SADDLE POINT PROBLEMS: STOKES AND EDDY CURRENTS

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Stokes problem

As it is well-known, the Stokes problem reads:

$$\begin{cases} -\mu\Delta\mathbf{v} + \mathbf{grad}\, p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

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where $\mathbf{v}$ and p are the velocity and the pressure of the fluid, respectively, $\mu > 0$ is the viscosity, and $\mathbf{f}$ is the applied force field. [Clearly, the pressure is defined up to an additive constant.]

In a weak form, in a constrained space, it can be written as

$$\begin{cases} \text{Find } \mathbf{v} \text{ with } \operatorname{div} \mathbf{v} = 0 \ \text{ in } \Omega\,,\ \mathbf{v} = \mathbf{0} \ \text{ on } \partial\Omega\ : \\ \qquad \mu \int_\Omega \nabla\mathbf{v} \cdot \nabla\mathbf{w} = \int_\Omega \mathbf{f} \cdot \mathbf{w} \\ \forall\ \mathbf{w} \text{ with } \operatorname{div} \mathbf{w} = 0 \ \text{ in } \Omega\,,\ \mathbf{w} = \mathbf{0} \ \text{ on } \partial\Omega\,. \end{cases} \qquad (2)$$

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It has the structure of a saddle-point problem:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix} , \tag{4}$$

Stokes problem (cont'd)

which is well-posed, for instance, if A is coercive in $\ker B$ and the following (necessary) inf–sup condition

$$\exists\, \beta > 0 \;:\; \sup_{\mathbf{v}} \frac{\int_\Omega p\, B\mathbf{v}}{\|\mathbf{v}\|} \geq \beta\, \|p\| \quad \forall\, p \tag{5}$$

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For Stokes problem, condition (5) is satisfied as for each $p \in L^2_0(\Omega)$ [the closed subspace of $L^2(\Omega)$ constituted by the functions with vanishing mean value] one can choose a velocity $\mathbf{v} \in H^1_0(\Omega)$ such that $\mathrm{div}\,\mathbf{v} = p$ and $\|\mathbf{v}\| \leq \beta^{-1}\|p\|$.

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Note that condition (5) is also saying that $\ker B^T = 0$.

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In this respect, the most known choices are

- P_2-P_0
  Crouzeix–Raviart (P_2 + bubble)-P_1
  for discontinuous pressure
- Taylor–Hood P_2-P_1
  Arnold–Brezzi–Fortin (P_1 + bubble)-P_1
  for continuous pressure.

Eddy current problems

Eddy current equations are obtained from Maxwell equations by disregarding the displacement currents:

$$\begin{cases} \mathbf{curl}\,\mathcal{H} = \boldsymbol{\sigma}\mathcal{E} + \mathcal{J}_e + \cancel{\boldsymbol{\epsilon}\frac{\partial \mathcal{E}}{\partial t}} & \text{(Ampère)} \\ \boldsymbol{\mu}\frac{\partial \mathcal{H}}{\partial t} + \mathbf{curl}\,\mathcal{E} = \mathbf{0} & \text{(Faraday).} \end{cases}$$

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Here

- $\mathcal{E}$ and $\mathcal{H}$ are the electric and magnetic fields, respectively, and $\mathcal{J}_e$ is the applied current density
- $\boldsymbol{\sigma}$ is the electric conductivity
- $\boldsymbol{\mu}$ is the magnetic permeability
- $\boldsymbol{\epsilon}$ is the electric permittivity.

Time-harmonic eddy current equations

When interested in time-periodic phenomena [alternating current], it is assumed that

$$\begin{aligned} \mathcal{J}_e(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{J}_e(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{E}(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{E}(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{H}(t, \mathbf{x}) &= \mathrm{Re}[\mathbf{H}(\mathbf{x}) \exp(i\omega t)] \ , \end{aligned}$$

where $\omega \neq 0$ is the assigned frequency, and one obtains

$$\begin{cases} \mathbf{curl}\, \mathbf{H} - \boldsymbol{\sigma} \mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\, \mathbf{E} + i\omega \boldsymbol{\mu} \mathbf{H} = \mathbf{0} & \text{in } \Omega \, . \end{cases}$$

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Here Ω is a bounded domain in $\mathbf{R}^3$, composed by two parts: Ω_C, an internal conductor, and Ω_I, its complementary part, an insulator, where the conductivity $\boldsymbol{\sigma}$ is vanishing.

Time-harmonic eddy current equations (cont'd)

Possible boundary conditions are

$$\mathbf{H} \times \mathbf{n} = \mathbf{0} \ \text{ on } \partial\Omega \quad \text{or} \quad \mathbf{E} \times \mathbf{n} = \mathbf{0} \ \text{ on } \partial\Omega \, .$$

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In the following we focus on the first one.

Rewriting the problem in terms of $\mathbf{H}$ only, in a weak form, in a constrained space we have [$\mathbf{H}_I := \mathbf{H}_{|\Omega_I}$ and so on...]:

$$\begin{cases} \text{Find } \mathbf{H} \text{ with } \mathbf{curl}\, \mathbf{H}_I = \mathbf{J}_{e,I} \text{ in } \Omega_I \, , \ \mathbf{H} \times \mathbf{n} = \mathbf{0} \ \text{ on } \partial\Omega \\ \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + i\omega \int_\Omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{w}} \\ = \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} \\ \forall \ \mathbf{w} \text{ with } \mathbf{curl}\, \mathbf{w}_I = \mathbf{0} \ \text{ in } \Omega_I \, , \ \mathbf{w} \times \mathbf{n} = \mathbf{0} \ \text{ on } \partial\Omega \, . \end{cases} \tag{6}$$

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Inserting a Lagrange multiplier we write

$$\left\{\begin{array}{l} \text{Find } \mathbf{H}, \mathbf{E}_I \text{ with } \mathbf{H}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \; : \\ \qquad \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} + i\omega \int_\Omega \boldsymbol{\mu}\mathbf{H}\cdot\overline{\mathbf{w}} \\ \qquad\qquad + \int_{\Omega_I} \mathbf{E}_I \cdot \mathbf{curl}\,\overline{\mathbf{w}_I} = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}\cdot\mathbf{curl}\,\overline{\mathbf{w}_C} \\ \qquad \int_{\Omega_I} \mathbf{curl}\,\mathbf{H}_I \cdot \overline{\mathbf{N}_I} = \int_{\Omega_I} \mathbf{J}_{e,I}\cdot\overline{\mathbf{N}_I} \\ \forall\, \mathbf{w}, \mathbf{N}_I \text{ with } \mathbf{w}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\,. \end{array}\right. \qquad (7)$$

Time-harmonic eddy current equations (cont'd)

This problem has the saddle-point structure $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$,

with A coercive in $\ker B$, but the inf–sup condition cannot be satisfied, as

$$\ker B^T = \{\mathbf{E}_I \mid \mathbf{curl}\,\mathbf{E}_I = \mathbf{0} \text{ in } \Omega_I, (\mathbf{E}_I \times \mathbf{n})_{|\partial\Omega_C} = \mathbf{0}\}$$

is different from $\mathbf{0}$ [indeed, is quite large...].

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- Here we see a difference between Stokes and eddy current problems: the duality $\mathrm{div}\,/\,\mathbf{grad}$ sounds different than the duality $\mathbf{curl}\,/\,\mathbf{curl}$.

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To restore well-posedness, we impose other conditions on the Lagrange multiplier $\mathbf{E}_I$: tipically, that $\mathrm{div}(\boldsymbol{\epsilon}_I \mathbf{E}_I) = 0$ in Ω_I and $\boldsymbol{\epsilon}_I \mathbf{E}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$ [in simple topology...].

Time-harmonic eddy current equations (cont'd)

This is done by using another Lagrange multiplier ϕ_I [which will turn out to be 0], obtaining

$$\left\{\begin{array}{l} \text{Find } \mathbf{H}, \mathbf{E}_I, \phi_I \\ \qquad \text{with } \mathbf{H}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \text{ and } \phi_I = 0 \text{ on } \partial\Omega_C \ : \\ \qquad \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}_C} + i\omega \int_\Omega \boldsymbol{\mu}\mathbf{H}\cdot\overline{\mathbf{w}} \\ \qquad\qquad + \int_{\Omega_I} \mathbf{E}_I \cdot \mathbf{curl}\,\overline{\mathbf{w}_I} = \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}\cdot\mathbf{curl}\,\overline{\mathbf{w}_C} \\ \qquad \int_{\Omega_I} \mathbf{curl}\,\mathbf{H}_I \cdot \overline{\mathbf{N}_I} \\ \qquad\qquad + \int_{\Omega_I} \boldsymbol{\epsilon}_I\, \mathbf{grad}\,\phi_I \cdot \overline{\mathbf{N}_I} = \int_{\Omega_I} \mathbf{J}_{e,I}\cdot\overline{\mathbf{N}_I} \\ \qquad \int_{\Omega_I} \boldsymbol{\epsilon}_I \mathbf{E}_I \cdot \mathbf{grad}\,\overline{\eta_I} = 0 \\ \forall\ \mathbf{w}, \mathbf{N}_I, \eta_I \\ \qquad \text{with } \mathbf{w}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \text{ and } \eta_I = 0 \text{ on } \partial\Omega_C\,. \end{array}\right. \tag{8}$$

Time-harmonic eddy current equations (cont'd)

The structure now is

$$\begin{pmatrix} A & B^T & 0 \\ B & 0 & C^T \\ 0 & C & 0 \end{pmatrix},$$

and the analysis can be done by following Chen, Du, Zou, SIAM J. Numer. Anal., 37 (2000), pp. 1542–1570.

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One needs:

- the inf–sup condition for C
- the inf–sup condition for B, but on the subspace $\ker C$
- the coerciveness of A, but on a space larger than $\ker B$

Chen-Du-Zou

The problem being set in the Hilbert spaces X, Q and M, define

$$Q^0 := \{\mathbf{N}_I \in Q \,|\, c(\mathbf{N}_I, \eta_I) = 0 \;\forall\; \eta_I \in M\} = \ker C$$

$$X^0 := \{\mathbf{w} \in X \,|\, b(\mathbf{w}, \mathbf{N}_I) = 0 \;\forall\; \mathbf{N}_I \in Q^0\} \supseteq \ker B\,.$$

The assumptions are

$$\exists\, \gamma > 0 \;:\; \sup_{\mathbf{N}_I} \frac{|c(\mathbf{N}_I, \eta_I)|}{\|\mathbf{N}_I\|} \geq \gamma\, \|\eta_I\| \quad \forall\; \eta_I \in M$$

$$\exists\, \beta > 0 \;:\; \sup_{\mathbf{w}} \frac{|b(\mathbf{w}, \mathbf{N}_I)|}{\|\mathbf{w}\|} \geq \beta\, \|\mathbf{N}_I\| \quad \forall\; \mathbf{N}_I \in Q^0$$

$$\exists\, \alpha > 0 \;:\; |a(\mathbf{w}, \mathbf{w})| \geq \alpha \|\mathbf{w}\|^2 \quad \forall\; \mathbf{w} \in X^0\,.$$

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The finite element approximation needs some attention, as we want to satisfy the coerciveness and the inf–sup conditions uniformly with respect to the mesh size.

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We use

- Nédélec lower order finite elements X_h^1 for $\mathbf{H}$ in Ω
- piecewise-constant finite elements Q_h for $\mathbf{E}_I$ in Ω_I
- Crouzeix–Raviart piecewise-linear discontinuous finite elements M_h for ϕ_I in Ω_I ,

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where the Crouzeix–Raviart finite elements are

$$M_h = \{\eta_{I,h} \in L^2(\Omega_I) \,|\, \eta_{I,h|K} \in P_1 \ \forall\ K \in \mathcal{T}_{I,h} \,, \ \eta_{I,h} \text{ is continuous at the centroid of each common face}\} \,.$$

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[see Monk, SIAM J. Numer. Anal., 28 (1991), pp. 1610–1634]

- the uniform Poincaré-like estimate

$$\|\mathbf{p}_{I,h}\|_{L^2(\Omega_I)} \leq C_0 \|\, \mathbf{curl}\, \mathbf{p}_{I,h}\|_{L^2(\Omega_I)}$$

for each $\mathbf{p}_{I,h} \in (V^0_{I,h})^\perp$, where

$$V^0_{I,h} := \{\mathbf{w}_{I,h} \in X^1_{I,h} \mid \mathbf{curl}\, \mathbf{w}_{I,h} = \mathbf{0} \text{ in } \Omega_I, (\mathbf{w}_{I,h} \times \mathbf{n})_{|\partial\Omega} = \mathbf{0}\} \,.$$

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Having proved these results, the error estimate is more or less standard.

Other saddle-point formulations

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- hybrid $\mathbf{E}_C/\mathbf{H}_I$ formulation: Ampère in Ω_C/Gauss in Ω_I + differential constraint $\mathbf{curl}\, \mathbf{H}_I = \mathbf{J}_{e,I}$ in Ω_I

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- $\mathbf{H}$ formulation

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