A unified FEM–BEM approach for electro–magnetostatics and eddy-current problems

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Time-harmonic eddy current problem

Maxwell equations + time-harmonic structure (for a given frequency ω) + low frequency lead to:

$$\begin{cases} \mathbf{curl}\,\mathbf{H}_C - \boldsymbol{\sigma}\mathbf{E}_C = \mathbf{J}_e & \text{in } \Omega_C \\ \mathbf{curl}\,\mathbf{H}_I = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_C} \\ \mathbf{curl}\,\mathbf{E}_C + i\omega\boldsymbol{\mu}_C\mathbf{H}_C = \mathbf{0} & \text{in } \Omega_C \\ \mathrm{div}(\boldsymbol{\mu}\mathbf{H}) = 0 & \text{in } \mathbb{R}^3 \\ \mathbf{H}_C \times \mathbf{n} - \mathbf{H}_I \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_C \\ \mathbf{H}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\,. \end{cases} \tag{1}$$

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[Here: $\mathbf{H}$ magnetic field; $\mathbf{E}$ electric field; $\boldsymbol{\sigma}$ conductivity; $\boldsymbol{\mu}$ magnetic permeability; $\mathbf{J}_e$ applied density current (in Ω_C); Ω_C conductor, a simply-connected bounded open set; $\mathbf{n}$ unit outward normal vector on $\partial\Omega_C$.]

Electro–magnetostatics problem

The electro–magnetostatics problem is obtained by setting $\omega = 0$ in (1). In that case equations are decoupled and one can find at first $\mathbf{E}_C$ from

$$\begin{cases} \mathbf{curl}\,\mathbf{E}_C = \mathbf{0} & \text{in } \Omega_C \\ \operatorname{div}(\boldsymbol{\sigma}\mathbf{E}_C) = -\operatorname{div}\mathbf{J}_e & \text{in } \Omega_C \\ \boldsymbol{\sigma}\mathbf{E}_C \cdot \mathbf{n} = -\mathbf{J}_e \cdot \mathbf{n} & \text{on } \partial\Omega_C \end{cases} \tag{2}$$

(as $\operatorname{div}\mathbf{curl}\,\mathbf{H}_C = 0$ in Ω_C and $\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega_C$).

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(as $\operatorname{div}\mathbf{curl}\,\mathbf{H}_C = 0$ in Ω_C and $\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega_C$). Then $\mathbf{H}$ is determined by solving

$$\begin{cases} \mathbf{curl}\,\mathbf{H} = \mathbf{J} & \text{in } \mathbb{R}^3 \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{H}) = 0 & \text{in } \mathbb{R}^3 \\ \mathbf{H}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\,, \end{cases} \tag{3}$$

where $\mathbf{J}_C = \boldsymbol{\sigma}\mathbf{E}_C + \mathbf{J}_e$, $\mathbf{J}_I = \mathbf{0}$.

Electro–magnetostatics problem (cont'd)

An even simpler approach consists in looking for $\mathbf{E}_C = \mathbf{grad}\,\varphi_C$, solution to

$$\begin{cases} \mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\varphi_C) = -\,\mathrm{div}\,\mathbf{J}_e & \text{in } \Omega_C \\ \boldsymbol{\sigma}\,\mathbf{grad}\,\varphi_C \cdot \mathbf{n} = -\mathbf{J}_e \cdot \mathbf{n} & \text{on } \partial\Omega_C\,, \end{cases} \qquad (4)$$

followed by the solution of (3) [when $\boldsymbol{\mu}$ is constant, via the Biot–Savart formula].

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followed by the solution of (3) [when $\boldsymbol{\mu}$ is constant, via the Biot–Savart formula].

- However, this simple and decoupled approach cannot be used for $\omega \neq 0$, as $\mathbf{curl}\,\mathbf{E}_C = -i\omega\boldsymbol{\mu}_C\mathbf{H}_C \neq 0$.

Unified approach via vector and scalar potentials

- Aim: devise a unified approach, suitable for both electro–magnetostatics ($\omega = 0$) and eddy-current problems ($\omega \neq 0$), employing a reduced number of degrees of freedom.

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- Tools: potentials. More precisely, a couple of magnetic vector and scalar potentials, and an electric scalar potential.

Potentials and the Coulomb gauge

This means: new unknowns $\mathbf{A}_C$, ψ_I and V_C such that

$$\mathbf{curl}\,\mathbf{A}_C = \boldsymbol{\mu}_C \mathbf{H}_C \ , \ \mathbf{grad}\,\psi_I = \mathbf{H}_I \ , \ -i\omega\mathbf{A}_C - \mathbf{grad}\,V_C = \mathbf{E}_C \ .$$

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Faraday equation in Ω_C and Ampère equation in $\mathbb{R}^3 \setminus \overline{\Omega_C}$ are then satisfied. Moreover, also $\mathrm{div}(\boldsymbol{\mu}_C \mathbf{H}_C) = 0$ in Ω_C follows.

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We choose the so-called Coulomb gauge

$$\mathrm{div}\,\mathbf{A}_C = 0 \ \text{ in } \ \Omega_C \quad , \quad \mathbf{A}_C \cdot \mathbf{n} = 0 \ \text{ on } \ \partial\Omega_C \ .$$

Strong formulation

We are thus left with:

$$\left\{\begin{array}{ll}\mathbf{curl}(\boldsymbol{\mu}_C^{-1}\,\mathbf{curl}\,\mathbf{A}_C) & \\ \qquad\qquad +i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathbf{grad}\,V_C = \mathbf{J}_e & \text{in } \Omega_C \\ \Delta\psi_I = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_C} \\ \operatorname{div}\mathbf{A}_C = 0 & \text{in } \Omega_C \\ \mathbf{A}_C \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_C \\ \boldsymbol{\mu}_C^{-1}\,\mathbf{curl}\,\mathbf{A}_C \times \mathbf{n} - \mathbf{grad}\,\psi_I \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_C \\ \mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n} - \mu_0\,\mathbf{grad}\,\psi_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_C \\ |\psi_I(\mathbf{x})| + |\,\mathbf{grad}\,\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \\ \int_{\Omega_C} V_C = 0 & \end{array}\right. \qquad (5)$$

(having assumed that $\boldsymbol{\mu} = \mu_0 > 0$ in $\mathbb{R}^3 \setminus \overline{\Omega_C}$).

Penalization

The divergence-free constraint can be inserted in the formulation, via penalization [$\mu_* > 0$ freely chosen]:

$$\left\{\begin{array}{ll}
\mathbf{curl}(\boldsymbol{\mu}_C^{-1}\,\mathbf{curl}\,\mathbf{A}_C) - \mu_*^{-1}\,\mathbf{grad}\,\mathrm{div}\,\mathbf{A}_C & \\
\qquad\qquad + i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathbf{grad}\,V_C = \mathbf{J}_e & \text{in } \Omega_C \\
\mathrm{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathbf{grad}\,V_C) = \mathrm{div}\,\mathbf{J}_e & \text{in } \Omega_C \\
\Delta\psi_I = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_C} \\
(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathbf{grad}\,V_C)\cdot\mathbf{n} = \mathbf{J}_e\cdot\mathbf{n} & \text{on } \partial\Omega_C \\
\mathbf{A}_C\cdot\mathbf{n} = 0 & \text{on } \partial\Omega_C \\
\boldsymbol{\mu}_C^{-1}\,\mathbf{curl}\,\mathbf{A}_C\times\mathbf{n} - \mathbf{grad}\,\psi_I\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_C \\
\mathbf{curl}\,\mathbf{A}_C\cdot\mathbf{n} - \mu_0\,\mathbf{grad}\,\psi_I\cdot\mathbf{n} = 0 & \text{on } \partial\Omega_C \\
|\psi_I(\mathbf{x})| + |\,\mathbf{grad}\,\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \\
\int_{\Omega_C} V_C = 0\,. &
\end{array}\right. \tag{6}$$

Coupling strategy

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An important improvement is due to the work of Costabel (1987), that shows how to arrive to a symmetric (or positive) problem.

Coupling strategy (cont'd)

Among others, extensions to nonlinear elasticity [Costabel and Stephan (1990)], nonlinear elliptic problems [Gatica and Hsiao (1989, 1992), Gatica and Wendland (1996, 1997), Carstensen and Wriggers (1997)], variational inequalities [Carstensen and Gwinner (1997)], transonic flows [Berger, Warnecke and Wendland (1994, 1997)], and Maxwell equations [Ammari and Nédélec (1998, 1999)] have been also considered.

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For the eddy-current problem, the first FEM–BEM couplings have been proposed by Bossavit and Vérité (1982) [for the magnetic field, and using the Steklov–Poincaré operator] and Mayergoyz, Chari and Konrad (1983) [for the electric field, and using special basis functions near $\partial\Omega_C$].

Coupling strategy (cont'd)

Symmetric formulations à la Costabel are due to Hiptmair (2002) [unknowns: $\mathbf{E}_C$ in Ω_C, $\mathbf{H} \times \mathbf{n}$ on $\partial\Omega_C$] and Meddahi and Selgas (2003) [unknowns: $\mathbf{H}_C$ in Ω_C, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n}$ on $\partial\Omega_C$].

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[Other related results for vector magnetic potentials (but no coupling FEM–BEM): Bíró and V. (2005), Acevedo and Rodríguez (2006).]

Single layer and double layer potentials

To reduce the number of unknowns, we want to transform the problem for ψ_I to a problem on the interface $\partial\Omega_C$. Let us introduce on $\partial\Omega_C$ the single layer and double layer potentials

$$\mathcal{S}_{\mathcal{L}}(\lambda)(\mathbf{x}) := \int_{\partial\Omega_C} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \lambda(\mathbf{y}) dS_y$$

$$\mathcal{D}_{\mathcal{L}}(\eta)(\mathbf{x}) := \int_{\partial\Omega_C} \frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^3} \eta(\mathbf{y}) dS_y ,$$

and the hypersingular integral operator

$$\mathcal{H}(\eta)(\mathbf{x}) := -\,\mathbf{grad}\left(\int_{\partial\Omega_C} \frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^3} \eta(\mathbf{y}) dS_y\right)\cdot\mathbf{n}(\mathbf{x}) .$$

Integral equations

Due to the matching condition

$$\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n} - \mu_0\,\mathbf{grad}\,\psi_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_C ,$$

from potential theory it is well-known that the trace $\psi := \psi_{I|\partial\Omega_C}$ satisfies

$$\frac{1}{2}\psi - \mathcal{D}_{\mathcal{L}}(\psi) + \frac{1}{\mu_0}\mathcal{S}_{\mathcal{L}}(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}) = 0 \text{ on } \partial\Omega_C$$

$$\frac{1}{2}\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n} + \mathcal{D}_{\mathcal{L}}{}'(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}) + \mu_0\mathcal{H}(\psi) = 0 \text{ on } \partial\Omega_C .$$

[These equations are the basis of the symmetric approach à la Costabel.]

Weak formulation

Coupled problem: look for $(\mathbf{A}_C, \psi, V_C)$ such that

$$\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1}\, \mathbf{curl}\, \mathbf{A}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + \mu_*^{-1}\, \mathrm{div}\, \mathbf{A}_C\, \mathrm{div}\, \overline{\mathbf{w}_C}$$
$$+ i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma}\, \mathbf{grad}\, V_C \cdot \overline{\mathbf{w}_C})$$
$$+ \int_{\partial\Omega_C} [-\psi + \tfrac{1}{2}\psi - \mathcal{D}_{\mathcal{L}}(\psi) + \tfrac{1}{\mu_0} \mathcal{S}_{\mathcal{L}}(\mathbf{curl}\, \mathbf{A}_C \cdot \mathbf{n})]\, \mathbf{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n}$$
$$= \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{w}_C}$$

$$\int_{\partial\Omega_C} [\tfrac{1}{2}\, \mathbf{curl}\, \mathbf{A}_C \cdot \mathbf{n} + \mathcal{D}_{\mathcal{L}}{}'(\mathbf{curl}\, \mathbf{A}_C \cdot \mathbf{n}) + \mu_0 \mathcal{H}(\psi)]\overline{\eta} = 0$$

$$\int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \mathbf{grad}\, \overline{Q_C} + \boldsymbol{\sigma}\, \mathbf{grad}\, V_C \cdot \mathbf{grad}\, \overline{Q_C})$$
$$= \int_{\Omega_C} \mathbf{J}_e \cdot \mathbf{grad}\, \overline{Q_C}$$

for suitable test functions $(\mathbf{w}_C, \eta, Q_C)$.

Weak formulation (cont'd)

Deriving the first equation one has used the matching condition

$$\mathbf{n} \times \boldsymbol{\mu}_C^{-1} \, \mathbf{curl} \, \mathbf{A}_C - \mathbf{n} \times \mathbf{grad} \, \psi_I = \mathbf{0} \text{ on } \partial\Omega_C$$

and the relation

$$\int_{\partial\Omega_C} \mathbf{n} \times \mathbf{grad} \, \psi_I \cdot \overline{\mathbf{w}_C} = \int_{\partial\Omega_C} -\psi \, \mathbf{curl} \, \overline{\mathbf{w}_C} \cdot \mathbf{n} \; .$$

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Moreover, since $\mathcal{D}_{\mathcal{L}}(1) = -\frac{1}{2}$ and $\mathcal{H}(1) = 0$, we can rewrite the preceding problem replacing ψ with $q := \psi - \psi_\sharp$, where $\psi_\sharp := [\text{meas}\,(\partial\Omega_C)]^{-1} \int_{\partial\Omega_C} \psi$.

Variational space

Therefore, we are looking for the solution $(\mathbf{A}_C, q, V_C)$ of the coupled problem in the space

$$W \times H^{1/2}_\sharp(\partial\Omega_C) \times H^1_\sharp(\Omega_C) \ ,$$

where

$$W := \{\mathbf{w}_C \in H(\mathbf{curl}; \Omega_C) \,|\, \mathrm{div}\, \mathbf{w}_C \in L^2(\Omega_C) \ , \ \mathbf{w}_C \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_C\}$$

$$H^{1/2}_\sharp(\partial\Omega_C) := \left\{\eta \in H^{1/2}(\partial\Omega_C) \,|\, \int_{\partial\Omega_C} \eta = 0\right\}$$

$$H^1_\sharp(\Omega_C) := \left\{Q_C \in H^1(\Omega_C) \,|\, \int_{\Omega_C} Q_C = 0\right\} \ ,$$

choosing the test functions in the same space.

Existence and uniqueness

Recalling that the operators $\mathcal{S}_{\mathcal{L}}$ and $\mathcal{H}$ satisfy

$$\int_{\partial\Omega_C} \mathcal{S}_{\mathcal{L}}(\lambda)\,\overline{\lambda} \geq c_1||\lambda||^2_{-1/2}$$

$$\int_{\partial\Omega_C} \mathcal{H}(\eta)\,\overline{\eta} \geq c_2||\eta||^2_{1/2}$$

for each $\lambda \in H^{-1/2}(\partial\Omega_C)$ and $\eta \in H^{1/2}_{\sharp}(\partial\Omega_C)$, it can be shown that the sesquilinear form associated to this weak formulation is continuous and coercive [for $\omega \neq 0$, one has to multiply the third equation for i/ω; for $\omega = 0$, one has to multiply the third equation for β, a parameter large enough].

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Therefore, there exists a unique solution $(\mathbf{A}_C, q, V_C)$ to the coupled problem.

Existence and uniqueness (cont'd)

Then the scalar magnetic potential ψ_I in Ω_I is given by

$$\psi_I = \mathcal{D}_{\mathcal{L}}(q) - \frac{1}{\mu_0}\mathcal{S}_{\mathcal{L}}(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}) \ .$$

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[Clearly, q is not the correct value of the trace $\psi = \psi_{I|\partial\Omega_C}$, as $q = \psi - \psi_\sharp$. Though not necessary to reconstruct ψ_I, the constant $\psi_\sharp$ can be determined as

$$\psi_\sharp := \frac{1}{\text{meas}\,(\partial\Omega_C)} \int_{\partial\Omega_C} \Big[-\frac{1}{2}q + \mathcal{D}_{\mathcal{L}}(q) - \frac{1}{\mu_0}\mathcal{S}_{\mathcal{L}}(\mathbf{curl}\,\mathbf{A}_C \cdot \mathbf{n}) \Big] \ .]$$

Behaviour with respect to ω

Thanks to the positiveness of the sesquilinear form associated to the coupled problem, an interesting feature of the proposed approach comes into play: it is suitable for the static limit $\omega \to 0$ (this was known to engineers and practitioners; however, to our knowledge, a mathematical proof was still missing).

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More precisely, we have

$$||\mathbf{A}_C^\omega - \mathbf{A}_C^0||_W + ||q^\omega - q^0||_{1/2,\partial\Omega_C} + ||V_C^\omega - V_C^0||_{1,\Omega_C} = O(|\omega|)$$

[in agreement with the asymptotic result obtained for $\mathbf{E}_C$ and $\mathbf{H}_C$ by Ammari, Buffa and Nédélec, 2000].

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$$||\mathbf{A}_C^\omega - \mathbf{A}_C^0||_W + ||q^\omega - q^0||_{1/2,\partial\Omega_C} + ||V_C^\omega - V_C^0||_{1,\Omega_C} = O(|\omega|)$$

[in agreement with the asymptotic result obtained for $\mathbf{E}_C$ and $\mathbf{H}_C$ by Ammari, Buffa and Nédélec, 2000].

Therefore, the coupled approach yields a unified and ω-stable procedure for electro–magnetostatics and eddy-current problems.

Discrete approximation

Numerical approximation is now standard: assume that Ω_C is a (convex) polyhedral domain, and use nodal finite elements in Ω_C for all the components of $\mathbf{A}_C$ and for V_C, and nodal boundary elements on $\partial\Omega_C$ for q.

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Error estimates follow directly from Céa lemma:

$$
\begin{aligned}
&||\mathbf{A}_{C,h} - \mathbf{A}_C||_W + ||q_h - q||_{1/2,\partial\Omega_C} + ||V_{C,h} - V_C||_{1,\Omega_C} \\
&\quad \leq C(||\mathbf{w}_{C,h} - \mathbf{A}_C||_W + ||\eta_h - q||_{1/2,\partial\Omega_C} + ||Q_{C,h} - V_C||_{1,\Omega_C})
\end{aligned}
$$

for each $(\mathbf{w}_{C,h}, \eta_h, Q_{C,h}) \in W_h \times B_h \times V_h$, where $W_h \subset W$, $B_h \subset H^{1/2}_\sharp(\partial\Omega_C)$ and $V_h \subset H^1_\sharp(\Omega_C)$ are the discrete subspaces.

Discrete approximation (cont'd)

We also note that the static limit $\omega \to 0$ holds in the discrete case as well, uniformly with respect to h:

$$||\mathbf{A}^{\omega}_{C,h} - \mathbf{A}^{0}_{C,h}||_W + ||q^{\omega}_h - q^{0}_h||_{1/2,\partial\Omega_C} + ||V^{\omega}_{C,h} - V^{0}_{C,h}||_{1,\Omega_C} \leq C\,|\omega| \quad \text{for all } h > 0 \,.$$

Final remarks

The assumption of convexity for Ω_C is motivated by the fact that in general $(H^1(\Omega_C))^3$ is a proper closed subspace of W [Costabel and Dauge, 1997]; therefore nodal finite elements are not the right choice for approximation if the solution $\mathbf{A}_C \notin (H^1(\Omega_C))^3$, and this can happen in the polyhedral non-convex case.

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Alternative (not yet fully analyzed) assumptions or procedures are:

- $\partial\Omega_C$ smooth: need of controlling the non-conformity error due to $\Omega_{C,h} \neq \Omega_C$ [as in Johnson-Nédélec, 1980];

Final remarks (cont'd)

- include the polyhedral domain Ω_C in a convex polyhedral domain Ω_C^* [clearly, in $\Omega_C^* \setminus \overline{\Omega_C}$ the conductivity $\boldsymbol{\sigma}$ is vanishing], and solve for ψ_I only in $\mathbb{R} \setminus \overline{\Omega_C^*}$ (assuming also that $\boldsymbol{\mu}$ is smooth in Ω_C^*) [see Acevedo and Rodríguez, 2006];

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- correct the penalization term in (6) by adapting it to the geometry of the domain [as in Costabel and Dauge, 2002].

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- correct the penalization term in (6) by adapting it to the geometry of the domain [as in Costabel and Dauge, 2002].

- impose the divergence-free constraint by means of a Lagrange multiplier, employing edge elements [as in Kuhn, Langer and Schöberl (2000), Kuhn and Steinbach (2002) for magnetostatics]; however, due to the presence of the additional unknown, this is more expensive from the computational point of view.