POTENTIAL FORMULATIONS FOR TIME-HARMONIC EDDY-CURRENT PROBLEMS

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Eddy-current equations

Eddy-current equations are obtained from Maxwell equations by disregarding the displacement currents:

$$\begin{cases} \mathbf{curl}\,\mathcal{H} = \boldsymbol{\sigma}\mathcal{E} + \boldsymbol{\epsilon}\cancel{\dfrac{\partial \mathcal{E}}{\partial t}} & \text{(Ampère)} \\ \boldsymbol{\mu}\,\dfrac{\partial \mathcal{H}}{\partial t} + \mathbf{curl}\,\mathcal{E} = \mathbf{0} & \text{(Faraday).} \end{cases}$$

Here

- $\mathcal{E}$ and $\mathcal{H}$ are the electric and magnetic fields, respectively
- $\boldsymbol{\sigma}$ is the electric conductivity
- $\boldsymbol{\mu}$ is the magnetic permeability
- $\boldsymbol{\epsilon}$ is the electric permittivity.

Time-harmonic eddy-current equations

When interested in time-periodic phenomena, it is assumed that

$$\begin{aligned} \mathcal{E}(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{E}(\mathbf{x})\exp(i\omega t)] \\ \mathcal{H}(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{H}(\mathbf{x})\exp(i\omega t)] \ , \end{aligned}$$

where $\omega \neq 0$ is the assigned frequency, and one obtains

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{0} & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \,. \end{cases} \tag{1}$$

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Here Ω is a bounded domain in $\mathbf{R}^3$, composed by two parts: Ω_C, a conductor, and Ω_I, its complementary part, an insulator, where the conductivity $\boldsymbol{\sigma}$ is vanishing.

"Gauge" conditions

Problem:

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$$\operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}) = 0 \qquad \text{in } \Omega_I \, .$$

[Depending on the geometrical properties of Ω_I as well as on the boundary conditions on $\partial\Omega$, other "gauge" conditions for $\mathbf{E}$ in Ω_I can be necessary: here we will not enter this aspect.]

A coupled problem

Since the conductivity $\boldsymbol{\sigma}$ is vanishing in Ω_I and $\mathrm{div}(\boldsymbol{\epsilon}\mathbf{E}) = 0$ is only imposed in Ω_I, the eddy-current problem is a coupled problem between equations of different (though similar) type, the coupling taking place through the interface Γ between Ω_C and Ω_I:

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plus $\mathbf{E} \times \mathbf{n}_\Gamma$ and $\mathbf{H} \times \mathbf{n}_\Gamma$ continuous on Γ ($\mathbf{n}_\Gamma$ unit normal vector on Γ).

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plus $\mathbf{E} \times \mathbf{n}_\Gamma$ and $\mathbf{H} \times \mathbf{n}_\Gamma$ continuous on Γ ($\mathbf{n}_\Gamma$ unit normal vector on Γ).

Another kind of coupling will arise from the choice of the excitation term (up to now all the considered equations have vanishing right-hand side).

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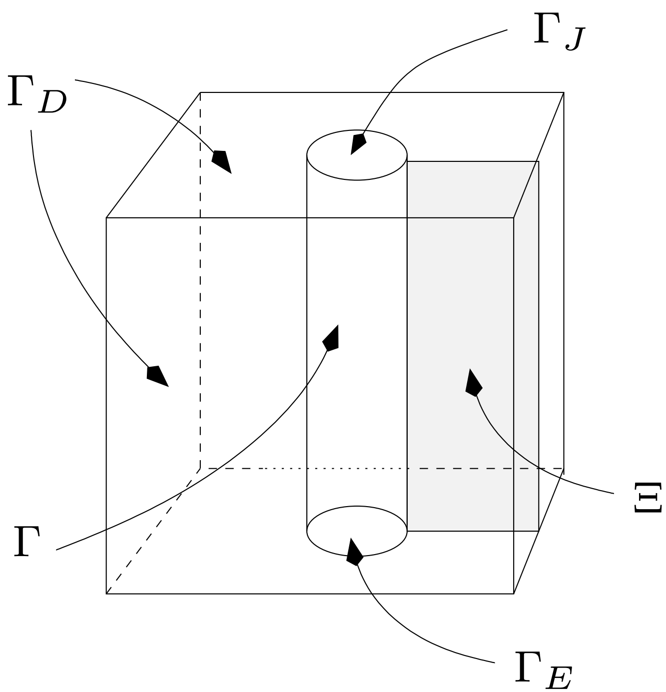
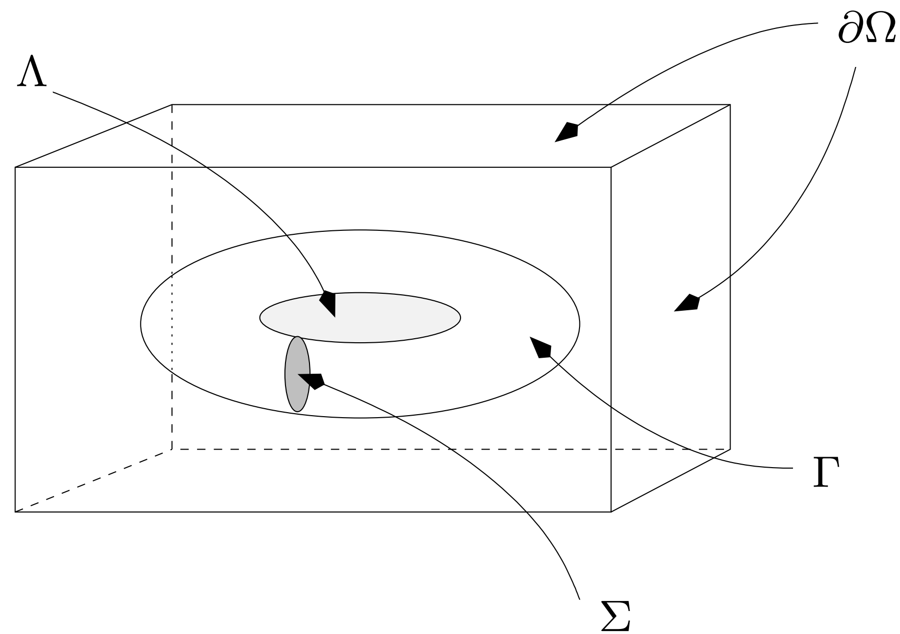
- Second geometrical case: internal conductor. The conductor Ω_C is strictly contained in Ω. For simplicity, Ω_C is a torus. Notation: $\partial\Omega_C = \Gamma$, $\partial\Omega_I = \partial\Omega \cup \Gamma$.

The geometrical configurations





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- Internal conductor

$$\begin{cases} \boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{3}$$

Coupling with circuits: voltage or current intensity excitation

We want to couple the eddy-current problem with a circuit problem, thus we have to consider, as the only external datum that determines the solution, a voltage V or a current intensity I^0.

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Question:

- how can we formulate the eddy-current problems when the excitation is given by a current intensity?

This can be a delicate point, as for the internal conductor case eddy-current problems have already a unique solution before a current intensity is assigned!

Poynting Theorem (energy balance)

In fact one has:

Uniqueness theorem. In the internal conductor case, for the solution of the eddy-current problem (1), (3) the magnetic field $\mathbf{H}$ in Ω and the electric field $\mathbf{E}_C$ in Ω_C are uniquely determined. [Adding the "gauge" conditions, also the electric field $\mathbf{E}_I$ in Ω_I is uniquely determined.]

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Proof. Multiply the Faraday equation by $\overline{\mathbf{H}}$, integrate in Ω and integrate by parts: it holds

$$\begin{aligned} 0 &= \int_\Omega \mathbf{curl}\, \mathbf{E} \cdot \overline{\mathbf{H}} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}} \\ &= \int_\Omega \mathbf{E} \cdot \mathbf{curl}\, \overline{\mathbf{H}} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}} \, . \end{aligned}$$

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Replacing $\mathbf{E}_C$ with $\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C$, and remembering that $\mathbf{curl}\,\mathbf{H}_I = \mathbf{0}$ in Ω_I, one has the Poynting Theorem (energy balance)

Poynting Theorem (energy balance) (cont'd)

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{H}_C} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{H}\cdot\overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n}\times\mathbf{E}\cdot\overline{\mathbf{H}} = \mathbf{0}\,.$$

Poynting Theorem (energy balance) (cont'd)

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \,\mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{H}_C} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}} = \mathbf{0}\,.$$

Since $\mathrm{div}_\tau(\mathbf{E} \times \mathbf{n}) = -i\omega\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$, one has

$$\mathbf{E} \times \mathbf{n} = \mathbf{grad}\, W \times \mathbf{n} \text{ on } \partial\Omega\ ,$$

and therefore

$$\begin{aligned} \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}} &= \int_{\partial\Omega} \overline{\mathbf{H}} \times \mathbf{n} \cdot \mathbf{grad}\, W = -\int_{\partial\Omega} \mathrm{div}(\overline{\mathbf{H}} \times \mathbf{n})\, W \\ &= -\int_{\partial\Omega} \mathbf{curl}\, \overline{\mathbf{H}} \cdot \mathbf{n}\, W = 0\ , \end{aligned}$$

as $\mathbf{curl}\, \mathbf{H}_I = \mathbf{0}$ in Ω_I and $\partial\Omega \subset \partial\Omega_I$. $\square$

Poynting Theorem (energy balance) (cont'd)

In the electric port case, instead, we can repeat the computation here above and find

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \, \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{H}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}} \\ = W_{|\Gamma_J} \int_{\Gamma_J} \mathbf{curl}\, \overline{\mathbf{H}_C} \cdot \mathbf{n} \ ,$$

where $W_{|\Gamma_J}$ is the (constant) value of the potential W on the electric port Γ_J (whereas $W_{|\Gamma_E} = 0$).

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where $W_{|\Gamma_J}$ is the (constant) value of the potential W on the electric port Γ_J (whereas $W_{|\Gamma_E} = 0$).

- In this case a degree of freedom is indeed still free (either the voltage $W_{|\Gamma_J}$, or else the current intensity $\int_{\Gamma_J} \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{n}$ in Ω_C).

A potential formulation

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Introduce the space of harmonic fields

$$\mathcal{H}(\Omega_I) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\,\mathbf{v}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\mu}\mathbf{v}_I) = 0, \\ \boldsymbol{\mu}\mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_I\} \,,$$

whose dimension is equal to 1, as in both geometrical configurations there is exactly one non-bounding cycle $\boldsymbol{\gamma}$ around Ω_C. We denote the basis function of $\mathcal{H}(\Omega_I)$ by $\boldsymbol{\rho}_I$, chosen in such a way that $\int_\gamma \boldsymbol{\rho}_I \cdot d\boldsymbol{\tau} = 1$.

We also introduce in Ω_C a function $\mathbf{R}_C$ that satisfies $\mathbf{R}_C \times \mathbf{n}_\Gamma = \boldsymbol{\rho}_I \times \mathbf{n}_\Gamma$ on Γ.

A potential formulation (cont'd)

This orthogonal decomposition result turns out to be useful: each vector function $\mathbf{v}_I$ with $\mathbf{curl}\,\mathbf{v}_I = \mathbf{0}$ can be written as

$$\mathbf{v}_I = \mathbf{grad}\,\phi_I + \alpha\boldsymbol{\rho}_I \, ,$$

where $\alpha = \int_{\partial\Gamma_J} \mathbf{v}_I \cdot d\boldsymbol{\tau}$.

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where $\alpha = \int_{\partial\Gamma_J} \mathbf{v}_I \cdot d\boldsymbol{\tau}$.

From the Stokes Theorem

$$I^0 = \int_{\Gamma_J} \mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{n}_C = \int_{\partial\Gamma_J} \mathbf{H}_C \cdot d\boldsymbol{\tau} = \int_{\partial\Gamma_J} \mathbf{H}_I \cdot d\boldsymbol{\tau} \,,$$

hence

$$\mathbf{H}_I = \mathbf{grad}\,\psi_I + I^0 \boldsymbol{\rho}_I \,.$$

A potential formulation (cont'd)

Then the magnetic field $\mathbf{H}$ can be written as

$$\mathbf{H} = \begin{cases} \mathbf{grad}\,\psi_I + I^0\,\boldsymbol{\rho}_I & \text{in } \Omega_I \\ \mathbf{T}_C + \mathbf{grad}\,\psi_C + I^0\,\mathbf{R}_C & \text{in } \Omega_C\ , \end{cases} \tag{4}$$

requiring on the interface Γ

$$\mathbf{T}_C \times \mathbf{n}_\Gamma = \mathbf{0}\ \ ,\ \ \psi_C = \psi_I\ . \tag{5}$$

[Instead, no conditions on $\mathbf{T}_C$, ψ_C and ψ_I are explicitly imposed on $\partial\Omega$.]

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Setting $\mathbf{E}_C := \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C$, the Ampère equation is satisfied in the whole Ω.

A potential formulation (cont'd)

Imposing the Faraday equation in Ω_C and the Gauss magnetic equation $\mathrm{div}(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω we find the following variational formulation [here $\sigma_* > 0$]:

$$\begin{aligned}
&\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{T}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} + \sigma_*^{-1}\int_{\Omega_C} \mathrm{div}\,\mathbf{T}_C\ \mathrm{div}\,\overline{\mathbf{w}_C} \\
&\quad + \int_{\Omega_C} i\omega\boldsymbol{\mu}_C(\mathbf{T}_C + \mathbf{grad}\,\psi_C)\cdot(\overline{\mathbf{w}_C} + \mathbf{grad}\,\overline{\phi_C}) \\
&\quad + \int_{\Omega_I} i\omega\boldsymbol{\mu}_I\,\mathbf{grad}\,\psi_I \cdot \mathbf{grad}\,\overline{\phi_I} \\
&= -I^0\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{R}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} \\
&\quad - I^0\int_{\Omega_C} i\omega\boldsymbol{\mu}_C\mathbf{R}_C\cdot(\overline{\mathbf{w}_C} + \mathbf{grad}\,\overline{\phi_C})
\end{aligned} \tag{6}$$

(for the reason of uniqueness, a penalization term for the divergence has been added; moreover, in the electric port case the condition $\mathbf{T}_C\cdot\mathbf{n} = 0$ on $\Gamma_E \cup \Gamma_J$ has been imposed).

Interpretation of the result

For both the electric port case and the internal conductor case it can be shown that this variational problem is well-posed, as the sesquilinear form at the left-hand side is coercive [Lax–Milgram lemma].

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However, an interpretation problem arises:

- for the electric port case a degree of freedom was available, therefore it has been possible to impose the current intensity I^0: this case is OK;
- for the internal conductor case we have proved an uniqueness result: thus what are we really solving when we also impose the current intensity I^0? What is the real effect of putting I^0 into the problem?

Don't forget the Faraday equation!

Since we have imposed the Faraday equation in Ω_C and the electric field $\mathbf{E}_I$ is determined by solving the Faraday equation in Ω_I (with $\mathbf{H}_I$ already known), it really seems that everything is all right...

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But let us see: the Faraday equation relates the flux of the magnetic induction through a surface with the line integral of the electric field on the boundary of that surface.

Since we know the magnetic field in the whole Ω, surfaces can stay everywhere in Ω; but at the moment we know the electric field only in Ω_C, therefore the boundary of the surface must stay in $\overline{\Omega_C}$.

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But the Faraday equation (in differential form) is satisfied in Ω_C, therefore for a surface contained in Ω_C everything is all right.

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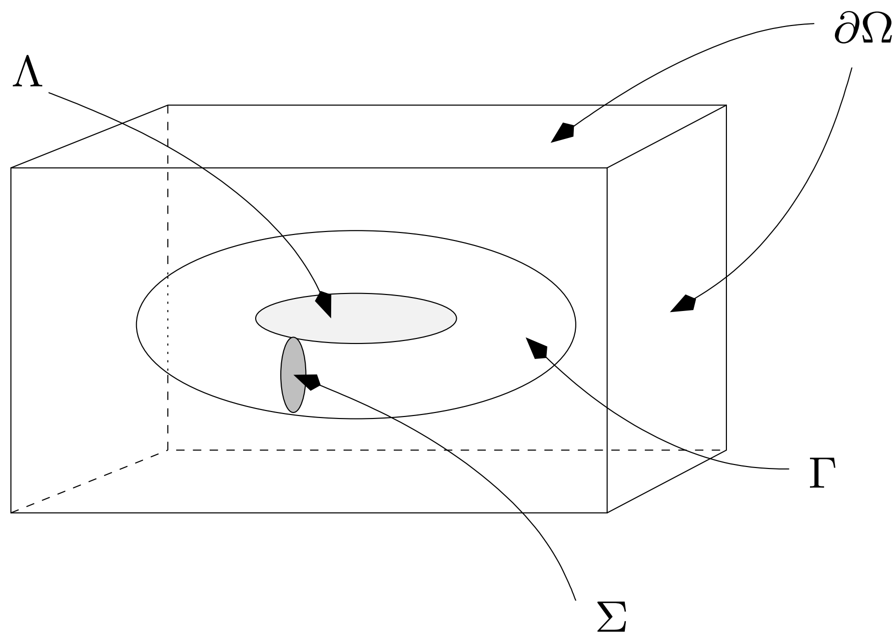
But the Faraday equation (in differential form) is satisfied in Ω_C, therefore for a surface contained in Ω_C everything is all right.

Thus we must verify if there are surfaces in Ω_I with boundary on Γ, and moreover such that this boundary is not the boundary of a surface in Ω_C [if this is not the case, the Divergence Theorem says that again everything is all right, as the magnetic induction is divergence free in Ω...].

Don't forget the Faraday equation! (cont'd)

- Claim: the Faraday equation is violated on the "cutting" surface Λ!



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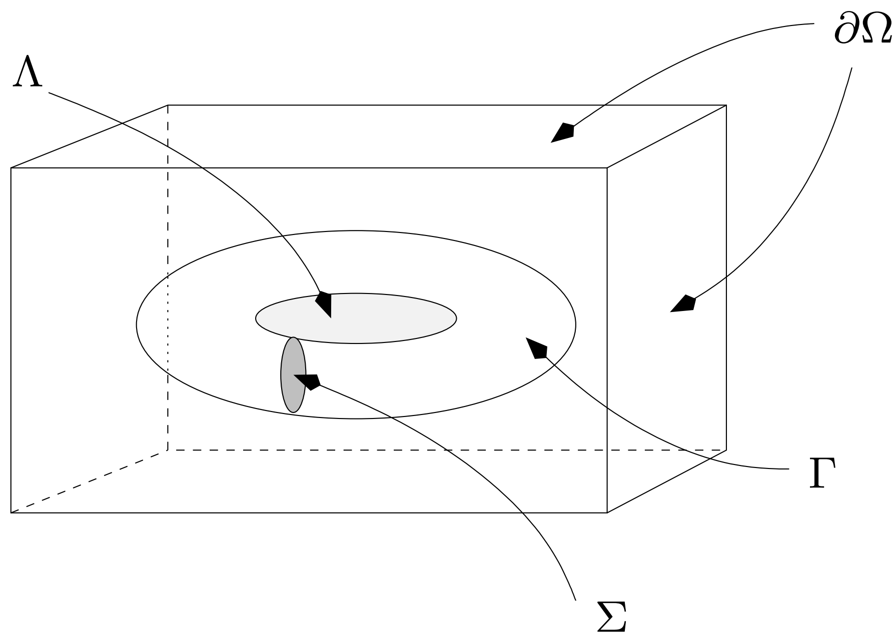

[Note. In the electric port case the cutting surface Σ has not the same properties: its boundary is not contained in Γ.]

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Let us see: the Faraday equation on Λ can be written as

$$\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I + \int_{\Gamma} (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}_I = 0 \ , \tag{7}$$

and this is not included in the variational formulation (6).

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and this is not included in the variational formulation (6). More precisely, we have

$$\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I = I^0 \int_{\Omega_I} i\omega \boldsymbol{\mu}_I \boldsymbol{\rho}_I \cdot \boldsymbol{\rho}_I$$

and

Don't forget the Faraday equation! (cont'd)

$$
\begin{aligned}
&\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I = \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{R}_C \\
&\quad = -\int_{\Omega_C} \mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{R}_C + \int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl}\,\mathbf{R}_C \\
&\quad = \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{R}_C + \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\mathbf{R}_C \\
&\quad = \int_{\Omega_C} i\omega \boldsymbol{\mu}_C (\mathbf{T}_C + \mathbf{grad}\,\psi_C) \cdot \mathbf{R}_C \\
&\qquad + I^0 \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{R}_C \cdot \mathbf{R}_C + I^0 \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{R}_C \cdot \mathbf{curl}\,\mathbf{R}_C \\
&\qquad + \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{T}_C \cdot \mathbf{curl}\,\mathbf{R}_C \,.
\end{aligned}
$$

Don't forget the Faraday equation! (cont'd)

$$
\begin{aligned}
&\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I = \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{R}_C \\
&\quad = - \int_{\Omega_C} \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{R}_C + \int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl}\, \mathbf{R}_C \\
&\quad = \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{R}_C + \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \mathbf{R}_C \\
&\quad = \int_{\Omega_C} i\omega \boldsymbol{\mu}_C (\mathbf{T}_C + \mathbf{grad}\, \psi_C) \cdot \mathbf{R}_C \\
&\qquad + I^0 \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{R}_C \cdot \mathbf{R}_C + I^0 \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{R}_C \cdot \mathbf{curl}\, \mathbf{R}_C \\
&\qquad + \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{T}_C \cdot \mathbf{curl}\, \mathbf{R}_C \,.
\end{aligned}
$$

Thus (7) is an additional equation for I^0 [and $\mathbf{T}_C$, ψ_C, ...]: I^0 cannot be a given quantity if we want to satisfy the Faraday equation on Λ.

Don't forget the Faraday equation! (cont'd)

$$
\begin{aligned}
&\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I = \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{R}_C \\
&\quad = -\int_{\Omega_C} \mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{R}_C + \int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl}\,\mathbf{R}_C \\
&\quad = \int_{\Omega_C} i\omega\boldsymbol{\mu}_C \mathbf{H}_C \cdot \mathbf{R}_C + \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\mathbf{R}_C \\
&\quad = \int_{\Omega_C} i\omega\boldsymbol{\mu}_C (\mathbf{T}_C + \mathbf{grad}\,\psi_C) \cdot \mathbf{R}_C \\
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&\qquad + \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{T}_C \cdot \mathbf{curl}\,\mathbf{R}_C \,.
\end{aligned}
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Thus (7) is an additional equation for I^0 [and $\mathbf{T}_C$, ψ_C, ...]: I^0 cannot be a given quantity if we want to satisfy the Faraday equation on Λ.

[Note. From another point of view: if (7) does not hold, a necessary compatibility condition on the data is not satisfied and we cannot find the electric field $\mathbf{E}_I$ such that $\mathbf{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}_I \mathbf{H}_I$ in Ω_I and $\mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C$ on Γ.]

Assigning the current density $\mathbf{J}_e$

Let us now consider a different situation: for the internal conductor case it is possible to solve the problem with an excitation given by an assigned current density $\mathbf{J}_e$ (for simplicity, supported in Ω_C). The problem reads:

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Let us now consider a different situation: for the internal conductor case it is possible to solve the problem with an excitation given by an assigned current density $\mathbf{J}_e$ (for simplicity, supported in Ω_C). The problem reads:

$$\begin{aligned}
&\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \,\mathbf{curl}\, \mathbf{T}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + \sigma_*^{-1} \int_{\Omega_C} \operatorname{div} \mathbf{T}_C \,\operatorname{div} \overline{\mathbf{w}_C} \\
&\quad + \int_{\Omega_C} i\omega\boldsymbol{\mu}(\mathbf{T}_C + \mathbf{grad}\,\psi_C) \cdot (\overline{\mathbf{w}_C} + \mathbf{grad}\,\overline{\phi_C}) \\
&\quad + \int_{\Omega_I} i\omega\boldsymbol{\mu}\,\mathbf{grad}\,\psi_I \cdot \mathbf{grad}\,\overline{\phi_I} + I^0 \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{R}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} \\
&\quad + I^0 \int_{\Omega_C} i\omega\boldsymbol{\mu}\mathbf{R}_C \cdot (\overline{\mathbf{w}_C} + \mathbf{grad}\,\overline{\phi_C}) = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{w}_C}
\end{aligned}$$

$$\begin{aligned}
&I^0 \int_{\Omega_I} i\omega\boldsymbol{\mu}_I\boldsymbol{\rho}_I \cdot \boldsymbol{\rho}_I + \int_{\Omega_C} i\omega\boldsymbol{\mu}_C(\mathbf{T}_C + \mathbf{grad}\,\psi_C) \cdot \mathbf{R}_C \\
&\quad + I^0 \int_{\Omega_C} i\omega\boldsymbol{\mu}_C\mathbf{R}_C \cdot \mathbf{R}_C + I^0 \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{R}_C \cdot \mathbf{curl}\,\mathbf{R}_C \\
&\quad + \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{T}_C \cdot \mathbf{curl}\,\mathbf{R}_C = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\mathbf{R}_C\,,
\end{aligned}$$

and also I^0 has to be determined.

Conclusion about solvability

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- Electric port case. The problem with a given current intensity is uniquely solvable. [The same is true for the problem with a given voltage.]

- Internal conductor case. The problem with a given current intensity is not solvable. [The same is true for the problem with a given voltage.] [The same is true for other boundary conditions, such as $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, or $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$.]

Conclusion about solvability

Summing up:

- Electric port case. The problem with a given current intensity is uniquely solvable. [The same is true for the problem with a given voltage.]
- Internal conductor case. The problem with a given current intensity is not solvable. [The same is true for the problem with a given voltage.] [The same is true for other boundary conditions, such as $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, or $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$.]
- Internal conductor case. Instead, the problem with a given current density $\mathbf{J}_e$ is uniquely solvable. [The same is true for other boundary conditions, such as $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, or $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$.]

Numerical approximation

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[For a more efficient implementation, it is possible to replace the functions $\boldsymbol{\rho}_I$ and $\mathbf{R}_C$ with two other functions that can be easily computed.]

Numerical approximation (cont'd)

However, when using vector nodal elements it is well-known that the convergence of the approximation scheme is not assured in non-convex domains. (This happens very often in real-life problems with electric ports.)

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- Therefore the $(\mathbf{T}_C, \psi)$-formulation has a limited range of application: its use is indicated for electric port problems or for problems in which excitation comes through an imposed current density $\mathbf{J}_e$, but only under the assumption that the conductor Ω_C is convex!

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- Therefore the $(\mathbf{T}_C, \psi)$-formulation has a limited range of application: its use is indicated for electric port problems or for problems in which excitation comes through an imposed current density $\mathbf{J}_e$, but only under the assumption that the conductor Ω_C is convex!

Since we do not want to end up with a somehow negative result, let us pass to...

A different formulation

A more efficient formulation for the electric port case is based on a different coupling: the one between the scalar magnetic potential ψ_I, so that $\mathbf{H}_I = \mathbf{grad}\,\psi_I + I^0 \boldsymbol{\rho}_I$, and the electric field $\mathbf{E_C}$, so that $\mathbf{H}_C = -(i\omega)^{-1} \boldsymbol{\mu}_C^{-1}\,\mathbf{curl}\,\mathbf{E}_C$ [see Alonso Rodríguez, Valli and Vázquez Hernández, Numer. Math., 2009].

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The potential ψ_I and $\mathbf{E}_C$ satisfy the Ampère equation in Ω_C

$$\begin{aligned} \int_{\Omega_C} \boldsymbol{\mu}_C^{-1}\,\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}_C} \\ -i\omega \int_\Gamma \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \mathbf{grad}\,\psi_I \\ = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C} + i\omega I^0 \int_\Gamma \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I\,, \end{aligned} \tag{8}$$

which also includes the no-jump condition for $\mathbf{H} \times \mathbf{n}$ on Γ,

A different formulation (cont'd)

and the Gauss magnetic equation in Ω_I

$$-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I\, \mathbf{grad}\, \psi_I \cdot \mathbf{grad}\, \overline{\varphi_I} = 0\,, \quad (9)$$

which also contains the no-jump condition for $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n}$ on Γ.

A different formulation (cont'd)

and the Gauss magnetic equation in Ω_I

$$-i\omega \int_{\Gamma} \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \,\mathbf{grad}\, \psi_I \cdot \mathbf{grad}\, \overline{\varphi_I} = 0\,, \quad (9)$$

which also contains the no-jump condition for $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n}$ on Γ.

This problem is well-posed

A different formulation (cont'd)

and the Gauss magnetic equation in Ω_I

$$-i\omega \int_{\Gamma} \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I\, \mathbf{grad}\, \psi_I \cdot \mathbf{grad}\, \overline{\varphi_I} = 0\,, \quad (9)$$

which also contains the no-jump condition for $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n}$ on Γ.

This problem is well-posed and can be approximated by using edge elements for $\mathbf{E}_C$ and scalar nodal elements for ψ_I, on meshes that do not need to match on the interface Γ.

A different formulation (cont'd)

and the Gauss magnetic equation in Ω_I

$$-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I\, \mathbf{grad}\, \psi_I \cdot \mathbf{grad}\, \overline{\varphi_I} = 0\,, \quad (9)$$

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This problem is well-posed and can be approximated by using edge elements for $\mathbf{E}_C$ and scalar nodal elements for ψ_I, on meshes that do not need to match on the interface Γ. Moreover, the convexity condition on the conductor Ω_C is not required.

A different formulation (cont'd)

and the Gauss magnetic equation in Ω_I

$$-i\omega \int_{\Gamma} \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \, \mathbf{grad}\, \psi_I \cdot \mathbf{grad}\, \overline{\varphi_I} = 0 \,, \quad (9)$$

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Finally, also the voltage excitation problem can be formulated in a similar way (in that case, the current intensity I^0 is a further unknown).