Finite element potentials

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Outline

The aim

Beyond a doubt, among the "stars" of vector calculus we have the operators

- **grad**
- **curl**
- div

Aim of this talk is to show how to determine in a constructive way the finite element solutions of $\mathbf{grad}\,\psi = \mathbf{H}$, $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$, $\mathrm{div}\,\mathbf{v} = G$.

First results

Determining the necessary and sufficient conditions for assuring that a function defined in a bounded domain $\Omega \subset \mathbb{R}^3$ is the gradient of a scalar potential, the curl of a vector potential or the divergence of a vector field is one of the most classical problem of vector analysis.

The answer is well-known, and shows an interesting interplay of differential calculus and topology (see, e.g., Cantarella et al. (2002)).

First results (cont'd)

- a vector field is the gradient of a scalar potential if and only if it is curl free and its line integral is vanishing on all the closed curves that give a basis of the first homology group of $\overline{\Omega}$;
- a vector field is the curl of a vector potential if and only if it is divergence free and its flux is vanishing across all the closed surfaces that give a basis of the second homology group of $\overline{\Omega}$, or, equivalently, across (all but one) the connected components of $\partial\Omega$;
- each scalar function is the divergence of a vector field [just take the gradient of the inverse of the Laplace operator...].

First results (cont'd)

However, a less clarified situation takes shape when we want to furnish an explicit and efficient procedure for constructing a finite element solution.

[Note: at this level the divergence case comes back on the table: in fact, the gradient of a (standard) finite element approximate solution of $\Delta\varphi = G$ has a distributional divergence which is not a function, and therefore this divergence cannot be equal to an assigned finite element.]

Tools

We suppose to have:

- for topological reasons
  - a basis σ_n, $n = 1, \ldots, g$, of the first homology group of $\overline{\Omega}$;
  - a basis $\widehat{\sigma}_n$, $n = 1, \ldots, g$, of the first homology group of $\mathbb{R}^3 \setminus \Omega$;
- for the efficiency of the solver
  - a spanning tree $\mathcal{S}_h$ of the graph given by the nodes and the edges of the mesh $\mathcal{T}_h$.

[Note: a suitable and easy way for constructing σ_n and $\widehat{\sigma}_n$ is presented in Hiptmair and Ostrowski (2002); the determination of a spanning tree is a standard procedure in graph theory.]

The grad problem

We want to solve $\mathbf{grad}\,\psi_h = \mathbf{H}_h$ in the finite element context.

The "right" finite elements are: ψ_h a nodal element, $\mathbf{H}_h$ an edge element.

More precisely, we know that:

- a Nédélec element of the lowest order is a vector field in $H(\mathbf{curl}\,;\Omega)$ that locally has the form $\mathbf{a}_K + \mathbf{b}_K \times \mathbf{x}$;
- a curl-free Nédélec element satisfies $\mathbf{b}_K = \mathbf{0}$ for each K (in fact, $\mathbf{curl}\,(\mathbf{a}_K + \mathbf{b}_K \times \mathbf{x}) = 2\,\mathbf{b}_K$);
- the gradient of a (globally continuous) piecewise-linear finite element is a vector field in $H(\mathbf{curl}\,;\Omega)$ that locally is constant (namely, a curl-free Nédélec element).

The solution of the grad problem

In other words, for solving $\mathbf{grad}\,\psi_h = \mathbf{H}_h$ we have to match two Nédélec edge elements of the lowest order, hence the line integral of $\mathbf{grad}\,\psi_h$ and $\mathbf{H}_h$ on each edge of the mesh $\mathcal{T}_h$ has to be the same.

The fundamental theorem of calculus says that

$$\psi_h(v_b) - \psi_h(v_a) = \int_e \mathbf{grad}\,\psi_h \cdot \boldsymbol{\tau} = \int_e \mathbf{H}_h \cdot \boldsymbol{\tau}$$

for an edge $e = [v_a, v_b]$. Hence the linear system associated to $\mathbf{grad}\,\psi_h = \mathbf{H}_h$ has exactly two non-zero values per row.

The solution of the grad problem (cont'd)

Starting from the root v_* of the spanning tree $\mathcal{S}_h$, where, for the sake of uniqueness, we impose $\psi_h(v_*) = 0$, we compute

$$\psi_h(\widehat{v}) = \psi_h(v_*) + \int_{e'} \mathbf{H}_h \cdot \boldsymbol{\tau} \tag{1}$$

for an edge $e' = [v_*, \widehat{v}] \in \mathcal{S}_h$; since $\mathcal{S}_h$ is a spanning tree, going on in this way we can visit all the nodes of $\mathcal{T}_h$.

The spanning tree is therefore a tool for selecting the rows for which, using the additional equation $\psi_h(v_*) = 0$, one can eliminate the unknowns one after the other.

We have thus found a nodal element ψ_h such that its gradient has line integral on all the edges of the spanning tree equal to that of $\mathbf{H}_h$.

The solution of the grad problem (cont'd)

What about the edges not belonging to the spanning tree?

For each node v_i, $v_i \neq v_*$ let us denote by C_{v_i} the set of edges in $\mathcal{S}_h$ joining v_* to v_i. Given an edge $e = [v_a, v_b]$ not belonging to $\mathcal{S}_h$, we define the cycle $D_e = C_{v_a} + e - C_{v_b}$.

Since $\mathbf{H}_h$ is a gradient (it is curl-free and its line integral on all the cycles σ_n vanishes), its line integral on D_e vanishes. Therefore we have

$$\begin{aligned} 0 &= \oint_{D_e} \mathbf{H}_h \cdot d\mathbf{s} = \psi_h(v_a) + \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} - \psi_h(v_b) \\ &= \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} - \int_e \mathbf{grad}\, \psi_h \cdot \boldsymbol{\tau}\,. \end{aligned}$$

The algorithm

The simple solution algorithm reads as follows. Let us denote by V the set of the nodes of the mesh $\mathcal{T}_h$, by R the set of the nodes where the value of ψ_h is already known, by P the set of edges $e \in \mathcal{S}_h$ with exactly one vertex, $v'(e)$, not in R, and by $E(v)$ the set of edges having v as a vertex.

Algorithm

1. $R = \{v_*\}$, $P = E(v_*) \cap \mathcal{S}_h$
2. *while* $R \neq V$
   1. *pick* $e \in P$
   2. *compute* $\psi_h(v'(e))$ *from (1)*
   3. *update* P: $P = [P \cup (E(v'(e)) \cap \mathcal{S}_h)] \setminus \{e\}$
   4. *update* R: $R = R \cup \{v'(e)\}$.

The curl problem

We want to solve $\mathbf{curl}\,\mathbf{A}_h = \mathbf{B}_h$ in the finite element context.

The "right" finite elements are: $\mathbf{A}_h$ an edge element, $\mathbf{B}_h$ a face element (namely, a Raviart–Thomas element).

More precisely, we know that:

- a Raviart–Thomas element of the lowest order is a vector field in $H(\mathrm{div};\Omega)$ that locally has the form $\mathbf{a}_K + b_K\mathbf{x}$;
- a divergence-free Raviart–Thomas element satisfies $b_K = 0$ for each K (in fact, $\mathrm{div}\,(\mathbf{a}_K + b_K\mathbf{x}) = 3\,b_K$);
- the curl of a (lowest order) Nédélec element is a vector field in $H(\mathrm{div};\Omega)$ that locally is constant (namely, a divergence-free Raviart–Thomas element).

The solution of the curl problem

In other words, for solving $\mathbf{curl}\,\mathbf{A}_h = \mathbf{B}_h$ we have to match two Raviart–Thomas face elements of the lowest order, hence the flux of $\mathbf{curl}\,\mathbf{A}_h$ and $\mathbf{B}_h$ on each face of the mesh $\mathcal{T}_h$ has to be the same.

Since the Stokes theorem assures that

$$\int_{e_1} \mathbf{A}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{A}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{A}_h \cdot \boldsymbol{\tau} = \int_f \mathbf{curl}\,\mathbf{A}_h \cdot \boldsymbol{\nu}_f = \int_f \mathbf{B}_h \cdot \boldsymbol{\nu}_f\,, \quad (2)$$

where $\partial f = e_1 \cup e_2 \cup e_3$ and $\boldsymbol{\nu}_f$ is the unit normal vector on f (with consistent orientation), we deduce that the linear system associated to $\mathbf{curl}\,\mathbf{A}_h = \mathbf{B}_h$ has exactly three non-zero values for each row.

With respect to the preceding case:

- need to work on the edges instead of on the nodes
- more important: three unknowns per row instead of two.

The solution of the curl problem (cont'd)

Having three unknows per row, in order to devise an efficient elimination algorithm it is useful to fix the value of other unknowns.

The best situation should occur when number of the new equations agrees with the dimension of the kernel of the curl operator.

Since this kernel is given by the gradients of nodal elements plus the basis of the first de Rham cohomology group of Ω, we see that its dimension is equal to $n_v - 1 + g$ (having denoted by n_v the number of the nodes in the mesh $\mathcal{T}_h$).

The solution of the curl problem (cont'd)

Having this in mind, we are led to the problem

$$\begin{aligned} &\mathbf{curl}\,\mathbf{A}_h = \mathbf{B}_h && \text{in } \Omega \\ &\oint_{\sigma_n} \mathbf{A}_h \cdot d\mathbf{s} = \rho_n && \forall\, n = 1, \ldots, g \\ &\int_{e'} \mathbf{A}_h \cdot \boldsymbol{\tau} = 0 && \forall\, e' \in \mathcal{S}_h\,, \end{aligned} \tag{3}$$

for suitable given constants ρ_n (made precise in the sequel).

[Note that the number of edges e' in $\mathcal{S}_h$ is $n_v - 1$; therefore $(3)_3$ can be seen as a "filter" for gradients. On the other hand, homology and cohomology are in duality, hence $(3)_2$ can be seen as a "filter" for cohomology fields.]

It is not difficult to prove that there exists a unique solution to (3).

Webb–Forghani algorithm

Webb and Forghani (1989) proposed this solution algorithm:

Algorithm

1. *set value 0 to the unknowns corresponding to an edge belonging to the spanning tree*
2. *take a face f for which at least one edge unknown has not yet been assigned*
   1. *if exactly one edge unknown is not determined, compute its value from the Stokes relation (2)*
   2. *if two or three edge unknowns are not determined, pass to another face*
3. *if the iterations stop, use* $\oint_{\sigma_n} \mathbf{A}_h \cdot d\mathbf{s} = \rho_n$ *to restart*

[In their case, step 3 was missing, as they considered the case of simple topology (namely, $g = 0$).]

Webb–Forghani algorithm (cont'd)

The Webb–Forghani algorithm is a simple elimination procedure for solving the linear system at hand, and it is quite efficient, as the computational costs is linearly dependent on the number of unknowns.

The weak point is that:

- it strongly depends on the choice of the spanning tree and it can stop without having determined all the edge unknowns (even in simple topological situations!)

(see Dłotko and Specogna (2010)).

An explicit formula for the vector potential

- Cure: devise an explicit formula for the solution to (3).

(We are able to do that if $\mathbf{B}_h \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$, a quite natural condition in the most interesting physical situations.)

The idea is the following. Define the Biot–Savart field

$$\mathbf{H}^{BS}(\mathbf{x}) = \frac{1}{4\pi} \int_\Omega \mathbf{B}_h(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y}\,,$$

and set $\rho_n = \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s}$ in (3).
One has $\mathbf{curl}\, \mathbf{H}^{BS} = \mathbf{B}_h$ in Ω (here the condition $\mathbf{B}_h \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$ has played a role). Hence the Nédélec interpolant $\Pi^{N_h}\mathbf{H}^{BS}$ satisfies $(3)_1$ and $(3)_2$.
To find the solution to (3), we can correct $\Pi^{N_h}\mathbf{H}^{BS}$ by a gradient, namely, construct the nodal element whose gradient has the same line integral of $\mathbf{H}^{BS}$ on the edges of the spanning tree $\mathcal{S}_h$.

An explicit formula for the vector potential (cont'd)

Being v_* the root of the spanning tree, in the nodes of the mesh define the finite element ϕ_h as $\phi_h(v_*) = 0$ and

$$\phi_h(v_b) = \phi_h(v_a) + \int_{e'} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} \quad \forall e' = [v_a, v_b] \in \mathbf{S}_h \,.$$

The Nédélec finite element $\mathbf{A}_h = \Pi^{N_h} \mathbf{H}^{BS} - \mathbf{grad}\, \phi_h$ is the solution to (3).

To express its degrees of freedom, we proceed as follows. For each edge $e \notin \mathcal{S}_h$, we define the cycle D_e as before (the edges from the root of the spanning tree to the first vertex of e, the edge e, the edges from the second vertex of e to the root of the spanning tree).

An explicit formula for the vector potential (cont'd)

The cycle D_e is constituted by edges all belonging to the spanning tree (except e): hence we have

$$\begin{aligned}
\int_e \mathbf{A}_h \cdot \boldsymbol{\tau} &= \int_e (\Pi^{N_h} \mathbf{H}^{BS} - \mathbf{grad}\, \phi_h) \cdot \boldsymbol{\tau} \\
&= \int_e \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - [\phi_h(v_b) - \phi_h(v_a)] \\
&= \int_e \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \left[\int_{C_{v_b}} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \int_{C_{v_a}} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} \right] \qquad (4) \\
&= \oint_{D_e} \mathbf{H}^{BS} \cdot d\mathbf{s} \\
&= \frac{1}{4\pi} \oint_{D_e} \left(\int_\Omega \mathbf{B}_h(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y} \right) \cdot d\mathbf{s}(\mathbf{x})\,.
\end{aligned}$$

Using (4), we can always restart the Webb–Forghani algorithm.

A basis of the first de Rham cohomology group

This algorithm permits to solve also the problem

$$\begin{aligned} &\mathbf{curl}\,\mathbf{A}_h = \mathbf{0} && \text{in } \Omega \\ &\oint_{\sigma_n} \mathbf{A}_h \cdot d\mathbf{s} = \kappa_n && \forall\, n = 1, \dots, g \\ &\int_{e'} \mathbf{A}_h \cdot \boldsymbol{\tau} = 0 && \forall\, e' \in \mathcal{S}_h\,, \end{aligned} \tag{5}$$

for any choice of the constants κ_n. In particular, taking κ_n equal to the entries $Q_{n,j}$ of a non-singular $g \times g$ square matrix Q, we find a basis $\mathbf{T}^{(j)}$ of the first de Rham cohomology group.
In fact, an explicit formula is available also in this case: the choice of $Q_{n,j} = \ell_K(\sigma_n, \widehat{\sigma}_j)$ gives

$$\int_e \mathbf{T}^{(j)} \cdot \boldsymbol{\tau} = \ell_K(D_e, \widehat{\sigma}_j)\,,$$

where ℓ_K is the linking number (and $\widehat{\sigma}_j$ has been chosen inside $\mathbb{R}^3 \setminus \overline{\Omega}$, namely, not intersecting $\partial\Omega$).

The linking number

The linking number between $\widehat{\sigma}_j$ and another disjoint cycle σ is given by:

$$\ell_{\mathcal{K}}(\sigma, \widehat{\sigma}_j) = \frac{1}{4\pi} \oint_{\sigma} \left(\oint_{\widehat{\sigma}_j} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x \, .$$

- The linking number (introduced by Gauss...) is an integer that represents the number of times that each cycle winds around the other.

The divergence problem

We want to solve $\operatorname{div} \mathbf{v}_h = G_h$ in the finite element context.

The "right" finite elements are: $\mathbf{v}_h$ a face element, G_h a nodal element.

More precisely, we know that:

- a nodal element of the lowest order is a function in $L^2(\Omega)$ that is locally constant;
- the divergence of a (lowest order) Raviart–Thomas finite element is a function in $L^2(\Omega)$ that is locally constant.

The solution of the divergence problem

In other words, for solving $\operatorname{div} \mathbf{v}_h = G_h$ we have to match two piecewise-constant elements, hence the integral of $\operatorname{div} \mathbf{v}_h$ and G_h on each element of the mesh $\mathcal{T}_h$ has to be the same.

The Gauss theorem says that

$$\begin{aligned} &\int_{f_1} \mathbf{v}_h \cdot \boldsymbol{\nu}_K + \int_{f_2} \mathbf{v}_h \cdot \boldsymbol{\nu}_K + \int_{f_3} \mathbf{v}_h \cdot \boldsymbol{\nu}_K + \int_{f_4} \mathbf{v}_h \cdot \boldsymbol{\nu}_K \\ &\qquad = \int_K \operatorname{div} \mathbf{v}_h = \int_K G_h \,, \end{aligned} \tag{6}$$

where $\partial K = f_1 \cup f_2 \cup f_3 \cup f_4$ and $\boldsymbol{\nu}_K$ is the unit outward normal vector on ∂K.

Hence the linear system associated to $\operatorname{div} \mathbf{v}_h = G_h$ has exactly four unknowns per row.

The solution of the divergence problem (cont'd)

In order to reduce the dimension of the system, we want to fix the value of some unknowns. Similarly to what done before we start by analyzing the dimension of the kernel of the divergence operator.

This kernel is given by the curl of the Nédélec elements plus the basis of the second de Rham cohomology group of Ω.

If we denote by $(\partial\Omega)_0, \ldots, (\partial\Omega)_p$ the connected components of $\partial\Omega$, we know that the dimension of the second de Rham cohomology group of Ω is equal to p.

The solution of the divergence problem (cont'd)

On the other hand, it is easy to check that the dimension of the space of the curl of the Nédélec elements is equal to the number of the edges minus the dimension of the kernel of the curl operator. Hence, it is equal to $n_e - n_v + 1 - g$.

By the Euler–Poincaré formula we have

$$n_v - n_e + n_f - n_t = 1 - g + p\,,$$

hence the dimension of the space of the curl can be rewritten as $n_f - n_t - p$.

In conclusion, besides the topological conditions

$$\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = c_r \quad , \quad r = 1, \ldots, p\,,$$

that are a filter for the cohomology fields, we could add $n_f - n_t - p$ equations.

A dual graph

To do that, let us note that an internal face connects two tetrahedra, while a boundary face connects a tetrahedron and a connected component of $\partial\Omega$.

We can therefore consider the following (connected) dual graph $\mathcal{G}_h$: the dual vertices are $W = T \cup \Sigma$, where the elements of T are the tetrahedra of the mesh and the elements of Σ are the $p+1$ connected components of $\partial\Omega$; the set of dual arcs is F, the set of the faces of the mesh.

A dual graph (cont'd)

The number of dual vertices is equal to $n_t + p + 1$, hence a spanning tree $\mathcal{M}_h$ of $\mathcal{G}_h$ has $n_t + p$ dual arcs (and consequently its cotree has $n_f - n_t - p$ dual arcs).

Therefore the linear system

$$\begin{aligned} &\operatorname{div} \mathbf{v}_h = G_h && \text{in } \Omega \\ &\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = c_r && \forall\, r = 1, \dots, p \\ &\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0 && \forall\, f \notin \mathcal{M}_h \end{aligned} \tag{7}$$

is a square linear system of n_f equations and unknowns.

Well-posedness of (7)

Now we show that it has a unique solution.

The procedure is constructive, similar in some sense to the elimination procedure used for the grad problem but now going along the dual spanning tree, starting from the leaves. (Let us recall that the leaves of a spanning tree $\mathcal{M}_h$ are the vertices of W that have only one arc of $\mathcal{M}_h$ incident to them.)

Remembering that we have imposed $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$ if $f \notin \mathcal{M}_h$, we can reduce the problem to the faces $f \in \mathcal{M}_h$.

Well-posedness of (7) (cont'd)

Given $w \in W$ (a tetrahedron or a connected component of $\partial\Omega$), let us set $\mathcal{F}(w) = \{f \in F \,:\, f \subset w\}$; the elements of this set are faces of the primal mesh, therefore dual arcs in the dual mesh.

The leaves of the spanning tree $\mathcal{M}_h$ are the vertices $w \in W$ such that $\mathcal{F}(w) \cap \mathcal{M}_h$ reduces to exactly one dual arc (namely, to a face).

Well-posedness of (7) (cont'd)

If w is a leave of $\mathcal{M}_h$ and $f(w)$ is the unique dual arc in $\mathcal{M}_h$ incident to w, we can easily compute the degree of freedom corresponding to $f(w)$, as we know that $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$ for all $f \notin \mathcal{M}_h$.

In fact we have

$$\int_{f(w)} \mathbf{v}_h \cdot \boldsymbol{\nu}_f = \begin{cases} \int_{\partial w} \mathbf{v}_h \cdot \boldsymbol{\nu}_f = \int_w G_h & \text{if } w \in T \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = c_r & \text{if } w = (\partial\Omega)_r\,, \\ & \quad r = 1, \dots, p \\ \int_{(\partial\Omega)_0} \mathbf{v}_h \cdot \boldsymbol{\nu} = \int_\Omega G_h - \sum_{r=1}^{p} c_r & \text{if } w = (\partial\Omega)_0\,, \end{cases}$$

having used the Gauss theorem in the first and the third line.

Well-posedness of (7) (cont'd)

Hence it is clear that, if $\mathbf{v}_h$ is a Raviart–Thomas element with $\operatorname{div} \mathbf{v}_h = 0$, $\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \boldsymbol{\nu} = 0$ for all $r = 1, \dots, p$, and $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$ for all $f \notin \mathcal{M}_h$, then $\int_{f(w)} \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$ for the faces $f(w)$ associated to all the leaves $w \in \mathcal{M}_h$.

We can iterate this argument: if we remove from the spanning tree $\mathcal{M}_h$ a leave and its corresponding incident arc, the remaining graph is still a tree.

Repeating the previous procedure, we can easily compute the degrees of freedom corresponding to the faces incident to the leaves of this new tree, finding that they are vanishing.

After a finite number of steps the remaining tree reduces to just on vertex, and the result is that $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$ for all $f \in F$. This proves that (7) has a unique solution.

The solution algorithm

We can also furnish an explicit way for computing the values of the degrees of freedom.

In fact, let us denote by U the set of vertices w and by N the set of arcs f of the reduced dual graph obtained at a step of the previous procedure (at the initial step, the set of arcs of the spanning tree is denoted by M).

Then:

- N is the set of faces where the degree of freedom $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f$ is still unknown;
- if w is a leave of (U, N), there exists exactly one face $f(w)$ incident to it and belonging to N.

The solution algorithm (cont'd)

Then, if w is a leave of (U, N), we have

$$\int_{f(w)} \mathbf{v}_h \cdot \boldsymbol{\nu}_f = A_w - \sum_{f \in \mathcal{F}(w) \setminus f(w)} \int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f \,, \tag{8}$$

where

$$A_w = \begin{cases} \int_w G_h & \text{if } w \in T \\ c_r & \text{if } w = (\partial\Omega)_r \,,\ r = 1, \ldots, p \\ \int_\Omega G_h - \sum_{r=1}^{p} c_r & \text{if } w = (\partial\Omega)_0 \,. \end{cases}$$

The solution algorithm (cont'd)

This can be rephrased as an elimination algorithm for the computation of $\mathbf{v}_h$.

Algorithm

1. $N = M$, $U = W$
2. *while* $N \neq \emptyset$
   1. *pick a leave* w *of the tree* (U, N)
   2. *compute* $\int_{f(w)} \mathbf{v}_h \cdot \boldsymbol{\nu}_f$ *from (8)*
   3. *update* U: $U = U \setminus \{w\}$
   4. *update* N: $N = N \setminus \{f(w)\}$.

Notice that at any step of the algorithm (U, N) is a tree, so while $N \neq \emptyset$ a leave w of (U, N) always exists.

A basis of the second de Rham cohomology group

- It is worth noting that the set of vector fields $\mathbf{W}^{(s)}$, $s = 1, \ldots, p$, solutions to problem (7) with $G_h = 0$ and $c_r = \delta_{r,s}$, $r = 1, \ldots, p$, is a basis of the second de Rham cohomology group of Ω.

Stability

A natural question is: how to construct stable finite element potentials?

(This means potentials whose natural norms can be estimated in terms of the norms of the data, uniformly with respect to the mesh size h.)

Stability (cont'd)

[However, let us note that, very often, the construction of finite element potentials is a preliminary step in the procedure aiming at solving a partial differential equation.

In this respect, the solution u_h will be written as $u_h = U_h + W_h$, W_h being the finite element potential and U_h the solution of an auxiliary problem in which W_h contributes at the right hand side.

In this situation, what is interesting is the stability of the solution u_h, and not that of W_h and U_h; in many cases, an unstable W_h produces an unstable U_h but a stable u_h.]

Stability (cont'd)

- Grad problem: project the scalar potential ψ_h on the space orthogonal to the constants, namely, take

$$\psi_h^\star = \psi_h - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \psi_h \,.$$

This is a stable scalar potential, satisfying

$$\|\psi_h^\star\|_{H^1(\Omega)} \leq K_G \|\mathbf{H}_h\|_{(L^2(\Omega))^3} \,.$$

Stability (cont'd)

- Curl problem: project the vector potential $\mathbf{A}_h$ (solution to (3) with $\rho_n = 0$) on the space orthogonal to the gradients of the piecewise-linear nodal finite elements, namely, take

$$\mathbf{A}_h^\star = \mathbf{A}_h - \mathbf{grad}\,\Phi_h\,,$$

where Φ_h is the solution to

$$\int_\Omega \mathbf{grad}\,\Phi_h \cdot \mathbf{grad}\,\eta_h = \int_\Omega \mathbf{A}_h \cdot \mathbf{grad}\,\eta_h \quad \forall\ \eta_h\,.$$

This is a stable vector potential, satisfying

$$\|\mathbf{A}_h^\star\|_{H(\mathbf{curl};\Omega)} \leq K_C \|\mathbf{B}_h\|_{(L^2(\Omega))^3}\,.$$

Stability (cont'd)

- Divergence problem: project the potential $\mathbf{v}_h$ (solution to (7) with $c_r = 0$) on the space orthogonal to the curls of the lowest order Nédélec finite elements, namely, take

$$\mathbf{v}_h^\star = \mathbf{v}_h - \mathbf{curl}\,\mathbf{q}_h\,,$$

where $\mathbf{q}_h$ is the solution to

$$\begin{aligned}
&\int_\Omega \mathbf{curl}\,\mathbf{q}_h \cdot \mathbf{curl}\,\mathbf{p}_h = \int_\Omega \mathbf{v}_h \cdot \mathbf{curl}\,\mathbf{p}_h && \forall\ \mathbf{p}_h \\
&\int_\Omega \mathbf{q}_h \cdot \mathbf{grad}\,\eta_h = 0 && \forall\ \eta_h \\
&\oint_{\sigma_n} \mathbf{q}_h \cdot d\mathbf{s} = 0 && \forall\ n = 1, \ldots, g\,.
\end{aligned}$$

This is a stable potential, satisfying

$$\|\mathbf{v}_h^\star\|_{H(\mathrm{div};\Omega)} \leq K_D \|G_h\|_{L^2(\Omega)}\,.$$

Stability (cont'd)

All these stability results are based on the fact that a Poincaré-like inequality is valid for functions orthogonal to constants, gradients and curls, respectively:

$$\|\psi_h^\star\|_{L^2(\Omega)} \leq K_G^\star \|\mathbf{grad}\, \psi_h^\star\|_{(L^2(\Omega))^3}$$
$$\|\mathbf{A}_h^\star\|_{(L^2(\Omega))^3} \leq K_C^\star \|\mathbf{curl}\, \mathbf{A}_h^\star\|_{(L^2(\Omega))^3}$$
$$\|\mathbf{v}_h^\star\|_{(L^2(\Omega))^3} \leq K_D^\star \|\mathrm{div}\, v_h^\star\|_{L^2(\Omega)}\,.$$

[To be precise, here the additional topological conditions $\oint_{\sigma_n} \mathbf{A}_h^\star \cdot d\mathbf{s} = 0$ for $n = 1, \ldots, g$, and $\int_{(\partial\Omega)_r} \mathbf{v}_h^\star \cdot \boldsymbol{\nu} = c_r$ for $r = 1, \ldots, p$, also play a role.]

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