Finite elements in electromagnetism and application to eddy current equations

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Outline

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2. Maxwell and eddy current equations
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What finite elements are?

Finite elements are piecewise-polynomials functions defined in a domain Ω.

- The word "piecewise" indicates that Ω has been split in a finite number of (non-overlapping) pieces, called elements. (In other words, a "triangulation" $\mathcal{T}_h$ of $\overline{\Omega}$ is available.)

Simplest case:

- all the elements have the same shape (triangles/tetrahedra, parallelograms/parallelepipeds)
- the polynomials are of the same type (and therefore of a fixed degree) for all the elements.

Thus

- a space of finite elements is a finite dimensional vector space.

Variational formulation of a PDE

Very often a partial differential equation (in variational form) can be written as:

$$u \in V \ : \ a(u,v) = \mathcal{F}(v) \quad \forall \ v \in V \,, \tag{1}$$

where V is an infinite dimensional vector space.

In the simplest case, $a(\cdot,\cdot)$ is a bilinear form, and $\mathcal{F}(\cdot)$ is a linear form.

Example of a variational formulation of a PDE

An example: given the problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \,, \end{cases}$$

for f such that $\int_\Omega f^2 < \infty$, multiply the equation by a test function v, integrate in Ω and integrate by parts. Using the boundary condition, you find that u is a solution of:

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega u\, v = \int_\Omega f\, v \quad \forall\, v \,.$$

The space V in this case is

$$H^1(\Omega) = \left\{ v : \Omega \to \mathbb{R} \,|\, \int_\Omega (|\nabla v|^2 + v^2) < \infty \right\}.$$

Discretization of a PDE

Denote by V_h a space of finite elements, where h is the maximum diameter of the pieces in which Ω has been split. A reasonable discretization of (1) is:

$$u_h \in V_h \ : \ a(u_h, v_h) = \mathcal{F}(v_h) \quad \forall \ v_h \in V_h \, . \tag{2}$$

In the simplest case one can choose $V_h \subset V$, therefore the consistency property

$$a(u, v_h) = a(u_h, v_h) \ \text{ for each } v_h \in V_h$$

holds.

If, in some sense, $V_h \to V$, we can expect that $u_h \to u$.

Discretization of a PDE (cont'd)

In fact, if

- $a(w, v) \leq \beta \|w\| \|v\|$ for all $w, v \in V$ [continuity]
- $a(v, v) \geq \alpha \|v\|^2$ for all $v \in V$ [coerciveness],

we obtain

$$\begin{aligned} \alpha \|u - u_h\|^2 &\leq a(u - u_h, u - u_h) = a(u - u_h, u) = a(u - u_h, u - v_h) \\ &\leq \beta \|u - u_h\| \|u - v_h\| \end{aligned}$$

for each $v_h \in V_h$, hence

$$\|u - u_h\| \leq \frac{\beta}{\alpha} \inf_{v_h \in V_h} \|u - v_h\| \, .$$

Matching conditions for finite elements

Assuming that $V_h \subset V$ has a consequence: we are implicitly requiring that, on the common boundary of two elements, the values of the two polynomials have to match in a suitable way. "Suitable" simply means that, as a function defined in Ω, v_h must belong to V.

Example: $v_h \in H^1(\Omega)$ if and only if v_h is continuous across the interelements (namely, if it is continuous in Ω).

Summarizing: a PDE is associated to a vector space V, and this vector space is dictating the matching conditions to the finite elements. Therefore, a PDE is dictating which types of finite elements are suitable for its discretization.

Examples of PDEs and variational spaces

- A second order linear elliptic equation

$$-\sum_{i,j} D_i(a_{ij} D_j u) + \sum_i b_i D_i u + cu = f$$

is associated to $V = H^1(\Omega)$ (or to a closed subspace of it).

The matching condition for finite elements is the continuity of u across interelements.

Examples of PDEs and variational spaces (cont'd)

- The time-harmonic Maxwell equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \omega^2 \varepsilon_0 \mu_0 \mathbf{u} = \mathbf{f}$$

  is associated to

$$V = H(\mathbf{curl}\,;\Omega) = \{\mathbf{v} : \Omega \to \mathbb{R}^3 \mid \int_\Omega (|\mathbf{curl}\,\mathbf{v}|^2 + |\mathbf{v}|^2) < \infty\}$$

  (or to a closed subspace of it).

It is easily seen that the matching condition for finite elements is the continuity of $\mathbf{u} \times \mathbf{n}$ across interelements ($\mathbf{n}$ being the unit normal vector on the common boundary).

Examples of PDEs and variational spaces (cont'd)

- The Darcy system

$$\begin{cases} K^{-1}\mathbf{u} + \nabla p = K^{-1}\mathbf{f} \\ \operatorname{div}\mathbf{u} = 0 \end{cases}$$

is associated to

$$V = H(\operatorname{div};\Omega) = \left\{ \mathbf{v} : \Omega \to \mathbb{R}^3 \mid \int_\Omega \left((\operatorname{div}\mathbf{v})^2 + |\mathbf{v}|^2\right) < \infty \right\}$$

(or to a closed subspace of it) for the velocity field $\mathbf{u}$, and to $L^2(\Omega) = \left\{ q : \Omega \to \mathbb{R} \mid \int_\Omega q^2 < \infty \right\}$ for the pressure p.

It is easily seen that the matching condition for finite elements is the continuity of $\mathbf{u} \cdot \mathbf{n}$ across interelements (and no matching for p).

Maxwell equations in electromagnetism

Let us focus now on Maxwell equations and electromagnetism. The complete system reads

$$\begin{cases} \dfrac{\partial \mathcal{D}}{\partial t} + \mathcal{J} = \mathbf{curl}\, \mathcal{H} & \text{Maxwell–Ampère equation} \\ \dfrac{\partial \mathcal{B}}{\partial t} + \mathbf{curl}\, \mathcal{E} = 0 & \text{Faraday equation} \\ \operatorname{div} \mathcal{D} = \rho & \text{Gauss electrical equation} \\ \operatorname{div} \mathcal{B} = 0 & \text{Gauss magnetic equation.} \end{cases}$$

- $\mathcal{H}$ and $\mathcal{E}$ are the magnetic field and electric field, respectively
- $\mathcal{B}$ and $\mathcal{D}$ are the magnetic induction and electric induction, respectively
- $\mathcal{J}$ and ρ are the (surface) electric current density and (volume) electric charge density, respectively.

Maxwell equations in electromagnetism (cont'd)

These fields are related through some constitutive equations: it is usually assumed a linear dependence like

$$\mathcal{D} = \varepsilon \mathcal{E} \quad , \quad \mathcal{B} = \boldsymbol{\mu} \mathcal{H} \quad , \quad \mathcal{J} = \boldsymbol{\sigma} \mathcal{E} + \mathcal{J}_e \ ,$$

where ε and $\boldsymbol{\mu}$ are the electric permittivity and magnetic permeability, respectively, and $\boldsymbol{\sigma}$ is the electric conductivity.

[In general, ε, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are symmetric and uniformly positive definite matrices. Clearly, the conductivity $\boldsymbol{\sigma}$ is only present in conductors, and is identically vanishing in any insulator.]

- $\mathcal{J}_e$ is the applied electric current density.

Eddy currents

As observed in experiments and stated by the Faraday law, a time-variation of the magnetic field generates an electric field. Therefore, in each conductor a current density $\mathbf{J}_{\text{eddy}} = \boldsymbol{\sigma}\mathbf{E}$ arises; this term expresses the presence in conducting media of the so-called eddy currents.

This phenomenon, and the related heating of the conductor, was observed and studied in the mid of the nineteenth century by the French physicist L. Foucault, and in fact the generated eddy currents are also known as Foucault currents.

Eddy current approximation

In many real-life applications, the time of propagation of the electromagnetic waves is very small with respect to some characteristic time scale, or, equivalently, their wave length is much larger than the diameter of the physical domain.

Therefore one can think that the speed of propagation is infinite, and take into account only the diffusion of the electromagnetic fields, neglecting electromagnetic waves, namely, neglecting both time derivatives or one of them, either $\frac{\partial \mathcal{D}}{\partial t}$ or $\frac{\partial \mathcal{B}}{\partial t}$.

Let us focus on the case in which the displacement current term $\frac{\partial \mathcal{D}}{\partial t}$ can be disregarded, while the time-variation of the magnetic induction is still important, as well as the related presence of eddy currents in the conductors.

- The resulting equations are called eddy current equations.

Time-harmonic Maxwell and eddy current equations

When interested in time-periodic phenomena, it is assumed that

$$\begin{array}{rcl} \mathcal{J}_e(t,\mathbf{x}) & = & \mathrm{Re}[\mathbf{J}_e(\mathbf{x})\exp(i\omega t)] \\ \mathcal{E}(t,\mathbf{x}) & = & \mathrm{Re}[\mathbf{E}(\mathbf{x})\exp(i\omega t)] \\ \mathcal{H}(t,\mathbf{x}) & = & \mathrm{Re}[\mathbf{H}(\mathbf{x})\exp(i\omega t)]\ . \end{array}$$

- $\omega \neq 0$ is the (angular) frequency.

Inserting these relations in the Maxwell equations one obtains the so-called time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - i\omega\varepsilon\mathbf{E} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega\,. \end{cases} \tag{3}$$

[Note that similar equations arise from the backward-Euler time-discretization of Maxwell equations: just substitute $i\omega$ by $\frac{1}{\Delta t}$...]

Time-harmonic Maxwell and eddy current equations (cont'd)

As a consequence one has $\mathrm{div}(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω (and the electric charge in conductors is defined by $\rho = \mathrm{div}(\boldsymbol{\varepsilon}\mathbf{E})$).

It can be proved that the time-harmonic Maxwell equations have a unique solution (provided that suitable boundary conditions are added, and that the conductor is not empty).

On the other hand, dropping the displacement current term, the time-harmonic eddy current equations are

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega\,. \end{cases} \tag{4}$$

[Since in an insulator one has $\boldsymbol{\sigma} = \mathbf{0}$, it follows that $\mathbf{E}$ is not uniquely determined in that region ($\mathbf{E} + \mathbf{grad}\,\psi$ is still a solution). Some additional conditions are thus necessary (typically, the conditions satisfied by the solution $\mathbf{E}$ of the Maxwell equations).]

Boundary conditions

From now on we will denote by Ω_I the insulator, namely, the region where $\sigma = 0$, by Ω_C the conductor, and by Γ the interface between Ω_I and Ω_C. We also assume that Ω_C is strictly contained in Ω, and that Ω_I is connected.

One of the following boundary conditions is usually imposed.

- Electric. One imposes $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. [As a consequence, one also has $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$.]
- Magnetic (Maxwell). One imposes $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. [As a consequence, one also has $\boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = -(i\omega)^{-1}\mathbf{J}_e \cdot \mathbf{n}$ on $\partial\Omega$.]
- Magnetic (eddy currents). One imposes $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$. [Note that $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ implies $\mathbf{J}_e \cdot \mathbf{n} = 0$ on $\partial\Omega$.]

We will focus on the electric boundary condition.

The spaces of harmonic fields

Now it is time to consider a couple of questions.

- If a vector field satisfies $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$ and $\mathrm{div}\,\mathbf{v} = 0$ in a domain, together with either the boundary condition $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ or the boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$, is it non-trivial, namely, not vanishing everywhere in the domain? [A field like that is called harmonic field.]
- If that is the case, do harmonic fields appear in electromagnetism?

Both questions have an affermative answer.

The spaces of harmonic fields (cont'd)

Let us start from the first question.

If the domain $\mathcal{O}$ is homeomorphic to a three-dimensional ball, a curl-free vector field $\mathbf{v}$ must be a gradient of a scalar function ψ, that must be harmonic due to the constraint on the divergence.

If the boundary condition is $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{O}$, which in this case is a connected surface, then it follows $\psi = \text{const}$ on $\partial\mathcal{O}$, and therefore $\psi = \text{const}$ in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

If the boundary condition is $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\mathcal{O}$, then ψ satisfies a homogeneous Neumann boundary condition and thus $\psi = \text{const}$ in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

The spaces of harmonic fields (cont'd)

However, the problem is different in a more general geometry.

In fact, take the magnetic field generated in the vacuum by a current of constant intensity I^0 passing along the x_3-axis: as it is well-known, for $x_1^2 + x_2^2 > 0$ it is given by

$$\mathbf{H}(x_1, x_2, x_3) = \frac{I^0}{2\pi} \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0 \right) .$$

It is easily checked that $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ and $\operatorname{div} \mathbf{H} = 0$.
Let us consider now the torus $\mathcal{T}$ obtained by rotating around the x_3-axis the disk (contained in the (x_1, x_3)-plane) of centre $(a, 0, 0)$ and radius b, with $0 < b < a$. One sees at once that $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}$; hence we have found a non-trivial harmonic field $\mathbf{H}$ in $\mathcal{T}$ satisfying $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}$.

The spaces of harmonic fields (cont'd)

On the other hand, consider now the electric field generated in the vacuum by a pointwise charge ρ_0 placed at the origin. For $\mathbf{x} \neq \mathbf{0}$ it is given by

$$\mathbf{E}(x_1, x_2, x_3) = \frac{\rho_0}{4\pi\varepsilon_0} \frac{\mathbf{x}}{|\mathbf{x}|^3} \ ,$$

where ε_0 is the electric permittivity of the vacuum.

It satisfies $\operatorname{div} \mathbf{E} = 0$ and $\mathbf{curl}\, \mathbf{E} = \mathbf{0}$, and moreover $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on the boundary of $\mathcal{C} := B_{R_2} \setminus \overline{B_{R_1}}$, where $0 < R_1 < R_2$ and $B_R := \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x}| < R\}$ is the ball of centre $\mathbf{0}$ and radius R. We have thus found a non-trivial harmonic field $\mathbf{E}$ in $\mathcal{C}$ satisfying $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{C}$.

The spaces of harmonic fields (cont'd)

These two examples show that the geometry of the domain and the type of boundary conditions play an essential role when considering harmonic fields.

What are the relevant differences between the set $\mathcal{O}$, homeomorphic to a ball, and the sets $\mathcal{T}$ and $\mathcal{C}$?

For the former, the point is that in $\mathcal{T}$ we have a non-bounding closed curve, namely, a cycle that is not the boundary of a surface contained in $\mathcal{T}$ (take, for instance, the circle of centre $\mathbf{0}$ and radius a in the (x_1, x_2)-plane).

In the latter case, in $\mathcal{C}$ we have a non-bounding closed surface, namely, a closed surface that is not the boundary of a subdomain contained in $\mathcal{C}$ (take, for instance, $\partial B_{R^\star}$, with $R_1 < R^\star < R_2$); or, equivalently, the boundary of $\mathcal{C}$ is not connected.

The spaces of harmonic fields (cont'd)

A couple of spaces of harmonic fields are coming into play.

- For the electric field

$$\mathcal{H}_I^{(e)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\,\mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\varepsilon}_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \cup \partial\Omega\}\,,$$

- For the magnetic field

$$\mathcal{H}_I^{(m)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\,\mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0 \\ \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma \cup \partial\Omega\}\,.$$

Both are finite dimensional! Their dimension is a topological invariant! [The dimension is also independent of $\boldsymbol{\varepsilon}_I$ and $\boldsymbol{\mu}_I$.]

The spaces of harmonic fields (cont'd)

The dimension of $\mathcal{H}_I^{(m)}$ is the first Betti number of Ω_I, or, equivalently, the dimension of the first homology group of $\overline{\Omega_I}$ (this is the quotient space between cycles in $\overline{\Omega_I}$ and bounding cycles in $\overline{\Omega_I}$).

- The number of handles of Ω_I!

The dimension of $\mathcal{H}_I^{(e)}$ is the second Betti number of Ω_I, or, equivalently, the dimension of the second homology group of $\overline{\Omega_I}$ (this is the quotient space between closed surfaces in $\overline{\Omega_I}$ and bounding surfaces in $\overline{\Omega_I}$).

- The number of connected components of $\partial\Omega_I$ minus 1!

Edge finite elements

We have seen that the Maxwell equations are associated to the operator **curl** , and that therefore electromagnetic problems can be approximated by means of vector finite elements for which only the continuity of the tangential components on the interelements is required (and not the continuity of all the components).

- These elements are called edge elements, and have been proposed by Nédélec (1980).

For $r \geq 1$ denote by $\widetilde{\mathbb{P}}_r$ the space of homogeneous polynomials of degree r and define

$$S_r := \{\mathbf{q} \in (\widetilde{\mathbb{P}}_r)^3 \,|\, \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0\} \ , \ R_r := (\mathbb{P}_{r-1})^3 \oplus S_r \, .$$

The Nédélec finite elements (for a tetrahedral mesh, say, $\mathcal{T}_h$) are

$$N_h^r := \{\mathbf{w}_h \in H(\mathbf{curl}\,;\Omega) \,|\, \mathbf{w}_{h|K} \in R_r \ \forall \ K \in \mathcal{T}_h\} \, . \qquad (5)$$

Lowest order edge finite elements

- Let us specify the form of Nédélec edge elements and their degrees of freedom for $r = 1$.

The condition $\mathbf{q} \cdot \mathbf{x} = 0$ for $\mathbf{q} \in (\widetilde{\mathbb{P}}_1)^3$ says that $\mathbf{q} = \mathbf{a} \times \mathbf{x}$ with $\mathbf{a} \in \mathbb{R}^3$. Hence the space R_1 is given by the polynomials of the form

$$\mathbf{q}(\mathbf{x}) = \mathbf{b} + \mathbf{a} \times \mathbf{x} \quad , \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \, . \tag{6}$$

The degrees of freedom are associated to the edges e of a tetrahedron K, and are given by the six line integrals

$$\int_e (\mathbf{b} + \mathbf{a} \times \mathbf{x}) \cdot d\mathbf{s} \, . \tag{7}$$

Lowest order edge finite elements (cont'd)

- Let us show that, if all the degrees of freedom of $\mathbf{q} = \mathbf{b} + \mathbf{a} \times \mathbf{x}$ on K are equal to 0, then $\mathbf{q} = \mathbf{0}$.

A direct computation shows that $\mathbf{curl}\,\mathbf{q} = 2\,\mathbf{a}$. Moreover, from Stokes theorem for each face f we have

$$\begin{aligned} 0 &= \sum_e \int_e \mathbf{q} \cdot d\mathbf{s} = \int_{\partial f} \mathbf{q} \cdot d\mathbf{s} \\ &= \int_f \mathbf{curl}\,\mathbf{q} \cdot \mathbf{n}_f \, dS = 2\,\mathbf{a} \cdot \mathbf{n}_f \,\mathrm{meas}(f)\,, \end{aligned}$$

hence $\mathbf{a} \cdot \mathbf{n}_f = 0$ on f. Since three of the vectors $\mathbf{n}_f$ are linearly independent, it follows $\mathbf{a} = \mathbf{0}$.

Lowest order edge finite elements (cont'd)

Then for each edge e

$$\begin{aligned} 0 &= \int_e \mathbf{q} \cdot d\mathbf{s} = \int_e \mathbf{b} \cdot d\mathbf{s} \\ &= \mathbf{b} \cdot \boldsymbol{\tau}_e \, \mathrm{length}(e) \, , \end{aligned}$$

and three of the unit tangent vectors $\boldsymbol{\tau}_e$ are linearly independent, so that $\mathbf{b} = \mathbf{0}$ and in conclusion $\mathbf{q} = \mathbf{0}$.

Lowest order edge finite elements (cont'd)

- Another point is to prove that if the three edge degrees of freedom of $\mathbf{q} = \mathbf{b} + \mathbf{a} \times \mathbf{x}$ on a face f are equal to 0 then $\mathbf{q} \times \mathbf{n}_f = \mathbf{0}$ on f.

We have already seen that $\mathbf{a} \cdot \mathbf{n}_f = 0$ on f. On the other hand,

$$\begin{aligned} \mathbf{q} \times \mathbf{n}_f &= \mathbf{b} \times \mathbf{n}_f + (\mathbf{a} \times \mathbf{x}) \times \mathbf{n}_f \\ &= \mathbf{b} \times \mathbf{n}_f + (\mathbf{a} \cdot \mathbf{n}_f)\,\mathbf{x} - (\mathbf{x} \cdot \mathbf{n}_f)\,\mathbf{a}\,. \end{aligned}$$

Since on a face one has $\mathbf{x} \cdot \mathbf{n}_f = \text{const}$, it follows that $\mathbf{q} \times \mathbf{n}_f$ is equal on f to a constant vector $\mathbf{c}_f$, with $\mathbf{c}_f \cdot \mathbf{n}_f = 0$.

Lowest order edge finite elements (cont'd)

Finally,

$$\begin{aligned} 0 &= \int_e \mathbf{q} \cdot d\mathbf{s} = \int_e (\mathbf{n}_f \times \mathbf{q} \times \mathbf{n}_f) \cdot d\mathbf{s} \\ &= (\mathbf{n}_f \times \mathbf{c}_f) \cdot \boldsymbol{\tau}_e \, \mathrm{length}(e) \, . \end{aligned}$$

Since two of the tangent vectors $\boldsymbol{\tau}_e$ are generating the plane containing f (and the vector $\mathbf{n}_f \times \mathbf{c}_f$), it follows $\mathbf{c}_f = \mathbf{0}$ and consequently $\mathbf{q} \times \mathbf{n}_f = \mathbf{0}$ on f.

Lowest order edge finite elements (cont'd)

- In particular, we have shown that the dimension of N_h^1 is equal to the total number of edges.

The basis functions are defined as follows: for each edge e_m we construct the vector function $\mathbf{\Phi}_m$ such that

$$\int_{e_l} \mathbf{\Phi}_m \cdot d\mathbf{s} = \begin{cases} 1 & \text{if } m = l \\ 0 & \text{if } m \neq l \,. \end{cases} \tag{8}$$

These basis functions have a "small" support: $\mathbf{\Phi}_m$ is non-vanishing only in the elements K of the triangulation that contain the edge e_m.

Lowest order edge finite elements (cont'd)

- The explicit construction of a basis for the edge element space N_h^1 is easily done.

In fact, it can be proved that the basis function $\mathbf{\Phi}_{i,j}$ associated to the edge $e_{i,j}$ joining the nodes $\mathbf{x}_i$ and $\mathbf{x}_j$ and satisfying $\int_{e_{i,j}} \mathbf{\Phi}_{i,j} \cdot d\mathbf{s} = 1$ is given by

$$\mathbf{\Phi}_{i,j} = \varphi_i \, \mathbf{grad} \, \varphi_j - \varphi_j \, \mathbf{grad} \, \varphi_i \,, \tag{9}$$

where φ_i is the piecewise-linear nodal basis function associated to the node $\mathbf{x}_i$.

Finite element spaces

To have a more complete picture of the most employed finite elements, we can look at the following list (for a triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ composed by tetrahedra):

- The space $L^1_h \subset H^1(\Omega)$ of (continuous) piecewise-linear nodal finite elements. Its dimension is the number of vertices in $\mathcal{T}_h$.
- The space $N^1_h \subset H(\mathbf{curl}\,;\Omega)$ of (tangentially continuous) Nédélec edge finite elements of degree 1 [locally: $\mathbf{a} + \mathbf{b} \times \mathbf{x}$]. Its dimension is the number of edges in $\mathcal{T}_h$.
- The space $RT^1_h \subset H(\mathrm{div};\Omega)$ of (normally continuous) Raviart–Thomas face finite elements of degree 1 [locally: $\mathbf{a} + b\,\mathbf{x}$]. Its dimension is the number of faces in $\mathcal{T}_h$.

We have $\mathbf{grad}\, L^1_h \subset N^1_h$ and $\mathbf{curl}\, N^1_h \subset RT^1_h$.

Variational formulation for Maxwell equations

The time-harmonic Maxwell equations (3)

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - i\omega\varepsilon\mathbf{E} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \end{cases}$$

can be rewritten in the vacuum (where $\boldsymbol{\sigma} = \mathbf{0}$) as

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \omega^2\varepsilon_0\mu_0\mathbf{E} = -i\omega\mu_0\mathbf{J}_e\,. \tag{10}$$

Splitting $\mathbf{J}_e$ into its real and imaginary parts, we can solve two real-valued problems of the same form for the real and imaginary parts of $\mathbf{E}$. Let us denote by $\mathbf{F}$ a general right hand side, and, for simplicity, let us introduce the wave number $\kappa = \sqrt{\mu_0\varepsilon_0}|\omega|$ (the reciprocal of the wave length).

Variational formulation for Maxwell equations (cont'd)

Adding the electrical boundary condition $\mathbf{E} \times \mathbf{n} = \mathbf{0}$, the variational formulation reads

$$\begin{aligned} &\text{find } \mathbf{E} \in H_0(\mathbf{curl}\,;\Omega) \ : \\ &\quad \int_\Omega [\mathbf{curl}\,\mathbf{E} \cdot \mathbf{curl}\,\mathbf{v} - \kappa^2\,\mathbf{E} \cdot \mathbf{v}] = \int_\Omega \mathbf{F} \cdot \mathbf{v} \\ &\text{for each } \mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)\ , \end{aligned} \tag{11}$$

where $H_0(\mathbf{curl}\,;\Omega) = \{\mathbf{v} \in H(\mathbf{curl}\,;\Omega) \,|\, \mathbf{v} \times \mathbf{n} = \mathbf{0}\}$.

- Problem (11) is associated to a bilinear form that is not coercive in $H(\mathbf{curl}\,;\Omega)$ [$-\kappa^2$ has the "wrong" sign...]. What we can say about existence and uniqueness of a solution?

Variational formulation for Maxwell equations (cont'd)

Consider the Maxwell eigenvalue problem

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \eta\,\mathbf{E} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases} \tag{12}$$

The classical Hilbert–Schmidt theory can be applied to obtain

- besides $\eta_0 = 0$, there exists a sequence of positive, increasing and diverging to ∞ eigenvalues η_m of problem (12).

Fredholm alternative theory can be used to prove

- when $\kappa \neq \sqrt{\eta_m}$, $m = 0, 1, 2, \ldots$, there exists a unique solution of problem (11).

Numerical approximation of Maxwell equations

Numerical approximation of (11) (and of (12)) is important in order to simulate real physical devices and obtain informations for shape optimization (for instance, an electromagnetic cavity is a model for microwave ovens).

The finite element approximation problem with edge elements reads

$$\begin{aligned} &\text{find } \mathbf{E}_h \in V_h \ : \\ &\quad \int_\Omega [\mathbf{curl}\,\mathbf{E}_h \cdot \mathbf{curl}\,\mathbf{v}_h - \kappa^2\,\mathbf{E}_h \cdot \mathbf{v}_h] = \int_\Omega \mathbf{F} \cdot \mathbf{v}_h \\ &\forall\ \mathbf{v}_h \in V_h\,, \end{aligned} \tag{13}$$

where

$$V_h := N_h^r \cap H_0(\mathbf{curl}\,;\Omega)\,.$$

Numerical approximation of Maxwell equations (cont'd)

It is possible to prove existence and uniqueness of the solution to the discrete problem (13), as well as its convergence to the solution of problem (11) and the associated error estimate. Precisely, for h small enough we have

$$\|\mathbf{E}-\mathbf{E}_h\|_{H(\mathbf{curl}\,,\Omega)} \leq C \inf_{\mathbf{v}_h \in V_h} \|\mathbf{E}-\mathbf{v}_h\|_{H(\mathbf{curl}\,,\Omega)} \,, \qquad (14)$$

therefore, for $\mathbf{E}$ smooth enough,

$$\|\mathbf{E}-\mathbf{E}_h\|_{H(\mathbf{curl}\,,\Omega)} = O(h^r)\,.$$

Variational formulation for the eddy current problem

A variational formulation of the eddy current problem

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,, \end{cases}$$

in terms of the sole magnetic field **H** can be derived as follows. Setting

$$W := \{\mathbf{w} \in H(\mathbf{curl}\,;\Omega)\,|\,\mathbf{curl}\,\mathbf{w}_I = \mathbf{0} \text{ in } \Omega_I\}\,,$$

multiplying the Faraday equation by $\overline{\mathbf{w}}$, with $\mathbf{w} \in W$, integrating in Ω and integrating by parts one finds

$$\int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} + \int_{\Omega_I} \mathbf{E}_I \cdot \mathbf{curl}\,\overline{\mathbf{w}_I} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{w}} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{w}} = 0\,,$$

Variational formulation for the eddy current problem (cont'd)

thus

$$\int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{w}} = 0 \,,$$

as $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and $\mathbf{curl}\, \mathbf{w}_I = \mathbf{0}$ in Ω_I.

Using the Ampère equation in Ω_C for expressing $\mathbf{E}_C$, we end up with the following problem

$$\text{find } \mathbf{H}_e \in H(\mathbf{curl}\,;\Omega) \ : \ \mathbf{curl}\, \mathbf{H}_{e,I} = \mathbf{J}_{e,I} \text{ in } \Omega_I$$

$$\begin{aligned} &\text{find } (\mathbf{H} - \mathbf{H}_e) \in W \ : \\ &\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{w}} \\ &\qquad = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} \end{aligned} \qquad (15)$$

for each $\mathbf{w} \in W$.

Variational formulation for the eddy current problem (cont'd)

The first step simply says that $\mathbf{J}_{e,I}$ satisfies the necessary conditions for solvability, namely,

$$\mathrm{div}\mathbf{J}_{e,I} = 0 \text{ in } \Omega_I \ , \ \int_{(\partial\Omega_I)_j} \mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \quad \forall\, j = 1, \ldots, p \,, \qquad (16)$$

where $(\partial\Omega_I)_j$, $j = 0, 1, \ldots, p$, are the connected components of Ω_I.

The second step is well-posed via the Lax–Milgram lemma, as the sesquilinear form

$$a_m(\mathbf{u}, \mathbf{w}) := \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{u}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + \int_{\Omega} i\omega \boldsymbol{\mu} \mathbf{u} \cdot \overline{\mathbf{w}}$$

is clearly continuous and coercive in W.

Numerical approximation of the eddy current problem

Having denoted by $\mathbf{J}_{e,I}^{(h)}$ a suitable (Raviart–Thomas) finite element approximation of $\mathbf{J}_{e,I}$, and having set $W_h = N_h^1 \cap W$, the approximate problem simply reads

$$
\begin{aligned}
&\text{find } \mathbf{H}_e^{(h)} \in N_h^1 \ : \ \mathbf{curl}\,\mathbf{H}_{e,I}^{(h)} = \mathbf{J}_{e,I}^{(h)} \text{ in } \Omega_I \\
&\text{find } (\mathbf{H}_h - \mathbf{H}_e^{(h)}) \in W_h \ : \\
&\quad \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{curl}\,\mathbf{H}_{C,h} \cdot \mathbf{curl}\,\overline{\mathbf{w}_{C,h}} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{H}_h \cdot \overline{\mathbf{w}_h} \\
&\qquad\qquad = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{w}_{C,h}} \\
&\text{for each } \mathbf{w}_h \in W_h \,.
\end{aligned}
\tag{17}
$$

Numerical approximation of the eddy current problem (cont'd)

It is possible to prove convergence of the approximate solution $\mathbf{H}_h$ of problem (17) to the solution $\mathbf{H}$ of problem (15) and the associated error estimate. Precisely, choosing in Ω_I as an approximation of the current density its Raviart–Thomas interpolant, namely, $\mathbf{J}_{e,I}^{(h)} = \Pi^{RT_h^1}\mathbf{J}_{e,I}$, and denoting by $\Pi^{N_h^1}$ the Nédélec interpolation operator, one finds

$$\begin{aligned}\|\mathbf{H}-\mathbf{H}_h\|^2_{H(\mathbf{curl};\Omega)} \leq C\,\big(&\|\mathbf{H}_C-\Pi^{N_h^1}\mathbf{H}_C\|^2_{H(\mathbf{curl};\Omega_C)}\\ &+\|\mathbf{H}_I-\Pi^{N_h^1}\mathbf{H}_I\|^2_{L^2(\Omega_I)}\\ &+\|\mathbf{J}_{e,I}-\Pi^{RT_h^1}\mathbf{J}_{e,I}\|^2_{L^2(\Omega_I)}\big)\,,\end{aligned} \tag{18}$$

therefore, for $\mathbf{H}$ smooth enough,

$$\|\mathbf{H}-\mathbf{H}_h\|_{H(\mathbf{curl};\Omega)} = O(h)\,.$$

Efficient implementation of the finite element eddy current problem

We can devise an efficient implementation of the finite element eddy current problem.
The ingredients are:

- compute an edge element source field $\mathbf{H}_e^{(h)}$, constructed in Ω_I in correspondence to $\mathbf{J}_{e,I}^{(h)}$ and extended by 0 on the edges internal to Ω_C;
- compute an edge element basis $\mathbf{T}_\alpha$ of the first de Rham cohomology group of Ω_I (the set of curl-free vector fields that are not gradients).

Efficient implementation of the finite element eddy current problem (cont'd)

Then a basis for the space W_h is constructed as follows:

- for all the edges internal to Ω_C (and not on Γ), the Nédélec basis function of the lowest degree, extended by 0 on the edges on Γ and internal to Ω_I;
- for all the nodes internal to Ω_I and on Γ (except one), the gradient of the Lagrange basis function of degree 1, extended by 0 on the edges internal to Ω_C;
- the de Rham cohomology basis functions $\mathbf{T}_\alpha$, $\alpha = 1, \ldots, g$, extended by 0 on the edges internal to Ω_C.

Edge element loop fields and source fields

For determining the source field $\mathbf{H}_e^{(h)}$ and the basis $\mathbf{T}_\alpha$ of discrete loop fields we propose the following procedure.

Tools:

- homology theory
  - generators of the first homology group of $\overline{\Omega_I}$ (denoted by σ_n, $n = 1, \ldots, g$, g being the first Betti number of Ω_I) and $\mathbb{R}^3 \setminus \Omega_I$ (denoted by $\widehat{\sigma}_\alpha$, $\alpha = 1, \ldots, g$)
- graph theory applied to the mesh
  - a spanning tree (denoted by $\mathcal{S}_h$) of the graph given by the edges of the mesh in $\overline{\Omega_I}$
- direct elimination procedure
  - a direct algorithm of Webb and Forghani and an explicit formula.

The fundamental discrete problem

The *main problem* now is: given $\mathbf{J}_h \in RT_h^1$ satisfying the necessary solvability conditions (16), find $\mathbf{Z}_h \in N_h^1$ such that

$$\begin{array}{ll} \mathbf{curl}\, \mathbf{Z}_h = \mathbf{J}_h & \text{in } \Omega_I \\ \oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n & \forall\, n = 1, \ldots, g \\ \int_{e'} \mathbf{Z}_h \cdot d\mathbf{s} = 0 & \forall\, e' \in \mathcal{S}_h\,, \end{array} \tag{19}$$

where $\kappa_1, \ldots, \kappa_g$ are real numbers.

Back to source fields and loop fields

Clearly,

- a discrete source field $\mathbf{H}_{e,I}^{(h)}$ can be computed by solving (19), for $\mathbf{J}_h = \mathbf{J}_{e,I}^{(h)}$ and any choice of κ_n.

But also

- a basis $\mathbf{T}_\alpha$ of the first de Rham cohomology group of Ω_I can be determined by solving (19) with $\mathbf{J}_h = \mathbf{0}$ and $\kappa_n = m_{n,\alpha}$, for any choice of a non-singular matrix $M = (m_{n,\alpha})$.

A further important result is that

- if M is the matrix of the linking numbers of σ_n and $\widehat{\sigma}_\alpha$, we have an explicit formula for the degrees of freedom of $\mathbf{T}_\alpha$.

Webb–Forghani algorithm

When solving the problem $\mathbf{curl}\, \mathbf{Z}_h = \mathbf{J}_h$ we have to match two Raviart–Thomas elements, hence their fluxes across each face of $\mathcal{T}_h$ have to be the same.
Since the Stokes theorem assures that

$$\int_{e_1} \mathbf{Z}_h \cdot d\mathbf{s} + \int_{e_2} \mathbf{Z}_h \cdot d\mathbf{s} + \int_{e_2} \mathbf{Z}_h \cdot d\mathbf{s} = \int_f \mathbf{J}_h \cdot \mathbf{n}_f \,, \qquad (20)$$

where $\partial f = e_1 \cup e_2 \cup e_3$ and $\mathbf{n}_f$ is the unit normal vector on f (with consistent orientation), we deduce that the corresponding linear system has exactly three non-zero values for each row.

- The Webb–Forghani algorithm is a simple elimination procedure for solving the linear system (19), and it is quite efficient, as the computational costs is linearly dependent on the number of unknowns.

Webb–Forghani algorithm (cont'd)

Let us describe the Webb–Forghani algorithm:

1. set value 0 to the unknowns corresponding to an edge belonging to the spanning tree
2. take a face f for which at least one edge unknown has not yet been assigned
    1. if exactly one edge unknown is not determined, compute its value from the Stokes theorem as indicated before
    2. if two or three edge unknowns are not determined, pass to another face
3. if the iterations stop before the end, check if one of the "homological" equations $\oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n$ permits to restart
4. if the iterations stop before the end, use the explicit formula to restart.

An explicit formula for the loop fields

If $\mathbf{J}_h = \mathbf{0}$ we devise an explicit formula for the solution to (19).

The idea is the following: the Biot–Savart law gives the magnetic field generated by a unitary density current concentrated along the edge cycle $\widehat{\sigma}_\alpha$ (a generator of the first homology group of $\mathbb{R}^3 \setminus \Omega_I$) by means of the formula:

$$\widehat{\mathbf{H}}(\mathbf{x}) = \frac{1}{4\pi} \oint_{\widehat{\sigma}_\alpha} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y \quad , \quad \mathbf{x} \notin \widehat{\sigma}_\alpha \, .$$

Since the cycle $\widehat{\sigma}_\alpha$ can be chosen external to $\overline{\Omega_I}$, one has $\mathbf{curl}\, \widehat{\mathbf{H}} = \mathbf{0}$ in Ω_I. Moreover, on each cycle $\gamma \subset \overline{\Omega_I}$ that is linking the current passing in $\widehat{\sigma}_\alpha$ one finds $\oint_\gamma \widehat{\mathbf{H}} \cdot d\mathbf{s} \neq 0$, hence $\widehat{\mathbf{H}}$ is a loop field in Ω_I.

[There are cycles γ with the required property: for instance, one of the generators of the first homology group of $\overline{\Omega_I}$.]

An explicit formula for the loop fields (cont'd)

The Nédélec interpolant $\Pi^{N_h^1}\widehat{\mathbf{H}}$ is a finite element loop field in Ω_I (curl and line integrals are conserved). For each $e \in \mathcal{T}_h$, its degrees of freedom are given by

$$\widehat{q}_e = \frac{1}{4\pi}\int_e \left(\oint_{\widehat{\sigma}_\alpha} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}_y\right) \cdot d\mathbf{s}_x \,.$$

This resembles the formula for computing the linking number between $\widehat{\sigma}_\alpha$ and another disjoint cycle σ:

$$\ell_{\mathcal{K}}(\sigma, \widehat{\sigma}_\alpha) = \frac{1}{4\pi}\oint_\sigma \left(\oint_{\widehat{\sigma}_\alpha} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}_y\right) \cdot d\mathbf{s}_x \,.$$

- The linking number is an integer that represents the number of times that each cycle winds around the other.

An explicit formula for the loop fields (cont'd)

- Is it possible to reduce the definition of the finite element loop field to the computation of suitable linking numbers?

Consider the spanning tree $\mathcal{S}_h$, its root v_*, and define in the vertices of $\mathcal{T}_h \cap \overline{\Omega_I}$ the scalar function $\phi_h \in L_h^1$ as $\phi_h(v_*) = 0$ and

$$\phi_h(v_b) = \phi_h(v_a) + \widehat{q}_{[v_a,v_b]} \quad \forall e' = [v_a, v_b] \in \mathcal{S}_h \, .$$

The Nédélec finite element $\mathbf{Z}_h = \Pi^{N_h^1} \widehat{\mathbf{H}} - \mathbf{grad}\, \phi_h$ is a loop field in Ω_I, and its degrees of freedom are equal to 0 for all the edges e' of the spanning tree $\mathcal{S}_h$.

For each $e \in \mathcal{T}_h \cap \overline{\Omega_I}$, define now by D_e the edge cycle constituted by: the edges from the root of the spanning tree $\mathcal{S}_h$ to the first vertex v_e^- of e, the edge e, the edges from the second vertex v_e^+ of e to the root of the spanning tree $\mathcal{S}_h$. In particular, $D_{e'}$ is a trivial cycle if $e' \in \mathcal{S}_h$.

An explicit formula for the loop fields (cont'd)

When $e \notin \mathcal{S}_h$ the cycle D_e is constituted by edges all belonging to the spanning tree (except e): hence we have

$$
\begin{aligned}
&\tfrac{1}{4\pi} \oint_{D_e} \left(\oint_{\widehat{\sigma}_\alpha} \tfrac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x \\
&= \widehat{q}_e + \textstyle\sum_{e' \in D_e \cap \mathcal{S}_h} \widehat{q}_{e'} \\
&= \widehat{q}_e + \textstyle\sum_{e' \in D_e \cap \mathcal{S}_h} \left(\phi_h(v_{e'}^+) - \phi_h(v_{e'}^-) \right) \\
&= \widehat{q}_e + \left(\phi_h(v_e^-) - \phi_h(v_e^+) \right) = \int_e \mathbf{Z}_h \cdot d\mathbf{s} \,,
\end{aligned}
$$

and thus the degrees of freedom of $\mathbf{Z}_h$ are given by

$$\int_e \mathbf{Z}_h \cdot d\mathbf{s} = \ell_{\mathcal{K}}(D_e, \widehat{\sigma}_\alpha) \,.$$

In particular, the loop field $\mathbf{Z}_h$ thus defined satisfies problem (19) with $\kappa_n = m_{n,\alpha} = \ell_{\mathcal{K}}(\sigma_n, \widehat{\sigma}_\alpha)$, a non-singular matrix.

- Selecting $\alpha = 1, \ldots, g$ we have an explicit formula for a basis of the first de Rham cohomology group.

Geometries

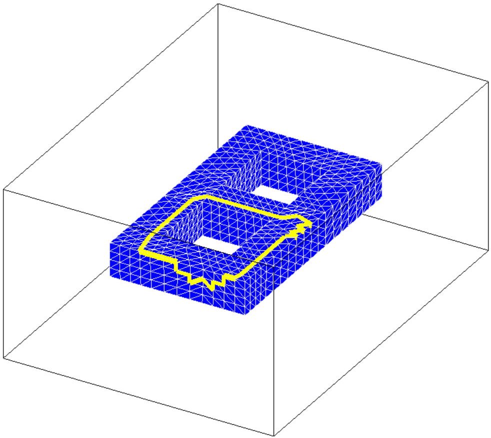

Figure: Case A: 2-torus (one homological cycle σ_n is drawn).

Geometries (cont'd)

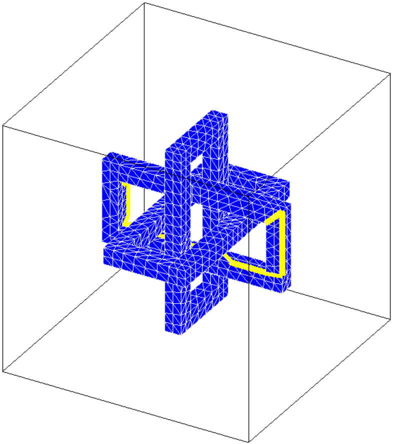

Figure: Case B: Borromean rings (one homological cycle σ_n is drawn).

Geometries (cont'd)

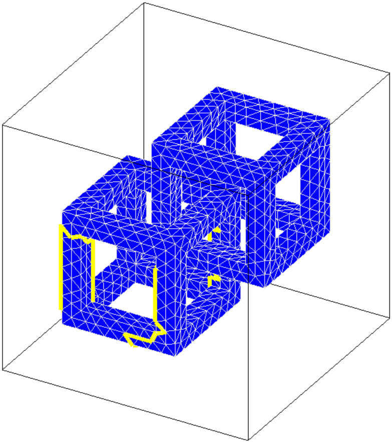

Figure: Case C: two-5-tori link (one homological cycle σ_n is drawn).

Geometries (cont'd)

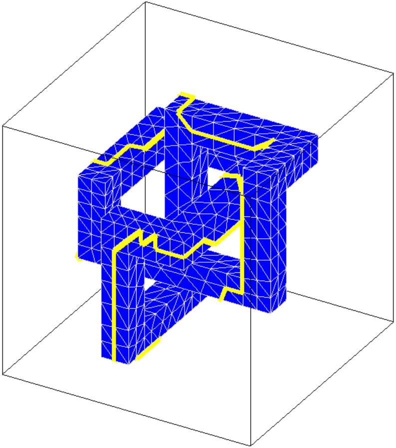

Figure: Case D: trefoil knot (one homological cycle σ_n is drawn).

Geometries (cont'd)

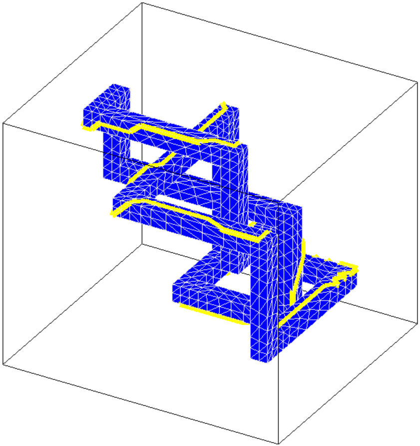

Figure: Case E: knot 4_1 (one homological cycle σ_n is drawn).

Geometries (cont'd)

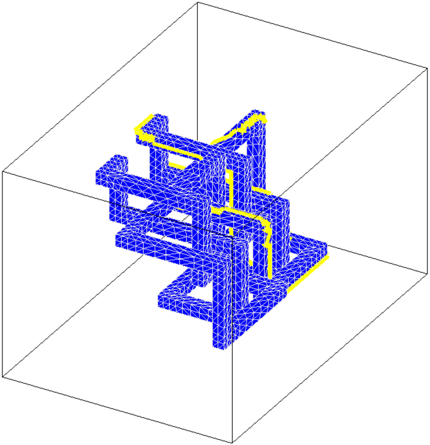

Figure: Case F: two-4_1-knots link (one homological cycle σ_n is drawn).

Computed loop fields

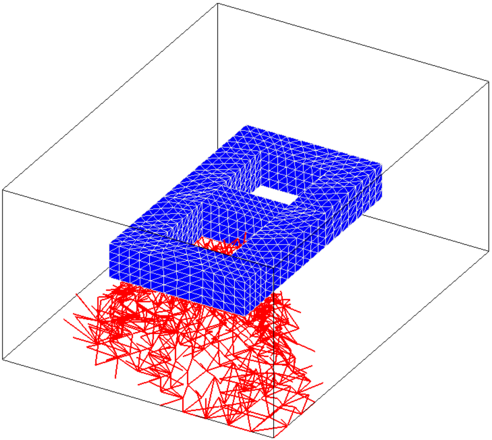

Figure: Support of a loop field. Case A: 2-torus.

Computed loop fields (cont'd)

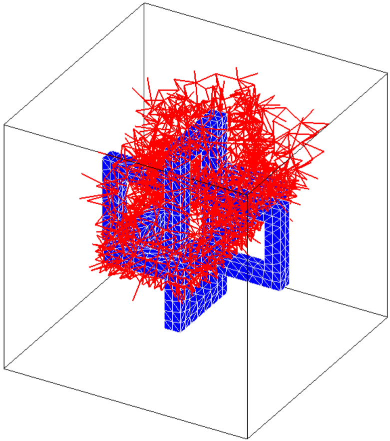

Figure: Support of a loop field. Case B: Borromean rings.

Computed loop fields (cont'd)

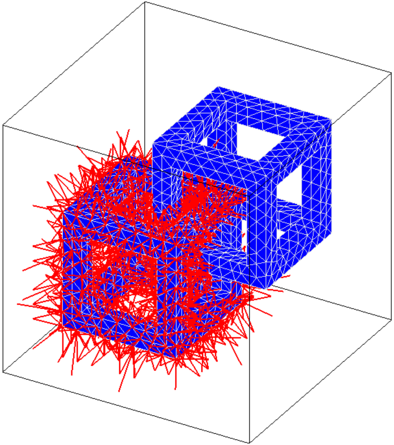

Figure: Support of a loop field. Case C: two-5-tori link.

Computed loop fields (cont'd)

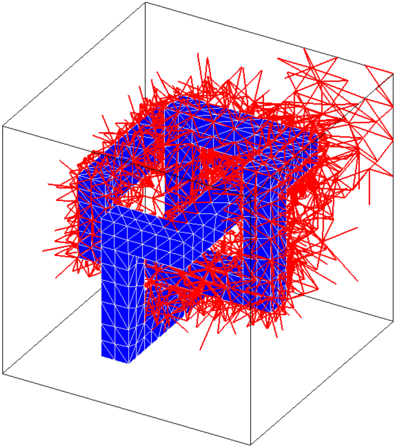

Figure: Support of a loop field. Case D: trefoil knot.

Computed loop fields (cont'd)

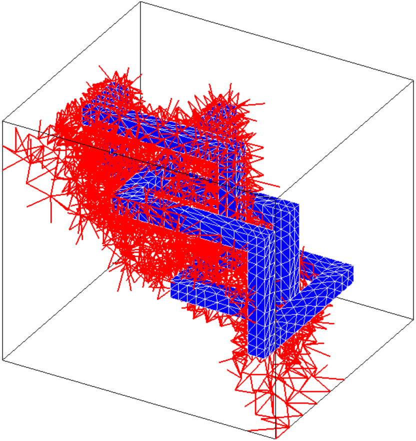

Figure: Support of a loop field. Case E: knot 4_1.

Computed loop fields (cont'd)

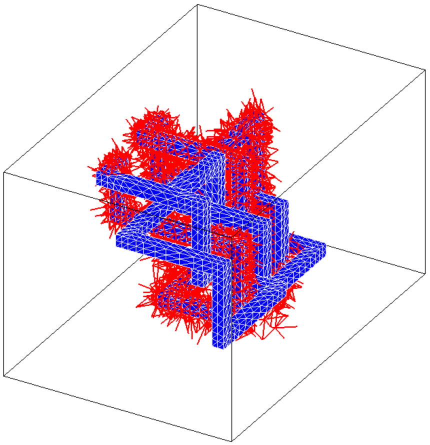

Figure: Support of a loop field. Case F: two-4_1-knots link.

Computed eddy current

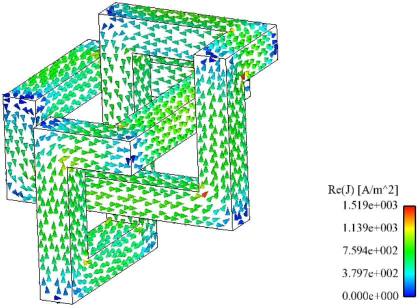

Figure: Eddy current in a trefoil knot.