\Gamma} \mathbf{E}_C \times \mathbf{n}_C \cdot \overline{\mathbf{v}_I}\,.$$

Setting

$$V_I(\mathbf{G}) := \{\mathbf{v}_I \in H(\text{curl};\Omega_I) \mid \text{curl}\,\mathbf{v}_I = \mathbf{G} \text{ in } \Omega_I\}\,,$$

we are thus looking for $\mathbf{E}_C \in H(\text{curl};\Omega_C)$ and $\mathbf{H}_I \in V_I(\mathbf{J}_{e,I})$ such that

Scalar potential formulation (cont'd)

$$\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \,\mathrm{curl}\, \mathbf{E}_C \cdot \mathrm{curl}\, \overline{\mathbf{z}_C} + i\omega \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C})$$
$$-i\omega \int_\Gamma \overline{\mathbf{z}_C} \times \mathbf{n}_C \cdot \mathbf{H}_I = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \qquad (85)$$
$$-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \overline{\mathbf{v}_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \overline{\mathbf{v}_I} = 0\,,$$

where $\mathbf{z}_C \in H(\mathrm{curl}; \Omega_C)$ and $\mathbf{v}_I \in V_I(\mathbf{0})$.

Using in (85) the orthogonal decompositions of $\mathbf{H}_I - \mathbf{H}_{e,I}$ and $\mathbf{v}_I$ one finds

$$\mathcal{K}((\mathbf{E}_C, \psi_I^*, \boldsymbol{\eta}_I^*), (\mathbf{z}_C, \chi_I^*, \boldsymbol{\theta}_I^*))$$
$$= -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} + i\omega \int_\Gamma \mathbf{H}_{e,I} \cdot \overline{\mathbf{z}_C} \times \mathbf{n}_C \qquad (86)$$
$$-\omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot (\mathrm{grad}\, \overline{\chi_I^*} + \sum_{\alpha=1}^{n_{\Omega_I}} \overline{\theta_{I,\alpha}^*} \boldsymbol{\rho}_{\alpha,I}^*)\ ,$$

Scalar potential formulation (cont'd)

where the sesquilinear form $\mathcal{K}(\cdot,\cdot)$, that can be proved to be continuous and coercive, is given by

$$\begin{aligned}
&\mathcal{K}((\mathbf{E}_C,\psi_I^*,\boldsymbol{\eta}_I^*),(\mathbf{z}_C,\chi_I^*,\boldsymbol{\theta}_I^*)) \\
&:= \int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\,\text{curl}\,\mathbf{E}_C\cdot\text{curl}\,\overline{\mathbf{z}_C}+i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{z}_C}) \\
&\quad -i\omega\int_\Gamma(\text{grad}\,\psi_I^*+\textstyle\sum_{\alpha=1}^{n_{\Omega_I}}\eta_{I,\alpha}^*\boldsymbol{\rho}_{\alpha,I}^*)\cdot\overline{\mathbf{z}_C}\times\mathbf{n}_C \\
&\quad -i\omega\int_\Gamma(\text{grad}\,\overline{\chi_I^*}+\textstyle\sum_{\alpha=1}^{n_{\Omega_I}}\overline{\theta_{I,\alpha}^*}\boldsymbol{\rho}_{\alpha,I}^*)\cdot\mathbf{E}_C\times\mathbf{n}_C \\
&\quad +\omega^2\int_{\Omega_I}\boldsymbol{\mu}_I\,\text{grad}\,\psi_I^*\cdot\text{grad}\,\overline{\chi_I^*} \\
&\quad +\omega^2[A^*\boldsymbol{\eta}_I^*,\boldsymbol{\theta}_I^*]\ .
\end{aligned}$$

Note that the interaction between $\mathbf{E}_C$ and $\mathbf{H}_I$ is driven in a weak way by boundary integrals, and no strong matching conditon on Γ has to be imposed: non-matching meshes can be employed!

Scalar potential formulation (cont'd)

- Domain decomposition approaches can be devised. Let us specify one of them for the formulation in terms of $\mathbf{E}_C$, ψ_I^* and $\boldsymbol{\eta}_I^*$.

  Given $\mathbf{e}_\Gamma^{\rm old}$ on Γ, find the solutions to

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} \psi_I^*) = -\operatorname{div}(\boldsymbol{\mu}_I \mathbf{H}_{e,I}) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \operatorname{grad} \psi_I^* \cdot \mathbf{n}_I = -i\omega^{-1} \operatorname{div}_\tau \mathbf{e}_\Gamma^{\rm old} & \\ \qquad\qquad -\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \mathbf{n}_I & \text{on } \Gamma \\ \boldsymbol{\mu}_I \operatorname{grad} \psi_I^* \cdot \mathbf{n} = -\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \mathbf{n} & \text{on } \partial\Omega \end{cases} \tag{87}$$

$$\begin{aligned} (A^* \boldsymbol{\eta}_I^*)_\beta = i\omega^{-1} \int_\Gamma \mathbf{e}_\Gamma^{\rm old} \cdot \boldsymbol{\rho}_{\beta,I}^* - \int_{\Omega_I} \boldsymbol{\mu}_I \operatorname{grad} \psi_I^* \cdot \boldsymbol{\rho}_{\beta,I}^* \\ - \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \boldsymbol{\rho}_{\beta,I}^* \quad \forall\, \beta = 1, \ldots, n_{\Omega_I} \end{aligned} \tag{88}$$

Scalar potential formulation (cont'd)

$$\begin{cases} \mathrm{curl}(\boldsymbol{\mu}_C^{-1}\,\mathrm{curl}\,\mathbf{E}_C) + i\omega\boldsymbol{\sigma}\mathbf{E}_C = -i\omega\mathbf{J}_{e,C} & \text{in } \Omega_C \\ (\boldsymbol{\mu}_C^{-1}\,\mathrm{curl}\,\mathbf{E}_C)\times\mathbf{n}_C = i\omega\,\mathrm{grad}\,\psi_I^*\times\mathbf{n}_I & \\ \qquad +i\omega\sum_{\alpha=1}^{n_{\Omega_I}}\eta_{I,\alpha}^*\,\boldsymbol{\rho}_{\alpha,I}^*\times\mathbf{n}_I + i\omega\mathbf{H}_{e,I}\times\mathbf{n}_I & \text{on } \Gamma\,, \end{cases} \tag{89}$$

finally set

$$\mathbf{e}_\Gamma^{\mathrm{new}} = (1-\delta)\mathbf{e}_\Gamma^{\mathrm{old}} + \delta\,\mathbf{E}_C\times\mathbf{n}_C \quad \text{on } \Gamma \tag{90}$$

and iterate until convergence ($\delta > 0$ is an acceleration parameter). At convergence one has $\mathbf{e}_\Gamma^\infty = \mathbf{E}_C\times\mathbf{n}_C$ on Γ, the right tangential value of the electric field on Γ.

This iteration-by-subdomain procedure has shown good convergence properties (convergence rate independent of the mesh size [Alonso and V. (1997)]).

Pros and cons

- *Pros*:
  - few degrees of freedom;
  - "positive definite" algebraic problem.
- *Cons*:
  - need of computing in advance a vector potential of the current density;
  - some difficulties coming from the topology of the computational domain, in particular of the conductor [construction of the "cutting" surfaces].

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