Coupling between circuit problems and eddy-current problems

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Maxwell equations

Maxwell equations can be written as:

$$\begin{cases} \boldsymbol{\epsilon}\,\dfrac{\partial \mathcal{E}}{\partial t} - \mathbf{curl}\,\mathcal{H} = -\boldsymbol{\sigma}\mathcal{E} - \mathcal{J}_e & \text{(Maxwell–Ampère)} \\ \boldsymbol{\mu}\,\dfrac{\partial \mathcal{H}}{\partial t} + \mathbf{curl}\,\mathcal{E} = \mathbf{0} & \text{(Faraday),} \end{cases}$$

where

- $\mathcal{E}$ and $\mathcal{H}$ are the electric and magnetic fields, respectively
- $\boldsymbol{\epsilon}$ is the electric permittivity
- $\boldsymbol{\mu}$ is the magnetic permeability
- $\boldsymbol{\sigma}$ is the conductivity
- $\mathcal{J}_e$ is the applied current density.

Time-harmonic Maxwell equations

When interested in time-periodic phenomena, it is assumed that

$$\begin{aligned} \mathcal{J}_e(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{J}_e(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{E}(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{E}(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{H}(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{H}(\mathbf{x}) \exp(i\omega t)] \ , \end{aligned}$$

where $\omega \neq 0$ is the assigned frequency, and one obtains

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - i\omega\boldsymbol{\epsilon}\mathbf{E} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} \, . \end{cases}$$

Time-harmonic eddy-current equations

If the frequency is small the displacement currents $\epsilon \dfrac{\partial \mathcal{E}}{\partial t}$ can be disregarded. Thus one finds the so-called eddy-current (or quasi-static) problem

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega\,. \end{cases} \tag{1}$$

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Here Ω is a bounded domain in $\mathbf{R}^3$, composed by two parts: Ω_C, a conductor, and Ω_I, its complementary part, an insulator, where the conductivity $\boldsymbol{\sigma}$ is vanishing.
We consider the case in which the geometry of Ω is simple (a “box"), while that of Ω_C can be of two different types: a cylinder that touches the boundary or an internal torus.

"Gauge" conditions

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[Depending on the geometrical properties of Ω_I as well as on the boundary conditions on $\partial\Omega$, other "gauge" conditions for $\mathbf{E}$ in Ω_I can be necessary: here we will not enter this aspect.]

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- First geometrical case: electric ports. The conductor Ω_C is not strictly contained in Ω. For simplicity, Ω_C is simply connected with $\partial\Omega_C \cap \partial\Omega = \Gamma_E \cup \Gamma_J$, where Γ_E and Γ_J are connected and disjoint surfaces on $\partial\Omega$ (“electric ports"). Notation: $\Gamma = \overline{\Omega_C} \cap \overline{\Omega_I}$, $\partial\Omega = \Gamma_E \cup \Gamma_J \cup \Gamma_D$, $\partial\Omega_C = \Gamma_E \cup \Gamma_J \cup \Gamma$, $\partial\Omega_I = \Gamma_D \cup \Gamma$.

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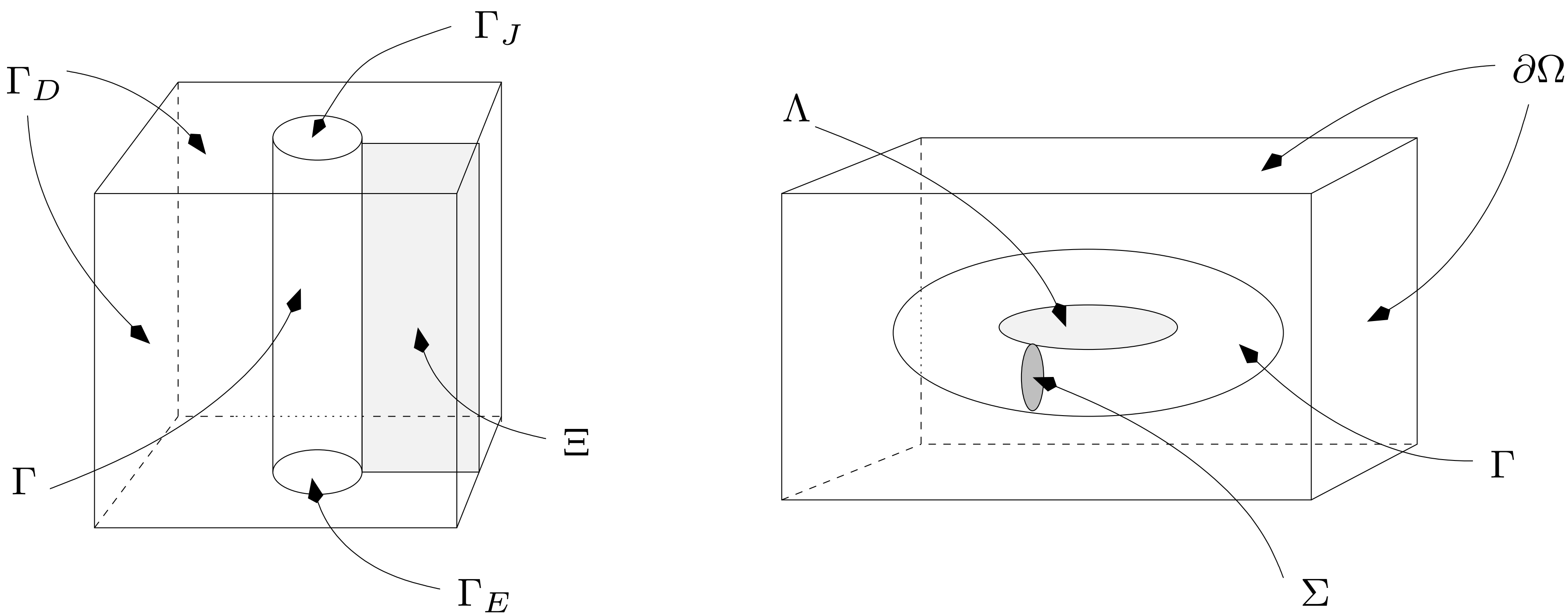
- Second geometrical case: internal conductor. The conductor Ω_C is strictly contained in Ω. For simplicity, Ω_C is a torus. Notation: $\partial\Omega_C = \Gamma$, $\partial\Omega_I = \partial\Omega \cup \Gamma$.

The geometrical configurations



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- Magnetic. One imposes $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\Gamma_E \cup \Gamma_J$, $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on Γ_D for the electric port case, while one requires $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$ for the internal conductor case.

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- Mixed [Bossavit, 2000]. One imposes $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\Gamma_E \cup \Gamma_J$, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on Γ_D for the electric port case, while one requires $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$ for the internal conductor case.

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- Case A. Electric ports, $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$
- Case B. Electric ports, $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\Gamma_E \cup \Gamma_J$, $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on Γ_D

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- Case C. Electric ports, $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\Gamma_E \cup \Gamma_J$, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on Γ_D

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- Case D. Internal conductor, $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$
- Case E. Internal conductor, $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$
- Case F. Internal conductor, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

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Question:

- how can we formulate the eddy-current problems when the excitation is given by a voltage or by a current intensity?

This is a delicate point, as eddy-current problems, for the five cases A, B, D, E, F, have a unique solution already before a voltage or a current intensity is assigned!

Poynting Theorem (energy balance)

In fact one has:
Uniqueness theorem. In the cases A, B, D, E, F, for the solution of the eddy-current problem (1) the magnetic field $\mathbf{H}$ in Ω and the electric field $\mathbf{E}_C$ in Ω_C are uniquely determined. [Adding the "gauge" conditions, also the electric field $\mathbf{E}_I$ in Ω_I is uniquely determined.]

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Proof. Multiply the Faraday equation by $\overline{\mathbf{H}}$, integrate in Ω and integrate by parts: it holds

$$\begin{aligned} 0 &= \int_\Omega \mathbf{curl}\,\mathbf{E} \cdot \overline{\mathbf{H}} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}} \\ &= \int_\Omega \mathbf{E} \cdot \mathbf{curl}\,\overline{\mathbf{H}} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}} \,. \end{aligned}$$

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Replacing $\mathbf{E}_C$ with $\boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C})$, and remembering that $\mathbf{curl}\,\mathbf{H}_I = \mathbf{J}_{e,I}$ in Ω_I, one has the Poynting Theorem (energy balance)

Poynting Theorem (energy balance) (cont'd)

$$
\begin{aligned}
&\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl}\, \overline{\mathbf{H}_C} - \int_{\Omega_I} \mathbf{E}_I \cdot \overline{\mathbf{J}_{e,I}} \\
&\quad = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{H}_C} + \int_{\Omega} i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}}\,.
\end{aligned}
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$$= \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{H}_C} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}}\,.$$

If $\mathbf{J}_e = \mathbf{0}$, we have only to take into account the term on $\partial\Omega$. This is clearly vanishing in the cases A, B, D ed E.

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$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{H}_C} - \int_{\Omega_I} \mathbf{E}_I \cdot \overline{\mathbf{J}_{e,I}}$$
$$= \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{H}_C} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}}\,.$$

If $\mathbf{J}_e = \mathbf{0}$, we have only to take into account the term on $\partial\Omega$. This is clearly vanishing in the cases A, B, D ed E. In the case F, since $\mathrm{div}_\tau(\mathbf{E} \times \mathbf{n}) = -i\omega\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$, one has

$$\mathbf{E} \times \mathbf{n} = \mathbf{grad}\,W \times \mathbf{n} \text{ on } \partial\Omega\ ,$$

and therefore

$$\begin{aligned}\int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}} &= \int_{\partial\Omega} \overline{\mathbf{H}} \times \mathbf{n} \cdot \mathbf{grad}\,W = -\int_{\partial\Omega} \mathrm{div}(\overline{\mathbf{H}} \times \mathbf{n})\,W \\ &= -\int_{\partial\Omega} \mathbf{curl}\,\overline{\mathbf{H}} \cdot \mathbf{n}\,W = 0\ ,\end{aligned}$$

as $\mathbf{curl}\,\mathbf{H}_I = \mathbf{0}$ in Ω_I and, for the case F, $\partial\Omega \subset \partial\Omega_I$. $\square$

Poynting Theorem for the case C

In the case C, instead, we can repeat the computation here above and find

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \,\mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{H}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}}$$
$$= W_{|\Gamma_J} \int_{\Gamma_J} \mathbf{curl}\, \overline{\mathbf{H}_C} \cdot \mathbf{n} \ ,$$

where $W_{|\Gamma_J}$ is the (constant) value of the potential W on the electric port Γ_J (whereas $W_{|\Gamma_E} = 0$).

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where $W_{|\Gamma_J}$ is the (constant) value of the potential W on the electric port Γ_J (whereas $W_{|\Gamma_E} = 0$).

- In this case a degree of freedom is indeed still free (either the voltage $W_{|\Gamma_J}$, or else the current intensity $\int_{\Gamma_J} \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{n}$ in Ω_C).

The case C: variational formulation

- Thus we start from the case C: how can we formulate the problem when the source $\mathbf{J}_e$ and the voltage or the current intensity are assigned?
[Alonso Rodríguez, Valli and Vázquez Hernández, 2008; Bermúdez, Rodríguez and Salgado, 2005]

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This orthogonal decomposition result turns out to be useful: each vector function $\mathbf{v}_I$ can be decomposed as

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1} \, \mathbf{curl} \, \mathbf{q}_I + \mathbf{grad} \, \psi_I + \alpha \boldsymbol{\rho}_I \, ,$$

where $\boldsymbol{\rho}_I$ is a harmonic field, namely, it belongs to the space

$$\mathcal{H}_{\mu_I}(\Omega_I) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 | \, \mathbf{curl} \, \mathbf{v}_I = \mathbf{0}, \, \mathrm{div}(\boldsymbol{\mu_I} \mathbf{v}_I) = 0, \\ \boldsymbol{\mu_I} \mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_I\} \, .$$

The case C: variational formulation (cont'd)

The harmonic field $\boldsymbol{\rho}_I$ is known from the data of the problem, and satisfies $\int_{\partial\Gamma_J} \boldsymbol{\rho}_I \cdot d\boldsymbol{\tau} = 1$; moreover, from $\mathbf{curl}\,\mathbf{v}_I = \mathbf{0}$ it follows $\mathbf{q}_I = \mathbf{0}$ and therefore $\alpha = \int_{\partial\Gamma_J} \mathbf{v}_I \cdot d\boldsymbol{\tau}$.

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$$I_0 = \int_{\Gamma_J} \mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{n}_C = \int_{\partial\Gamma_J} \mathbf{H}_C \cdot d\boldsymbol{\tau} = \int_{\partial\Gamma_J} \mathbf{H}_I \cdot d\boldsymbol{\tau} = \alpha\,,$$

hence

$$\mathbf{H}_I = \mathbf{grad}\,\psi_I + I_0\boldsymbol{\rho}_I\,. \qquad (3)$$

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hence

$$\mathbf{H}_I = \mathbf{grad}\, \psi_I + I_0\boldsymbol{\rho}_I\,. \tag{3}$$

We want to provide a "coupled" variational formulation, in terms of $\mathbf{E}_C$ in Ω_C and of $\mathbf{H}_I$ in Ω_I.

The case C: variational formulation (cont'd)

Inserting the Faraday equation into the Ampère equation in Ω_C we find

$$\int_{\Omega_C} \boldsymbol{\mu}_C^{-1} \,\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}_C} \\ - i\omega \int_{\Gamma} \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \mathbf{H}_I = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C}\,. \tag{4}$$

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Instead, the Ampère equation in Ω_I gives

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{grad}\, \overline{\varphi_I} + \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} = 0 \quad (5)$$

The case C: variational formulation (cont'd)

Inserting the Faraday equation into the Ampère equation in Ω_C we find

$$\begin{aligned}\int_{\Omega_C} \boldsymbol{\mu}_C^{-1}\,\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}_C} \\ -i\omega \int_\Gamma \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \mathbf{H}_I = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C}\,.\end{aligned} \tag{4}$$

Instead, the Ampère equation in Ω_I gives

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{grad}\,\overline{\varphi_I} + \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\,\overline{\varphi_I} = 0 \tag{5}$$

and

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I + \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I = V\ . \tag{6}$$

The case C: variational formulation (cont'd)

Here we have to note that

$$\int_{\Gamma_D} \mathbf{E}_I \times \mathbf{n}_I \cdot \boldsymbol{\rho}_I = \int_{\Gamma_D} \mathbf{grad}\, W \times \mathbf{n}_I \cdot \boldsymbol{\rho}_I$$
$$= \int_{\Gamma_D} \mathrm{div}_\tau(\boldsymbol{\rho}_I \times \mathbf{n}_I) W + V \int_{\partial\Gamma_J} \boldsymbol{\rho}_I \cdot d\boldsymbol{\tau} = V \ .$$

The case C: variational formulation (cont'd)

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$$= \int_{\Gamma_D} \mathrm{div}_\tau(\boldsymbol{\rho}_I \times \mathbf{n}_I) W + V \int_{\partial\Gamma_J} \boldsymbol{\rho}_I \cdot d\boldsymbol{\tau} = V\ .$$

Using (3) in (4), (5) and (6) one has

$$\int_{\Omega_C} \boldsymbol{\mu}_C^{-1}\, \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{w}_C}$$
$$-i\omega \int_\Gamma \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \mathbf{grad}\, \psi_I - i\omega I_0 \int_\Gamma \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I \qquad (7)$$
$$= -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C}$$

$$-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I\, \mathbf{grad}\, \psi_I \cdot \mathbf{grad}\, \overline{\varphi_I} = 0 \quad (8)$$

$$-i\omega \overline{Q} \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I + \omega^2 I_0 \overline{Q} \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_I \cdot \boldsymbol{\rho}_I = -i\omega V \overline{Q}\ . \quad (9)$$

The case C: existence and uniqueness

- If V is given, one solves (7), (8), (9) and determines $\mathbf{E}_C$, ψ_I and I_0 (hence $\mathbf{H}_C$ and $\mathbf{H}_I$).

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Moreover, it is simple to propose an approximation method based on finite elements, of "edge" type for $\mathbf{E}_C$ in Ω_C and of (scalar) nodal type for ψ_I in Ω_I. Convergence is assured by the Céa Lemma. [However, an efficient implementation demands to replace the harmonic field $\boldsymbol{\rho}_I$ with an easily computable function.]

The cases A, B, D, E, F

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The case C comes back to help us.

The cases A, B, D, E, F (cont'd)

In fact, let ϕ_C be the solution to

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\phi_C) = 0 & \text{in } \Omega_C \\ \phi_C = 1 & \text{on } \Gamma_J \\ \phi_C = 0 & \text{on } \Gamma_E \\ \boldsymbol{\sigma}\,\mathbf{grad}\,\phi_C \cdot \mathbf{n} = 0 & \text{on } \Gamma\ . \end{cases}$$

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One easily verifies that $\mathbf{E}_C = V\,\mathbf{grad}\,\phi_C$ and $\mathbf{H} = \mathbf{0}$ is the solution to the problem C with $\mathbf{J}_{e,C} = -V\boldsymbol{\sigma}\,\mathbf{grad}\,\phi_C$ and assigned voltage V.

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$$\begin{aligned} \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{H}_C} &= \int_{\Omega_C} (-V\,\mathbf{grad}\,\phi_C) \cdot \mathbf{curl}\,\overline{\mathbf{H}_C} \\ &= -V \int_{\Gamma\cup\Gamma_E\cup\Gamma_J} \phi_C\,\mathbf{curl}\,\overline{\mathbf{H}_C} \cdot \mathbf{n}_C \\ &= -V \int_{\Gamma_J} \mathbf{curl}\,\overline{\mathbf{H}_C} \cdot \mathbf{n}\ , \end{aligned}$$

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and from the Poynting Theorem

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \, \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{H}_C} + \int_{\Omega} i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}}$$
$$= V \int_{\Gamma_J} \mathbf{curl}\, \overline{\mathbf{H}_C} \cdot \mathbf{n} + \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl}\, \overline{\mathbf{H}_C} = 0 \, ,$$

so that $\mathbf{H} = \mathbf{0}$, and, moreover, from the Ampère equation $\mathbf{E}_C = -\boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} = V \, \mathbf{grad}\, \phi_C$.

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Thus, by linearity, the magnetic field $\mathbf{H}$ solution to problem (7), (8), (9) with data $\mathbf{J}_{e,C} = \mathbf{0}$ and $W_{|\Gamma_J} = V$ is the same than the one with data $\mathbf{J}_{e,C} = V \boldsymbol{\sigma} \,\mathbf{grad}\, \phi_C$ and $W_{|\Gamma_J} = 0$.

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[Instead, for the electric field one has that the difference in Ω_C is given by $V \,\mathbf{grad}\, \phi_C$.]

The cases A, B, D, E, F (cont'd)

For the cases A, B (electric ports), for which the "electric" voltage cannot be assigned, one is thus led to consider a "source" voltage V, that is the factor appearing in the current density $\mathbf{J}_{e,C} = V \boldsymbol{\sigma}\, \mathbf{grad}\, \phi_C$, and to solve eddy-current problems with this source.

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Note that $\mathbf{grad}\, \phi_C$ is the basis function of the space of harmonic fields

$$\widehat{\mathcal{H}}(\Omega_C) := \{\widehat{\boldsymbol{\eta}}_C \in (L^2(\Omega_C))^3 \,|\, \mathbf{curl}\, \widehat{\boldsymbol{\eta}}_C = \mathbf{0}, \mathrm{div}(\boldsymbol{\sigma}\widehat{\boldsymbol{\eta}}_C) = 0, \\ \boldsymbol{\sigma}\widehat{\boldsymbol{\eta}}_C \cdot \mathbf{n}_C = 0 \text{ on } \Gamma, \widehat{\boldsymbol{\eta}}_C \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_E \cup \Gamma_J\}\ ,$$

normalized by the condition $\int_{\widehat{\gamma}} \widehat{\boldsymbol{\eta}}_C \cdot d\boldsymbol{\tau} = 1$, where $\widehat{\gamma}$ is (any) path connecting Γ_E to Γ_J.

The cases A, B, D, E, F (cont'd)

Then, for the cases D, E, F (internal conductor) we define $\boldsymbol{\rho}_C$ the basis function of the space of harmonic fields

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normalized by the condition $\int_\gamma \boldsymbol{\rho}_C \cdot d\boldsymbol{\tau} = 1$, where the closed cycle γ runs internally along the whole torus Ω_C.

Similarly to the cases A,B (electric ports), for the cases D, E, F (internal conductor) one can thus consider a "source" voltage V, associated with the current density $\mathbf{J}_{e,C} = V\boldsymbol{\sigma}\boldsymbol{\rho}_C$.

- The voltage rule.
  Having to impose a voltage V, modify Ohm law in Ω_C adding to the current density $\boldsymbol{\sigma}\mathbf{E}_C$ the "applied" current density $\mathbf{J}_{e,C} = V\boldsymbol{\sigma}\mathbf{Q}_C$, where $\mathbf{Q}_C = \mathbf{grad}\,\phi_C$ for the electric port case, and $\mathbf{Q}_C = \boldsymbol{\rho}_C$ for the internal conductor case. Thus Ampère law becomes

$$\mathbf{curl}\,\mathbf{H}_C - \boldsymbol{\sigma}\mathbf{E}_C = V\boldsymbol{\sigma}\mathbf{Q}_C \ .$$

  In the former case, we intend that the voltage passes from 0 on Γ_E to V on Γ_J; in the latter case, the voltage passes from 0 to V along the internal cycle γ.

The current intensity rule

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  Having to impose a current intensity I_0, modify Ohm law in Ω_C adding to the current density $\boldsymbol{\sigma}\mathbf{E}_C$ the "applied" current density $\mathbf{J}_{e,C} = V\boldsymbol{\sigma}\mathbf{Q}_C$, where $\mathbf{Q}_C$ is as in the “voltage rule" and V has to be determined. Thus the Ampère law reads

$$\mathbf{curl}\,\mathbf{H}_C - \boldsymbol{\sigma}\mathbf{E}_C - V\boldsymbol{\sigma}\mathbf{Q}_C = 0 \; .$$

Then determine the field quantities $\mathbf{H}$ and $\mathbf{E}_C$ and the voltage V in such a way that also the additional constraint

$$\int_S \mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{n} = I_0$$

is satisfied.

The current intensity rule (cont'd)

In this constraint one has $S = \Gamma_J$ for the electric port case, and $S = \Sigma$, a section of Ω_C, for the internal conductor case. In the former case, the unit vector $\mathbf{n}$ is the outward normal on Γ_J; in the latter case, the unit vector $\mathbf{n}$ on Σ has the same orientation of the internal cycle γ.

Caso F: variational formulation

As an example, let us give the variational formulation for the case F: given a voltage $V \neq 0$, the problem to solve is

$$\begin{aligned} \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \,\mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{w}} \\ = V \int_{\Omega_C} \boldsymbol{\rho}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} \end{aligned} \tag{10}$$

for all $\mathbf{w} \in X$, where

$$X := \{\mathbf{w} \in H(\mathbf{curl}; \Omega) \,|\, \mathbf{curl}\, \mathbf{w}_I = \mathbf{0} \text{ in } \Omega_I\} \ .$$

Then one computes $I_0 = \int_{\Omega_C} \boldsymbol{\rho}_C \cdot \mathbf{curl}\, \mathbf{H}_C \neq 0$ [note that $\overline{I_0} = V^{-1}(\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \,\mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{H}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}})$...] and defines $\mathbf{E}_C = \boldsymbol{\sigma}^{-1} \,\mathbf{curl}\, \mathbf{H}_C - V \boldsymbol{\rho}_C$.

Caso F: variational formulation (cont'd)

Instead, given the current intensity $I_0 \neq 0$, the problem is

$$\begin{cases} \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \operatorname{\mathbf{curl}} \mathbf{H}_C \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{w}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{w}} \\ \qquad\qquad -V \int_{\Omega_C} \boldsymbol{\rho}_C \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{w}_C} = 0 \\ \int_{\Omega_C} \boldsymbol{\rho}_C \cdot \operatorname{\mathbf{curl}} \mathbf{H}_C = I_0 \end{cases}$$

for all $\mathbf{w} \in X$, and the voltage $V \neq 0$ [note that $V = \overline{I_0}^{-1} (\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \operatorname{\mathbf{curl}} \mathbf{H}_C \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{H}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}})$...] turns out to be a Lagrange multiplier associated with the constraint requiring that the intensity current is equal to I_0.
Then, as usual, one defines $\mathbf{E}_C = \boldsymbol{\sigma}^{-1} \operatorname{\mathbf{curl}} \mathbf{H}_C - V \boldsymbol{\rho}_C$.

Don't forget the Faraday law!

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Since

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and this term is vanishing for a test function $\mathbf{w}_C$ with a compact support in Ω_C, one verifies that the Faraday equation in Ω_C is satisfied, and, having set $\mathbf{E}_C = \boldsymbol{\sigma}^{-1} \,\mathbf{curl}\, \mathbf{H}_C$, the same clearly holds for the Ampère equation (without sources) in the whole Ω.

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[Note: since the electric field $\mathbf{E}_I$ is determined by solving the Faraday equation in Ω_I (with $\mathbf{H}_I$ already known), one is led to believe that everything is all right...]

Don't forget the Faraday law! (cont'd)

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Let us see: the Faraday law relates the flux of the magnetic induction through a surface with the line integral of the electric field on the boundary of that surface.
Since we know the magnetic field in the whole Ω, surfaces can stay everywhere; but at the moment we know the electric field only in Ω_C, therefore the boundary of the surface must stay in $\overline{\Omega_C}$.

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But the Faraday law (in differential form) is satisfied in Ω_C.

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But the Faraday law (in differential form) is satisfied in Ω_C. Thus we must verify if there are surfaces in Ω_I with boundary on Γ, and moreover such that this boundary is not the boundary of a surface in Ω_C [if this is not the case, the Divergence Theorem says that again everything is all right, as the magnetic induction is divergence free in Ω...].

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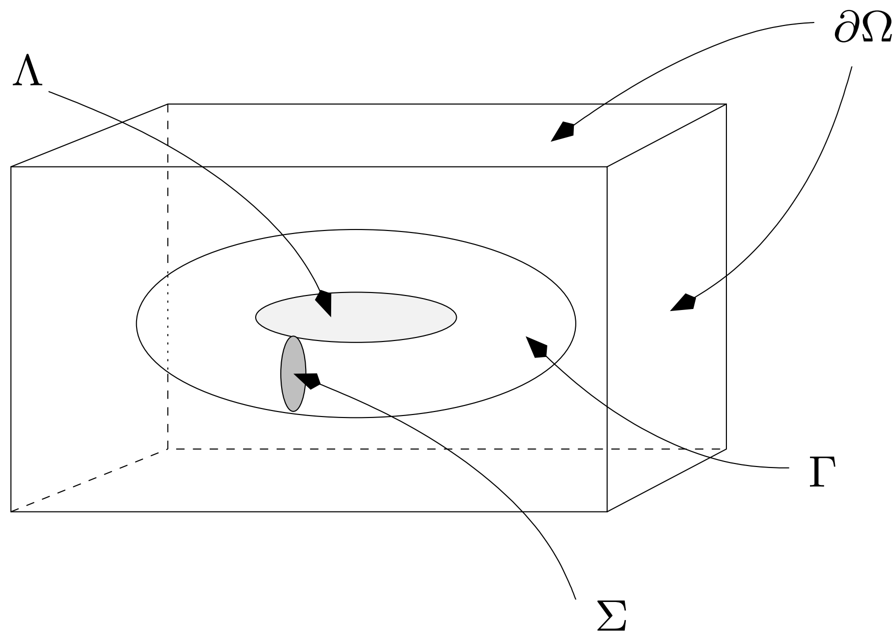
But the Faraday law (in differential form) is satisfied in Ω_C. Thus we must verify if there are surfaces in Ω_I with boundary on Γ, and moreover such that this boundary is not the boundary of a surface in Ω_C [if this is not the case, the Divergence Theorem says that again everything is all right, as the magnetic induction is divergence free in Ω...].

- Claim: the Faraday law is violated on the "cutting" surface Λ!



Don’t forget the Faraday law! (cont’d)

In fact, the Faraday law on Λ can be written as

$$\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I + \int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I = 0 \ ,$$

and from (10) we have

$$\begin{aligned} \textstyle\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I &= -\textstyle\int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{R}_C \\ &\quad + V \textstyle\int_{\Omega_C} \boldsymbol{\rho}_C \cdot \mathbf{curl}\, \mathbf{R}_C - \textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \mathbf{R}_C \ , \end{aligned}$$

where $\mathbf{R}_C$ is any (real) extension of $\boldsymbol{\rho}_I$ in Ω_C giving a global function that belongs to the space X.

Don't forget the Faraday law! (cont'd)

Setting $\mathbf{E}_C = \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C$ and integrating by parts one has

$$V\int_{\Omega_C}\boldsymbol{\rho}_C\cdot\mathbf{curl}\,\mathbf{R}_C - \int_{\Omega_C}\mathbf{E}_C\cdot\mathbf{curl}\,\mathbf{R}_C = V\int_\Gamma(\boldsymbol{\rho}_C\times\mathbf{n}_C)\cdot\boldsymbol{\rho}_I + \int_{\Omega_C} i\omega\boldsymbol{\mu}_C\mathbf{H}_C\cdot\mathbf{R}_C - \int_\Gamma(\mathbf{E}_C\times\mathbf{n}_C)\cdot\boldsymbol{\rho}_I\ ,$$

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$$
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&+ \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{R}_C - \int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I \ ,
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I + \int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I \\
= V \int_\Gamma (\boldsymbol{\rho}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I = V \neq 0 \ .
\end{aligned}
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$$V\int_{\Omega_C}\boldsymbol{\rho}_C\cdot\mathbf{curl}\,\mathbf{R}_C-\int_{\Omega_C}\mathbf{E}_C\cdot\mathbf{curl}\,\mathbf{R}_C=V\int_{\Gamma}(\boldsymbol{\rho}_C\times\mathbf{n}_C)\cdot\boldsymbol{\rho}_I$$
$$+\int_{\Omega_C}i\omega\boldsymbol{\mu}_C\mathbf{H}_C\cdot\mathbf{R}_C-\int_{\Gamma}(\mathbf{E}_C\times\mathbf{n}_C)\cdot\boldsymbol{\rho}_I\ ,$$

so that

$$\int_{\Omega_I}i\omega\boldsymbol{\mu}_I\mathbf{H}_I\cdot\boldsymbol{\rho}_I+\int_{\Gamma}(\mathbf{E}_C\times\mathbf{n}_C)\cdot\boldsymbol{\rho}_I$$
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Instead, everything works well if we define
$\mathbf{E}_C = \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C - V\boldsymbol{\rho}_C$.

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Setting $\mathbf{E}_C = \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C$ and integrating by parts one has

$$V \int_{\Omega_C} \boldsymbol{\rho}_C \cdot \mathbf{curl}\,\mathbf{R}_C - \int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl}\,\mathbf{R}_C = V \int_\Gamma (\boldsymbol{\rho}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I \\ + \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{R}_C - \int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I \ ,$$

so that

$$\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I + \int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I \\ = V \int_\Gamma (\boldsymbol{\rho}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I = V \neq 0 \ .$$

Instead, everything works well if we define
$\mathbf{E}_C = \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C - V\boldsymbol{\rho}_C$.

[Note: what is wrong in the previous argument? We cannot find the electric field $\mathbf{E}_I$ such that $\mathbf{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}_I\mathbf{H}_I$ in Ω_I and $\mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C$ on Γ: a necessary compatibility condition on the data is not satisfied!]

Cases A, B, D, E, F: existence and uniqueness

Summing up:

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Summing up:

- The problem with a given voltage is therefore a standard eddy-current problem, but with a particular assigned current density $\mathbf{J}_{e,C}$, hence it has a unique solution.
- The problem with a given current intensity is instead a saddle-point problem, and it needs a deeper analysis. In conclusion, however, it turns out to have a unique solution, too.

Cases A, B, D, E, F: numerical approximation

- For the voltage problem one can use any numerical approximation method that is suitable for eddy-current problems. [For a more efficient implementation, it is better to replace the functions $\mathbf{grad}\,\phi_C$ or $\boldsymbol{\rho}_C$ with a term that can be easily computed.]

Cases A, B, D, E, F: numerical approximation

- For the voltage problem one can use any numerical approximation method that is suitable for eddy-current problems. [For a more efficient implementation, it is better to replace the functions $\mathbf{grad}\,\phi_C$ or $\boldsymbol{\rho}_C$ with a term that can be easily computed.]
- For the current intensity problem, one has to use those numerical approximation methods that are suitable for saddle-point problems. [However, note that the current intensity contraint is associated with only one degree of freedom, therefore one is facing a rather simple extension of usual eddy-current problems.]

Numerical results for the Case C

Coming back to the case C and to its variational formulation (7), (8), (9), we use edge finite elements of the lowest degree ($\mathbf{a} + \mathbf{b} \times \mathbf{x}$ in each element) for approximating $\mathbf{E}_C$, and scalar piecewise linear elements for approximating ψ_I.

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The problem description is the following: the conductor Ω_C and the whole domainΩ are two coaxial cylinders of radius R_C and R_D, respectively, and height L. Assuming that $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are scalar constants, the exact solution for an assigned current intensity I_0 is known (through suitable Bessel functions), and also the basis function $\boldsymbol{\rho}_I$ is known, thus from (9) one easily computes the voltage V, too.

Numerical results for the Case C (cont'd)

We have the following data:

$$
\begin{aligned}
R_C &= 0.25 \text{ m} \\
R_D &= 0.5 \text{ m} \\
L &= 0.25 \text{ m} \\
\sigma &= 151565.8 \ \Omega^{-1}\text{m}^{-1} \\
\mu &= 4\pi \times 10^{-7} \text{ Hm}^{-1} \\
\omega &= 50 \times 2\pi \text{ rad/s}
\end{aligned}
$$

and

$$I_0 = 10^4 \text{ A} \quad \text{or} \quad V = 0.08979 + 0.14680i$$

[the voltage corresponds to the current intensity $I_0 = 10^4$ A].

Numerical results for the Case C (cont'd)

The relative errors (for $\mathbf{E}_C$ in $H(\mathbf{curl};\Omega_C)$ and for $\mathbf{H}_I$ in $L^2(\Omega_I)$) with respect to the number of degrees of freedom are given by:

Numerical results for the Case C (cont'd)

The relative errors (for $\mathbf{E}_C$ in $H(\mathbf{curl};\Omega_C)$ and for $\mathbf{H}_I$ in $L^2(\Omega_I)$) with respect to the number of degrees of freedom are given by:

Elements	DoF	e_E	e_H	e_V
2304	1684	0.2341	0.1693	0.0312
18432	11240	0.1132	0.0847	0.0089
62208	35580	0.0750	0.0567	0.0048
147456	81616	0.0561	0.0425	0.0018

Numerical results for the Case C (cont'd)

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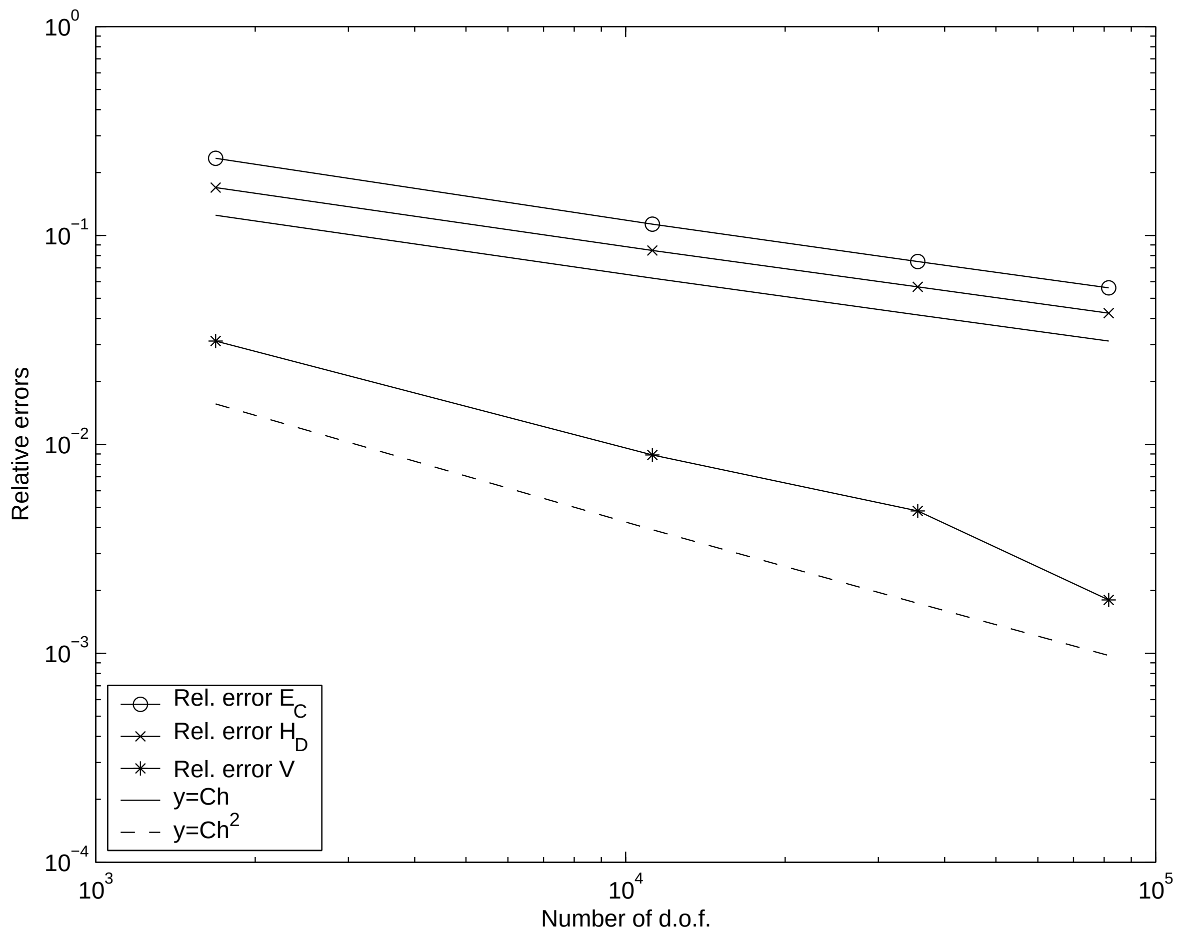
Elements	DoF	e_E	e_H	e_I
2304	1685	0.2336	0.1685	0.0274
18432	11241	0.1132	0.0847	0.0085
62208	35581	0.0750	0.0566	0.0041
147456	81617	0.0561	0.0425	0.0024

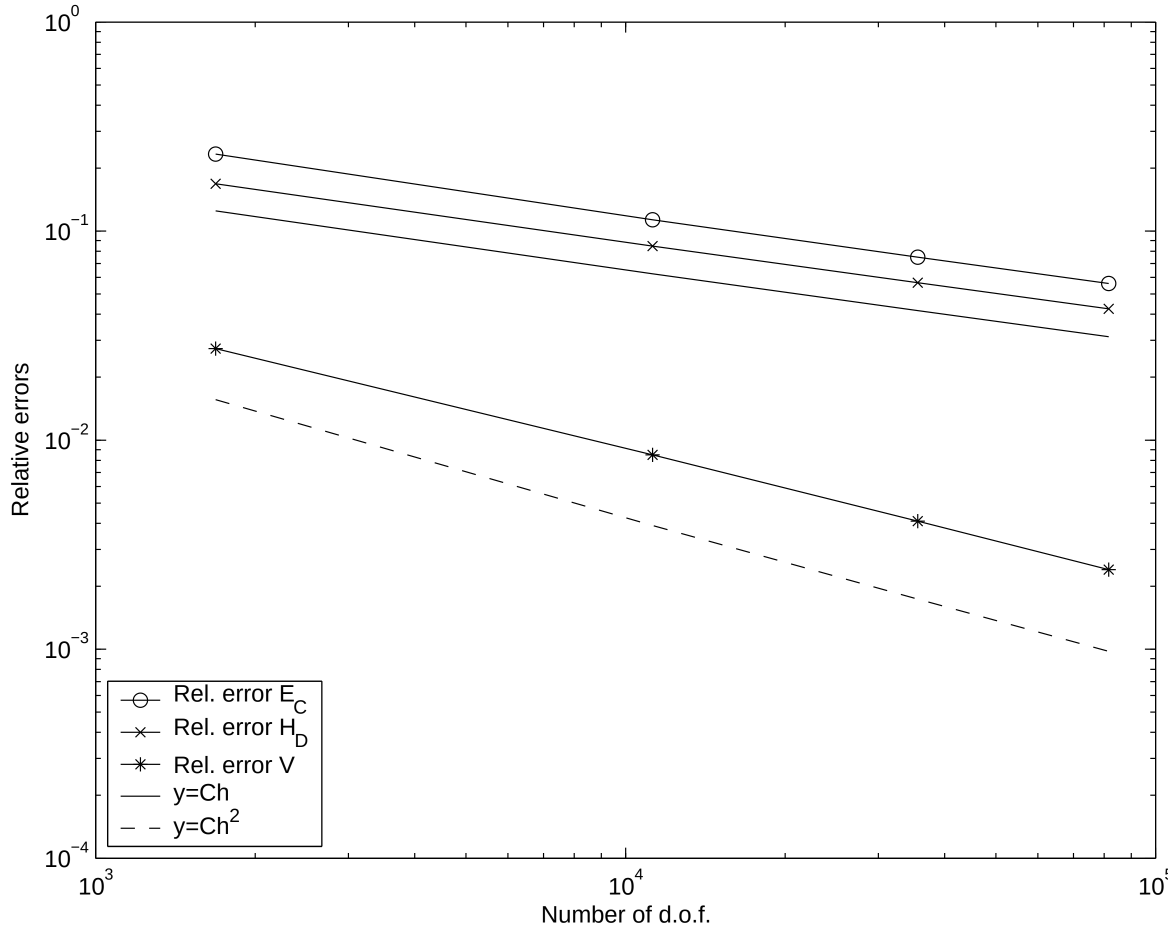
Numerical results for the Case C (cont'd)

On a graph: for assigned current intensity



Numerical results for the Case C (cont'd)

for assigned voltage



Numerical results for the Case C (cont'd)

A more realistic problem, considered by Bermúdez, Rodríguez and Salgado, 2005, is that of a cylindircal electric furnace with three electrodes ELSA [dimensions: furnace height 2 m.; furnace diameter 8.88 m.; electrode height 1.25 m.; electrode diameter 1 m.; distance of the center of the electrode from the wall 3 m.].

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The three electrodes ELSA are constituted by a graphite core of 0.4 m. of diameter, and by an outer part of Söderberg paste. The electric current enters the electrodes through horizontal copper bars of rectangular section (0.07 m. $\times$ 0.25 m.), connecting the top of the electrode with the external boundary.

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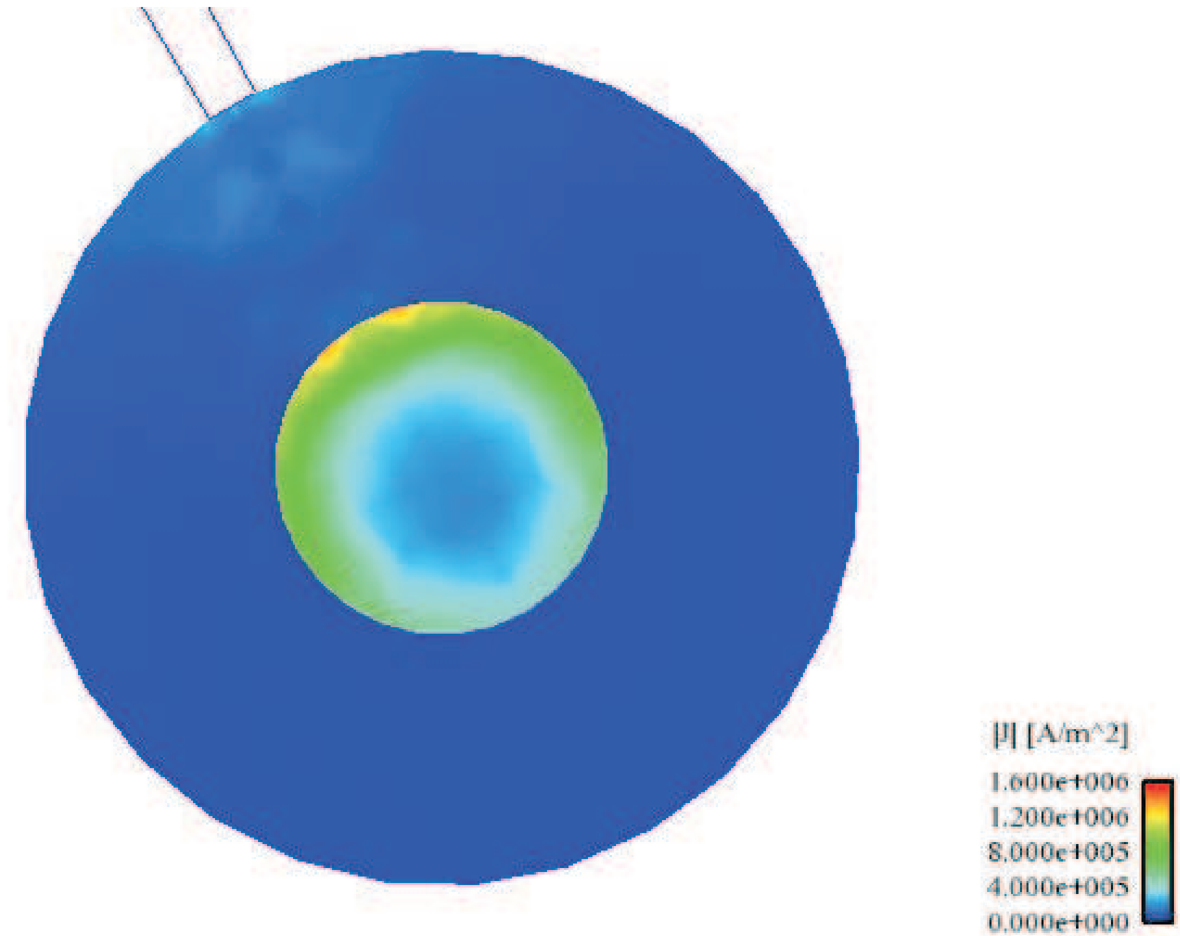
Data: $\sigma = 10^6\ \Omega^{-1}\mathrm{m}^{-1}$ for graphite, $\sigma = 10^4\ \Omega^{-1}\mathrm{m}^{-1}$ for Söderberg paste, $\sigma = 5 \times 10^6\ \Omega^{-1}\mathrm{m}^{-1}$ for copper, $\mu = 4\pi \times 10^{-7}\ \mathrm{Hm}^{-1}$, $\omega = 50 \times 2\pi$ rad/s, $I_0 = 7 \times 10^4$ A for each electrode.

Numerical results for the Case C (cont'd)

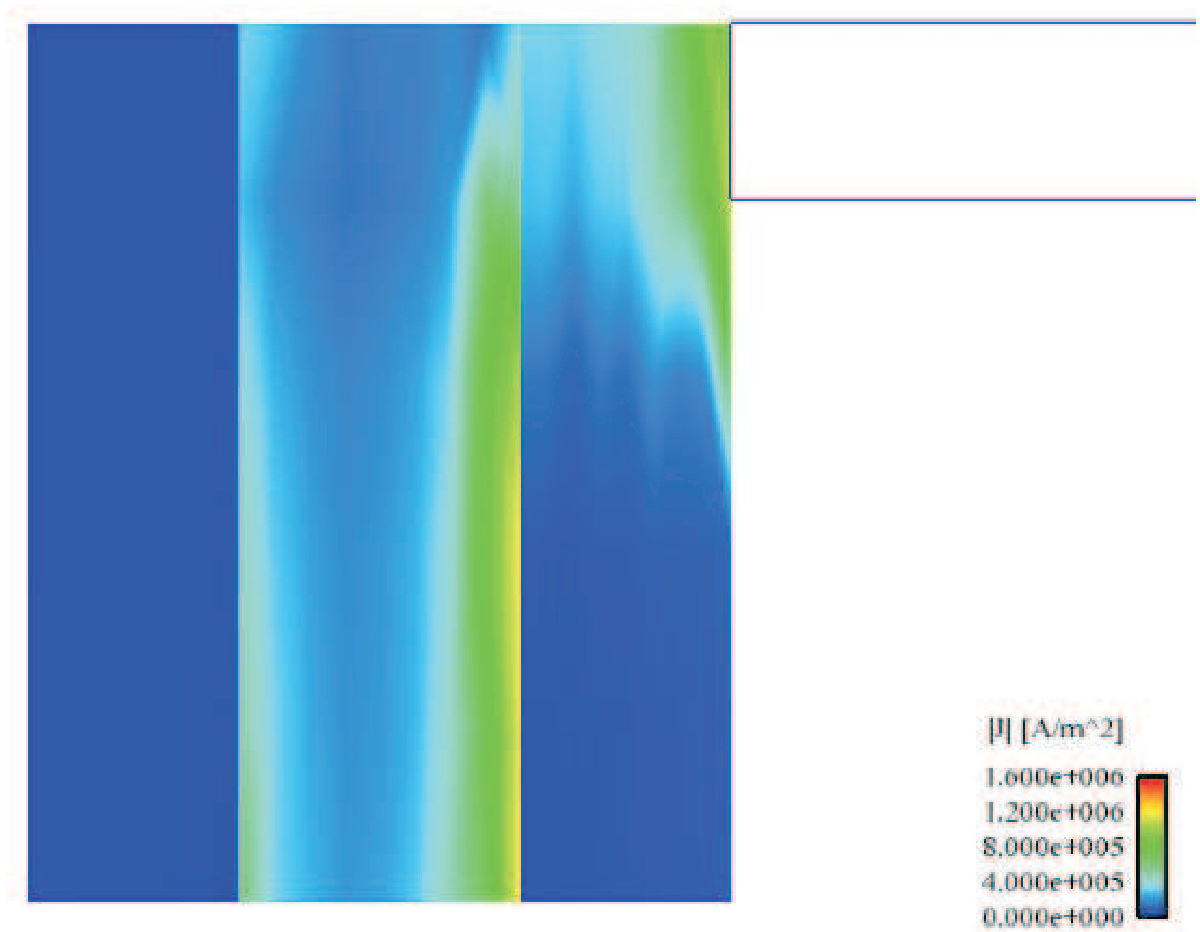
The value of the magnetic "potential" in the insulator: the magnetic field is the gradient of the represented function (not taking into account the jump surfaces).

Numerical results for the Case C (cont'd)



The magnitude of the current density $\mathbf{J}_{e,C} = \sigma \mathbf{E}_C$ on a horizontal section of one electrode.

Numerical results for the Case C (cont'd)



The magnitude of the current density $\mathbf{J}_{e,C} = \sigma \mathbf{E}_C$ on a vertical section of one electrode.

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