Divergence-free or curl-free finite elements for solving the curl-div system

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Outline

The aim

Aim of this talk is the analysis of the following three problems and of their mutual relations:

(a) finding finite element potentials, namely, solving by means of finite elements the problems $\mathbf{grad}\,\psi = \mathbf{H}$, $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$, $\mathrm{div}\,\mathbf{v} = G$;

(b) finding suitable basis functions for the spaces of curl-free or divergence-free finite elements;

(c) based on (a) and (b), devising simple finite element schemes for the solution of the curl–div system, which reads

$$\begin{cases} \mathbf{curl}\,\mathbf{u} = \mathbf{B} & \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} = G & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = \mathbf{a} \quad (\text{or } \mathbf{u} \cdot \mathbf{n} = b) & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

First results

Determining the necessary and sufficient conditions for assuring that a function defined in a bounded domain $\Omega \subset \mathbb{R}^3$ is the gradient of a scalar potential, or the curl of a vector potential, or the divergence of a vector field is one of the most classical problem of vector analysis.

The answer is well-known, and shows an interesting interplay of differential calculus and topology (see, e.g., Cantarella et al. (2002)).

First results (cont'd)

- a vector field is the gradient of a scalar potential if and only if it is curl free and its line integral is vanishing on all the closed curves that furnish a basis of the first homology group of $\overline{\Omega}$;
- a vector field is the curl of a vector potential if and only if it is divergence free and its flux is vanishing across all the closed surfaces that furnish a basis of the second homology group of $\overline{\Omega}$, or, equivalently, across (all but one) the connected components of $\partial\Omega$;
- each scalar function is the divergence of a vector field.

First results (cont'd)

However, this theoretical result only clarifies when the answer is positive, and does not say how to determine an explicit and efficient procedure for constructing finite element potentials.

Our approach is based on (simple) tools from algebraic topology and graph theory. We suppose to have:

- a basis σ_n, $n = 1, \dots, g$, of the first homology group of $\overline{\Omega}$;
- a basis $\widehat{\sigma}_n$, $n = 1, \dots, g$, of the first homology group of $\mathbb{R}^3 \setminus \Omega$;
- a spanning tree $\mathcal{S}_h$ of the graph given by the nodes and the edges of the mesh $\mathcal{T}_h$.

[Note: an easy way for constructing σ_n and $\widehat{\sigma}_n$ is presented in Hiptmair and Ostrowski (2002); the determination of a spanning tree is a standard procedure in graph theory.]

First results (cont'd)

Let us also introduce the finite element spaces we will use:

- the space L_h of continuous piecewise-linear elements, with dimension n_v, the number of vertices in $\mathcal{T}_h$;
- the space N_h of Nédélec edge elements of degree 1, with dimension n_e, the number of edges in $\mathcal{T}_h$;
- the space RT_h of Raviart-Thomas elements of degree 1, with dimension n_f, the number of faces in $\mathcal{T}_h$;
- the space PC_h of piecewise-constant elements, with dimension n_t, the number of tetrahedra in $\mathcal{T}_h$.

The grad problem

We want to solve $\mathbf{grad}\,\psi_h = \mathbf{H}_h$ in the finite element context. This is an easy problem, and the only reason for considering it is that it is useful for understanding better the procedures needed for the other two problems.

The "right" finite elements are: $\psi_h \in L_h$ a piecewise-linear nodal element, $\mathbf{H}_h \in N_h$ a lowest order Nédélec edge element, and we only have to impose that the line integral of $\mathbf{grad}\,\psi_h$ and $\mathbf{H}_h$ on each edge of the mesh $\mathcal{T}_h$ is the same.

The fundamental theorem of calculus says that

$$\psi_h(v_b) - \psi_h(v_a) = \int_e \mathbf{grad}\,\psi_h \cdot \boldsymbol{\tau} = \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} \qquad (2)$$

for an edge $e = [v_a, v_b]$. Hence the linear system associated to $\mathbf{grad}\,\psi_h = \mathbf{H}_h$ has exactly two non-zero values per row.

The grad problem (cont'd)

Starting from a root v_* of the spanning tree $\mathcal{S}_h$, where, for the sake of uniqueness, we impose $\psi_h(v_*) = 0$, for an edge $e' = [v_*, \widehat{v}] \in \mathcal{S}_h$ we compute

$$\psi_h(\widehat{v}) = \psi_h(v_*) + \int_{e'} \mathbf{H}_h \cdot \boldsymbol{\tau} \, ;$$

since $\mathcal{S}_h$ is a spanning tree, going on in this way we can visit all the nodes of $\mathcal{T}_h$.

The spanning tree is therefore a tool for selecting the rows for which, using the additional equation $\psi_h(v_*) = 0$, one can eliminate the unknowns one after the other.

We have thus found a nodal element ψ_h such that its gradient has line integral on all the edges of the spanning tree equal to that of $\mathbf{H}_h$.

The grad problem (cont'd)

What about the edges not belonging to the spanning tree?

For each node v_i, $v_i \neq v_*$, let us denote by C_{v_i} the set of edges in $\mathcal{S}_h$ joining v_* to v_i. Given an edge $e = [v_a, v_b]$ not belonging to $\mathcal{S}_h$, we define the cycle $D_e = C_{v_a} + e - C_{v_b}$.

Since $\mathbf{H}_h$ is a gradient (it is curl-free and its line integral on all the cycles σ_n vanishes), its line integral on D_e vanishes. Therefore we have

$$\begin{aligned} 0 &= \oint_{D_e} \mathbf{H}_h \cdot d\mathbf{s} = \psi_h(v_a) + \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} - \psi_h(v_b) \\ &= \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} - \int_e \mathbf{grad}\, \psi_h \cdot \boldsymbol{\tau} \,. \end{aligned}$$

The curl problem

We want to solve $\mathbf{curl}\,\mathbf{A}_h = \mathbf{B}_h$ in the finite element context.

The "right" finite elements are: $\mathbf{A}_h \in N_h$ a lowest order Nédélec edge element, $\mathbf{B}_h \in RT_h$ a lowest order Raviart–Thomas face element, and we only have to impose that the flux of $\mathbf{curl}\,\mathbf{A}_h$ and $\mathbf{B}_h$ on each face of the mesh $\mathcal{T}_h$ is the same.

The Stokes theorem assures that

$$\int_{e_1} \mathbf{A}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{A}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{A}_h \cdot \boldsymbol{\tau} = \int_f \mathbf{curl}\,\mathbf{A}_h \cdot \boldsymbol{\nu}_f = \int_f \mathbf{B}_h \cdot \boldsymbol{\nu}_f \,, \quad (3)$$

where $\partial f = e_1 \cup e_2 \cup e_3$, hence the linear system associated to $\mathbf{curl}\,\mathbf{A}_h = \mathbf{B}_h$ has exactly three non-zero values for each row.

The curl problem (cont'd)

With respect to the preceding case:

- three unknowns per row instead of two.

Therefore, in order to devise an efficient elimination algorithm, it is useful to fix the value of other unknowns.

The best situation should occur when the number of the new equations agrees with the dimension of the kernel of the curl operator.

Since this kernel is given by the gradients of nodal elements plus the space generated by the basis of the first de Rham cohomology group of Ω, we see that its dimension is equal to $n_v - 1 + g$.

The curl problem (cont'd)

Having this in mind, we are led to the problem

$$\begin{cases} \mathbf{curl}\, \mathbf{A}_h = \mathbf{B}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{A}_h \cdot ds = \rho_n & \forall\, n = 1, \dots, g \\ \int_{e'} \mathbf{A}_h \cdot \boldsymbol{\tau} = 0 & \forall\, e' \in \mathcal{S}_h\,, \end{cases} \tag{4}$$

for arbitrarily given constants ρ_n.

[Note that the number of edges e' in $\mathcal{S}_h$ is $n_v - 1$; therefore $(4)_3$ can be seen as a "filter" for gradients. On the other hand, homology and cohomology are in duality, hence $(4)_2$ can be seen as a "filter" for cohomology fields.]

It is not difficult to prove that there exists a unique solution to (4).

Webb–Forghani algorithm

Webb and Forghani (1989) proposed this solution algorithm:

Algorithm

1. *take a face f for which at least one edge unknown has not yet been assigned*
   1. *if exactly one edge unknown is not determined, compute its value from the Stokes relation (3)*
   2. *if two or three edge unknowns are not determined, pass to another face.*

This is a simple elimination procedure for solving the linear system at hand, and it is quite efficient, as the computational cost is linearly dependent on the number of unknowns.

The weak point is that:

- it can stop without having determined all the edge unknowns (even in simple topological situations!)

An explicit formula for the vector potential

- Cure: devise an explicit formula for the solution to (4).

(We are able to do that if $\mathbf{B}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$, a quite natural condition in the most interesting physical situations, and for a suitable choice of the constants ρ_n.)

The idea is the following. Define the Biot–Savart field

$$\mathbf{H}^{BS}(\mathbf{x}) = \frac{1}{4\pi} \int_\Omega \mathbf{B}_h(\mathbf{y}) \times \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{y} \,,$$

and set $\rho_n = \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s}$ in (4).
One has $\mathbf{curl}\, \mathbf{H}^{BS} = \mathbf{B}_h$ in Ω (here the condition $\mathbf{B}_h \cdot \mathbf{n} = 0$ on $\partial\Omega$ has played a role). Hence the Nédélec interpolant $\Pi^{N_h}\mathbf{H}^{BS}$ satisfies $(4)_1$ and $(4)_2$.

An explicit formula for the vector potential (cont'd)

To find the solution to (4), we can correct $\Pi^{N_h}\mathbf{H}^{BS}$ by a gradient, namely, construct the nodal element ϕ_h whose gradient has the same line integral of $\mathbf{H}^{BS}$ on the edges of the spanning tree $\mathcal{S}_h$.

The Nédélec finite element $\mathbf{A}_h = \Pi^{N_h}\mathbf{H}^{BS} - \mathbf{grad}\,\phi_h$ is the solution to (4).

To express its degrees of freedom, we proceed as follows. For each edge $e \notin \mathcal{S}_h$, we define the cycle D_e as before (the edges from the root of the spanning tree to the first vertex of e, the edge e, the edges from the second vertex of e to the root of the spanning tree).

An explicit formula for the vector potential (cont'd)

The cycle D_e is constituted by edges all belonging to the spanning tree (except e): hence we have

$$\begin{aligned}
\int_e \mathbf{A}_h \cdot \boldsymbol{\tau} &= \int_e (\Pi^{N_h} \mathbf{H}^{BS} - \mathbf{grad}\, \phi_h) \cdot \boldsymbol{\tau} \\
&= \int_e \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - [\phi_h(v_b) - \phi_h(v_a)] \\
&= \int_e \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \left[\int_{C_{v_b}} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \int_{C_{v_a}} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} \right] \\
&= \oint_{D_e} \mathbf{H}^{BS} \cdot d\mathbf{s} \\
&= \frac{1}{4\pi} \oint_{D_e} \left(\int_\Omega \mathbf{B}_h(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y} \right) \cdot d\mathbf{s}(\mathbf{x}) \,.
\end{aligned} \tag{5}$$

Using (5), we can always restart the Webb–Forghani algorithm.

A basis of the first de Rham cohomology group

This algorithm permits to solve also the problem

$$\begin{cases} \mathbf{curl}\,\mathbf{A}_h = \mathbf{0} & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{A}_h \cdot d\mathbf{s} = \kappa_n & \forall\, n = 1, \dots, g \\ \int_{e'} \mathbf{A}_h \cdot \boldsymbol{\tau} = 0 & \forall\, e' \in \mathcal{S}_h \,, \end{cases} \tag{6}$$

for any choice of the constants κ_n.

Taking κ_n equal to $\ell_{\mathcal{K}}(\sigma_n, \widehat{\sigma}_j)$, $j = 1, \dots, g$, ($\ell_{\mathcal{K}}$ denotes the linking number) we find a basis $\mathbf{T}^{(j)}$ of the first de Rham cohomology group, and we have also an explicit formula like (5) for expressing the degrees of freedom of each $\mathbf{T}^{(j)}$.

The div problem

We want to solve $\operatorname{div} \mathbf{v}_h = G_h$ in the finite element context.

The "right" finite elements are: $\mathbf{v}_h \in RT_h$ a lowest order Raviart–Thomas face element, $G_h \in PC_h$ a piecewise-constant nodal element, and we have only to impose that the integral of $\operatorname{div} \mathbf{v}_h$ and of G_h on each element of the mesh $\mathcal{T}_h$ is the same.

The Gauss theorem says that

$$\begin{aligned} \int_{f_1} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_2} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_3} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_4} \mathbf{v}_h \cdot \boldsymbol{\nu}_f \\ = \int_K \operatorname{div} \mathbf{v}_h = \int_K G_h \,, \end{aligned} \tag{7}$$

where $\partial K = f_1 \cup f_2 \cup f_3 \cup f_4$, hence the linear system associated to $\operatorname{div} \mathbf{v}_h = G_h$ has exactly four unknowns per row.

The div problem (cont'd)

In order to reduce the dimension of the system, we want to fix the value of some unknowns. Similarly to what done before we start by analyzing the dimension of the kernel of the divergence operator.

This kernel is given by the curl of the Nédélec elements plus the space generated by the basis of the second de Rham cohomology group of Ω.

If we denote by $(\partial\Omega)_0, \dots, (\partial\Omega)_p$ the connected components of $\partial\Omega$, we know that the dimension of the second de Rham cohomology group of Ω is equal to p.

The div problem (cont'd)

On the other hand, it is easy to check that the dimension of the space of the curl of the Nédélec elements is equal to the number of the edges minus the dimension of the kernel of the curl operator. Hence, it is equal to $n_e - n_v + 1 - g$.

By the Euler–Poincaré formula we have

$$n_v - n_e + n_f - n_t = 1 - g + p\,,$$

hence the dimension of the space of the curl of the Nédélec elements can be rewritten as $n_f - n_t - p$.

In conclusion, besides the topological conditions

$$\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = c_r \quad , \quad r = 1, \dots, p\,,$$

that are a filter for the cohomology fields, we could add $n_f - n_t - p$ equations.

A dual graph

To do that, let us note that an internal face connects two tetrahedra, while a boundary face connects a tetrahedron and a connected component of $\partial\Omega$.

We can therefore consider the following (connected) dual graph $\mathcal{G}_h$: the dual vertices are $W = T \cup \Sigma$, where the elements of T are the tetrahedra of the mesh and the elements of Σ are the $p+1$ connected components of $\partial\Omega$; the set of dual arcs is F, the set of the faces of the mesh.

A dual graph (cont'd)

The number of dual vertices is equal to $n_t + p + 1$, hence a spanning tree $\mathcal{M}_h$ of $\mathcal{G}_h$ has $n_t + p$ dual arcs (and consequently its cotree has $n_f - n_t - p$ dual arcs).

Therefore the linear system

$$\begin{cases} \operatorname{div} \mathbf{v}_h = G_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = c_r & \forall\, r = 1, \ldots, p \\ \int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0 & \forall\, f \notin \mathcal{M}_h \end{cases} \tag{8}$$

is a square linear system of n_f equations and unknowns.

It can be shown that this system has a unique solution.

Well-posedness of (8)

The procedure is constructive, similar in some sense to the elimination procedure used for the grad problem but now going along the dual spanning tree, starting from the leaves. (Let us recall that the leaves of a spanning tree $\mathcal{M}_h$ are the vertices of W that have only one arc of $\mathcal{M}_h$ incident to them.)

Remembering that we have imposed $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$ if $f \notin \mathcal{M}_h$, we can reduce the problem to the faces $f \in \mathcal{M}_h$. If w (a tetrahedron or a connected component) is a leaf of $\mathcal{M}_h$, then on it there is only one face $f(w)$ belonging to the spanning tree $\mathcal{M}_h$, therefore the value of the flux of $\mathbf{v}_h$ on $f(w)$ can be computed by the Gauss theorem (recall that we know that $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$ for all $f \notin \mathcal{M}_h$).

Well-posedness of (8) (cont'd)

We can iterate this argument: if we remove from the spanning tree $\mathcal{M}_h$ a leaf and its corresponding incident arc, the remaining graph is still a tree. After a finite number of steps the remaining tree reduces to just on vertex, and the result is that $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f$ is known for all $f \in F$.

A basis of the second de Rham cohomology group

It can be also noted that the solutions $\mathbf{W}^{(s)}$, $s = 1, \ldots, p$, of the problem

$$\begin{cases} \operatorname{div} \mathbf{v}_h = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = \delta_{r,s} & \forall\, r = 1, \ldots, p \\ \int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0 & \forall\, f \notin \mathcal{M}_h \end{cases} \tag{9}$$

furnish a basis of the second de Rham cohomology group of Ω.

Curl-free finite elements

The problem of describing in a suitable way curl-free finite elements is quite easy. In fact, it is straightforward to find a basis of the finite element space

$$\mathcal{V}_{0,h} = \{\mathbf{v}_h \in N_h \,|\, \mathbf{curl}\,\mathbf{v}_h = \mathbf{0} \text{ in } \Omega, \oint_{\sigma_n} \mathbf{v}_h \cdot d\mathbf{s} = 0 \ \forall\ n = 1, \ldots, g\}\,, \tag{10}$$

as this space is coincident with $\mathbf{grad}\, L_h$ (indeed, the conditions $\oint_{\sigma_n} \mathbf{v}_h \cdot d\mathbf{s} = 0$ are filtering all the curl-free vector fields that are not gradients, namely, the fields belonging to the first de Rham cohomology group).

Thus we have only to identify and eliminate the kernel of the gradient operator: the constants. In conclusion, a basis for $\mathcal{V}_{0,h}$ is simply given by $\mathbf{grad}\, \Phi_h^i$, $i = 1, \ldots, n_v - 1$, where Φ_h^i, $i = 1, \ldots, n_v$, are the standard nodal basis functions of L_h.

Divergence-free finite elements

A more complicated situation arises for divergence-free finite elements. In fact, we start considering the space

$$\mathcal{W}_{0,h} = \{\mathbf{v}_h \in RT_h \,|\, \mathrm{div}\,\mathbf{v}_h = 0 \text{ in } \Omega, \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = 0 \ \forall \ r = 1, \ldots, p\}, \tag{11}$$

and it is easy to check that $\mathcal{W}_{0,h} = \mathbf{curl}\, N_h$.

However, the problem is that

- the kernel of the curl operator is large: it contains all the gradients and the fields belonging to the first de Rham cohomology group, and has dimension equal to $n_v - 1 + g$.

Divergence-free finite elements (cont'd)

Thus we need:

- to devise a strategy for selecting $n_e - n_v + 1 - g$ edges in order that the associated edge element basis functions have linearly independent curls.

Results in this direction were obtained by Hecht (1981), Dubois (1990) and Scheichl (2002) for a simply-connected domain, and by Rapetti et al. (2003) for a κ-fold torus.

Here we present a general procedure for the determination of a set of basis functions of $\mathcal{W}_{0,h}$, together with an easy proof of its effectiveness.

Divergence-free finite elements (cont'd)

Let us assume for a while that Ω is simply-connected (therefore we have $g = 0$). Consider all the edges not belonging to the spanning tree $\mathcal{S}_h$, namely, belonging to the cotree $\mathcal{C}_h$; their number is $n_e - n_v + 1$. The result is:

- A basis of $\mathcal{W}_{0,h}$ is given by $\mathbf{curl}\,\mathbf{w}_h^j$, for the indices j such that the corresponding edges e_j belong to the cotree $\mathcal{C}_h$ (say, $j = 1, \ldots, n_e - n_v + 1$).

Divergence-free finite elements (cont'd)

The proof is as follows: from

$$0 = \sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{curl}\, \mathbf{w}_h^j = \mathbf{curl} \left(\sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j \right)$$

we can conclude that $\sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j$ is a gradient, say, $\mathbf{grad}\, \varphi_h$. For all $i = 2, \ldots, n_v$ consider the unique path C_{v_i}, composed by edges belonging to the spanning tree, connecting v_1 and v_i. One clearly has $\int_{C_{v_i}} \mathbf{w}_h^j \cdot \boldsymbol{\tau} = 0$, thus

$$\varphi_h(v_i) - \varphi_h(v_1) = \int_{C_{v_i}} \mathbf{grad}\, \varphi_h \cdot \boldsymbol{\tau} = \sum_{j=1}^{n_e - n_v + 1} \alpha_j \int_{C_{v_i}} \mathbf{w}_h^j \cdot \boldsymbol{\tau} = 0 \, .$$

Consequently $\sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j = \mathbf{grad}\, \varphi_h = \mathbf{0}$ and $\alpha_j = 0$ for all $j = 1, \ldots, n_e - n_v + 1$, since $\{\mathbf{w}_h^j\}_{j=1}^{n_e}$ is a basis of N_h.

Divergence-free finite elements (cont'd)

The general topological case needs the identification of g additional edges to discard: a possible option is to select one edge for each basis element σ_n of the first homology group of $\overline{\Omega}$, having constructed the spanning tree in such a way that all the other edges of σ_n belong to it. (For definiteness, suppose these edges are associated to the indices $j = 1, \dots, g$: the union of the spanning tree $\mathcal{S}_h$ and these additional g edges was called belted tree in Bossavit (1998), Rapetti et al. (2003).)

With this choice we have that the line integral of $\sum_{j=g+1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j$ over σ_n vanishes for each $n = 1, \dots, g$ (all the edges contained in σ_n belong to the belted tree, namely, they correspond to indices smaller than $g + 1$ or larger than $n_e - n_v + 1$). Therefore $\sum_{j=g+1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j$ is a gradient, and the argument develops as before.

First case: $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$

The problem at hand (slightly more general than the one previously presented) reads

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\eta}\mathbf{u}) = \mathbf{B} & \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} = G & \text{in } \Omega \\ (\boldsymbol{\eta}\mathbf{u}) \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega \\ \int_{(\partial\Omega)_r} \mathbf{u} \cdot \mathbf{n} = \alpha_r & \forall\ r = 1, \ldots, p\,, \end{cases} \tag{12}$$

where $\boldsymbol{\eta}$ is a symmetric matrix, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$, $\mathbf{B} \in (L^2(\Omega))^3$, $G \in L^2(\Omega)$, $\mathbf{a} \in H^{-1/2}(\mathrm{div}_\tau; \partial\Omega)$ [the space of tangential traces of vector fields belonging to $H(\mathbf{curl}\,; \Omega)$], $\boldsymbol{\alpha} \in \mathbb{R}^p$.

First case: $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The data satisfy the necessary conditions $\operatorname{div} \mathbf{B} = 0$ in Ω, $\int_\Omega \mathbf{B} \cdot \boldsymbol{\rho} + \int_{\partial\Omega} \mathbf{a} \cdot \boldsymbol{\rho} = 0$ for each $\boldsymbol{\rho} \in \mathcal{H}(m)$, and $\mathbf{B} \cdot \mathbf{n} = \operatorname{div}_\tau \mathbf{a}$ on $\partial\Omega$. Here $\mathcal{H}(m)$ is the space of Neumann harmonic fields, namely,

$$\mathcal{H}(m) = \{\boldsymbol{\rho} \in (L^2(\Omega))^3 \,|\, \mathbf{curl}\, \boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \operatorname{div} \boldsymbol{\rho} = 0 \text{ in } \Omega, \\ \boldsymbol{\rho} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$$

The first step of the procedure is to find a vector field $\mathbf{u}^\star \in (L^2(\Omega))^3$ satisfying

$$\begin{cases} \operatorname{div} \mathbf{u}^\star = G & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{u}^\star \cdot \mathbf{n} = \alpha_r & \forall\ r = 1, \dots, p\,. \end{cases} \tag{13}$$

First case: $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The vector field $\mathbf{W} = \mathbf{u} - \mathbf{u}^\star$ satisfies

$$\begin{cases} \mathbf{curl}\,(\eta \mathbf{W}) = \mathbf{B} - \mathbf{curl}\,(\eta \mathbf{u}^\star) & \text{in } \Omega \\ \operatorname{div} \mathbf{W} = 0 & \text{in } \Omega \\ (\eta \mathbf{W}) \times \mathbf{n} = \mathbf{a} - (\eta \mathbf{u}^\star) \times \mathbf{n} & \text{on } \partial\Omega \\ \int_{(\partial\Omega)_r} \mathbf{W} \cdot \mathbf{n} = 0 & \forall\ r = 1, \dots, p\,. \end{cases} \tag{14}$$

The second step is to devise a variational formulation of (14).

A variational formulation for the first case

Multiplying the first equation by a test function $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$, integrating in Ω and integrating by parts we find:

$$\begin{aligned}\int_\Omega \mathbf{B}\cdot\mathbf{v} &= \int_\Omega \mathbf{curl}\,[\eta(\mathbf{W}+\mathbf{u}^\star)]\cdot\mathbf{v} \\ &= \int_\Omega \eta(\mathbf{W}+\mathbf{u}^\star)\cdot\mathbf{curl}\,\mathbf{v} - \int_{\partial\Omega}[\eta(\mathbf{W}+\mathbf{u}^\star)\times\mathbf{n}]\cdot\mathbf{v} \\ &= \int_\Omega \eta\mathbf{W}\cdot\mathbf{curl}\,\mathbf{v} + \int_\Omega \eta\mathbf{u}^\star\cdot\mathbf{curl}\,\mathbf{v} - \int_{\partial\Omega}\mathbf{a}\cdot\mathbf{v}\,.\end{aligned}$$

Let us introduce the space

$$\begin{aligned}\mathcal{W}_0 = \{\mathbf{v}\in H(\mathrm{div}\,;\Omega)\,|\,\mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega,\\ \int_{(\partial\Omega)_r}\mathbf{v}\cdot\mathbf{n} = 0\ \forall\ r=1,\ldots,p\}\,.\end{aligned} \tag{15}$$

It is readily seen that $\mathcal{W}_0 = \mathbf{curl}\,[H(\mathbf{curl}\,;\Omega)]$.

A variational formulation for the first case (cont'd)

The vector field $\mathbf{W}$ is thus a solution to

$$\begin{aligned} &\mathbf{W} \in \mathcal{W}_0 \ : \\ &\quad \int_\Omega \eta \mathbf{W} \cdot \mathbf{curl}\, \mathbf{v} = \int_\Omega \mathbf{B} \cdot \mathbf{v} - \int_\Omega \eta \mathbf{u}^\star \cdot \mathbf{curl}\, \mathbf{v} + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v} \\ &\forall\ \mathbf{v} \in H(\mathbf{curl}\, ; \Omega) \, . \end{aligned} \tag{16}$$

More precisely, $\mathbf{W}$ is the unique solution of that problem: in fact, assuming $\mathbf{B} = \mathbf{u}^\star = \mathbf{a} = \mathbf{0}$, and taking $\mathbf{v}$ such that $\mathbf{curl}\, \mathbf{v} = \mathbf{W}$, it follows at once $\int_\Omega \eta \mathbf{W} \cdot \mathbf{W} = 0$, hence $\mathbf{W} = \mathbf{0}$.

Finite element approximation of the first case

The finite element approximation follows the same steps.

The first one is finding a finite element potential $\mathbf{u}_h^\star \in RT_h$ such that

$$\begin{cases} \operatorname{div} \mathbf{u}_h^\star = G_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{u}_h^\star \cdot \mathbf{n} = \alpha_r & \forall\ r = 1, \ldots, p\,, \end{cases} \tag{17}$$

where $G_h \in PC_h$ is the piecewise-constant interpolant $I_h^{PC} G$ of G. This can be done as in (8).

Finite element approximation of the first case (cont'd)

The second step concerns the numerical approximation of problem (16). The natural choice for the finite element space is clearly the space $\mathcal{W}_{0,h}$ introduced in (11). The finite element approximation of (16) reads as follows:

$$
\begin{aligned}
&\mathbf{W}_h \in \mathcal{W}_{0,h} \quad : \\
&\quad \int_\Omega \eta \mathbf{W}_h \cdot \mathbf{curl}\, \mathbf{v}_h \\
&\qquad\qquad = \int_\Omega \mathbf{B} \cdot \mathbf{v}_h - \int_\Omega \eta \mathbf{u}_h^\star \cdot \mathbf{curl}\, \mathbf{v}_h + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v}_h \\
&\forall\ \mathbf{v}_h \in N_h^\star \,,
\end{aligned}
\tag{18}
$$

where

$$
N_h^\star = \mathrm{span}\{\mathbf{w}_h^j\}_{j=g+1}^{n_e-n_v+1} \,. \tag{19}
$$

Finite element approximation of the first case (cont'd)

The corresponding algebraic problem is a square linear system of dimension $n_e - n_v + 1 - g$, and it is uniquely solvable. In fact, we note that $\mathcal{W}_{0,h} = \mathbf{curl}\, N_h^\star$, hence we can choose $\mathbf{v}_h^\star \in N_h^\star$ such that $\mathbf{curl}\, \mathbf{v}_h^\star = \mathbf{W}_h$; from (18) we find at once $\mathbf{W}_h = \mathbf{0}$, provided that $\mathbf{G} = \mathbf{u}_h^\star = \mathbf{a} = \mathbf{0}$.

The convergence of this finite element scheme is easily shown by standard arguments. For the sake of completeness, let us present the proof.

Convergence of the approximation for the first case

Theorem A. Let $\mathbf{W} \in \mathcal{W}_0$ and $\mathbf{W}_h \in \mathcal{W}_{0,h}$ be the solutions of problem (16) and (18), respectively. Set $\mathbf{u} = \mathbf{W} + \mathbf{u}^\star$ and $\mathbf{u}_h = \mathbf{W}_h + \mathbf{u}_h^\star$, where $\mathbf{u}^\star \in H(\mathrm{div}\,;\Omega)$ and $\mathbf{u}_h^\star \in RT_h$ are solutions to problem (13) and (17), respectively. Assume that $\mathbf{u}$ is regular enough, so that the interpolant $I_h^{RT}\mathbf{u}$ is defined. Then the following error estimate holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{div}\,;\Omega)} \leq c_0\big(\|\mathbf{u} - I_h^{RT}\mathbf{u}\|_{L^2(\Omega)} + \|G - I_h^{PC}G\|_{L^2(\Omega)}\big)\,. \quad (20)$$

Convergence of the approximation for the first case (cont'd)

Proof. Since $N_h^\star \subset H(\mathbf{curl}\,;\Omega)$, we can choose $\mathbf{v} = \mathbf{v}_h \in N_h^\star$ in (16). By subtracting (18) from (16) we end up with

$$\int_\Omega \eta[(\mathbf{W} + \mathbf{u}^\star) - (\mathbf{W}_h + \mathbf{u}_h^\star)] \cdot \mathbf{curl}\,\mathbf{v}_h = 0 \qquad \forall\ \mathbf{v}_h \in N_h^\star\,,$$

namely, the consistency property

$$\int_\Omega \eta(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{curl}\,\mathbf{v}_h = 0 \qquad \forall\ \mathbf{v}_h \in N_h^\star\,. \tag{21}$$

Then from $\mathcal{W}_{0,h} = \mathbf{curl}\, N_h^\star$ we can write $\mathbf{W}_h = \mathbf{curl}\,\mathbf{v}_h^\star$ for a suitable $\mathbf{v}_h^\star \in N_h^\star$, and using (21) we find

Convergence of the approximation for the first case (cont'd)

$$
\begin{aligned}
c_1 \|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(\Omega)} &\leq \int_\Omega \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{u}_h) \\
&= \int_\Omega \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{W}_h - \mathbf{u}_h^\star) \\
&= \int_\Omega \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{curl}\, \mathbf{v}_h^\star - \mathbf{u}_h^\star) \\
&= \int_\Omega \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{curl}\, \mathbf{v}_h - \mathbf{u}_h^\star) \\
&\leq c_2 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \|\mathbf{u} - \boldsymbol{\Phi}_h - \mathbf{u}_h^\star\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{\Phi}_h \in \mathcal{W}_{0,h}\,.
\end{aligned}
$$

We can choose $\boldsymbol{\Phi}_h = (I_h^{RT}\mathbf{u} - \mathbf{u}_h^\star) \in \mathcal{W}_{0,h}$; in fact, $\mathrm{div}\,(I_h^{RT}\mathbf{u}) = I_h^{PC}(\mathrm{div}\,\mathbf{u}) = I_h^{PC} G = G_h$ and $\int_{(\partial\Omega)_r} I_h^{RT}\mathbf{u} \cdot \mathbf{n} = \int_{(\partial\Omega)_r} \mathbf{u} \cdot \mathbf{n} = \alpha_r$ for each $r = 1, \ldots, p$. Then it follows at once $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq \frac{c_2}{c_1} \|\mathbf{u} - I_h^{RT}\mathbf{u}\|_{L^2(\Omega)}$.

Finally, $\mathrm{div}\,(\mathbf{u} - \mathbf{u}_h) = G - G_h = G - I_h^{PC} G$. □

Convergence of the approximation for the first case (cont'd)

Note that a sufficient condition for defining the interpolant of $\mathbf{u}$ is that $\mathbf{u} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$. This is satisfied if, e.g., $\boldsymbol{\eta}$ is a scalar Lipschitz function in $\overline{\Omega}$ and $\mathbf{a} \in (H^{\gamma}(\partial\Omega))^3$, $\gamma > 0$.

The algebraic problem for the first case

The solution $\mathbf{W}_h \in \mathcal{W}_{0,h}$ can be written in terms of the basis as $\mathbf{W}_h = \sum_{j=g+1}^{n_e-n_v+1} W_j \mathbf{curl}\,\mathbf{w}_{h,j}$. Hence the finite dimensional problem (18) can be rewritten as

$$\begin{aligned}\sum_{j=g+1}^{n_e-n_v+1} W_j \int_\Omega \eta\, \mathbf{curl}\,\mathbf{w}_{h,j} \cdot \mathbf{curl}\,\mathbf{w}_{h,m} \\ = \int_\Omega \mathbf{B} \cdot \mathbf{w}_{h,m} - \int_\Omega \eta\, \mathbf{u}_h^\star \cdot \mathbf{curl}\,\mathbf{w}_{h,m} + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{w}_{h,m}\,,\end{aligned} \tag{22}$$

for each $m = g+1, \dots, n_e - n_v + 1$.

The matrix $\mathbf{K}^\star$ with entries

$$K^\star_{mj} = \int_\Omega \eta\, \mathbf{curl}\,\mathbf{w}_{h,j} \cdot \mathbf{curl}\,\mathbf{w}_{h,m}$$

is clearly symmetric and positive definite, as the vector fields $\mathbf{curl}\,\mathbf{w}_{h,j}$ are linearly independent.

Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$

The problem at hand reads

$$\begin{cases} \mathbf{curl}\,\mathbf{u} = \mathbf{B} \\ \mathrm{div}\,(\boldsymbol{\mu}\mathbf{u}) = G \\ \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n} = b \\ \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n \quad \forall\ n = 1, \dots, g\,, \end{cases} \tag{23}$$

where $\boldsymbol{\mu}$ is a symmetric matrix, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$, $\mathbf{B} \in (L^2(\Omega))^3$, $G \in L^2(\Omega)$, $b \in H^{-1/2}(\partial\Omega)$, $\boldsymbol{\beta} \in \mathbb{R}^g$.

Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The data satisfy the necessary conditions $\operatorname{div} \mathbf{B} = 0$ in Ω, $\int_\Omega G = \int_{\partial\Omega} b$; moreover, in order that the line integral of $\mathbf{u}$ on σ_n has a meaning, we also assume that $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial\Omega$ (which is more restrictive than the necessary condition $\int_{(\partial\Omega)_r} \mathbf{B} \cdot \mathbf{n} = 0$ for each $r = 1, \ldots, p$).

The first step of the procedure is to find a vector field $\mathbf{u}^* \in (L^2(\Omega))^3$ satisfying

$$\begin{cases} \mathbf{curl}\, \mathbf{u}^* = \mathbf{B} & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}^* \cdot d\mathbf{s} = \beta_n & \forall\ n = 1, \ldots, g\,. \end{cases} \tag{24}$$

Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The vector field $\mathbf{V} = \mathbf{u} - \mathbf{u}^*$ satisfies

$$\begin{cases} \mathbf{curl}\, \mathbf{V} = \mathbf{0} & \text{in } \Omega \\ \operatorname{div}(\mu \mathbf{V}) = G - \operatorname{div}(\mu \mathbf{u}^*) & \text{in } \Omega \\ (\mu \mathbf{V}) \cdot \mathbf{n} = b - (\mu \mathbf{u}^*) \cdot \mathbf{n} & \text{on } \partial\Omega \\ \oint_{\sigma_n} \mathbf{V} \cdot d\mathbf{s} = 0 & \forall\ n = 1, \dots, g\,, \end{cases} \tag{25}$$

The second step is to devise a variational formulation of (25).

A variational formulation for the second case

Multiplying the second equation by a test function $\varphi \in H^1(\Omega)$, integrating in Ω and integrating by parts we find:

$$
\begin{aligned}
\int_\Omega G\,\varphi &= \int_\Omega \mathrm{div}\,[\boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)]\,\varphi \\
&= -\int_\Omega \boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)\cdot \mathbf{grad}\,\varphi + \int_{\partial\Omega}[\boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)\cdot \mathbf{n}]\,\varphi \\
&= -\int_\Omega \boldsymbol{\mu}\mathbf{V}\cdot \mathbf{grad}\,\varphi - \int_\Omega \boldsymbol{\mu}\mathbf{u}^*\cdot \mathbf{grad}\,\varphi + \int_{\partial\Omega} b\,\varphi\,.
\end{aligned}
$$

Let us introduce the space

$$
\begin{aligned}
\mathcal{V}_0 = \{\mathbf{v}\in H(\mathbf{curl}\,;\Omega)\,|\,\mathbf{curl}\,\mathbf{v} = \mathbf{0} \text{ in } \Omega, \\
\oint_{\sigma_n} \mathbf{v}\cdot d\mathbf{s} = 0\ \forall\ n = 1,\ldots,g\}\,.
\end{aligned}
\tag{26}
$$

Note that $\mathcal{V}_0 = \mathbf{grad}\,[H^1(\Omega)]$.

A variational formulation for the second case (cont'd)

The vector field $\mathbf{V}$ is thus a solution to

$$\mathbf{V} \in \mathcal{V}_0 \ : \quad \int_\Omega \boldsymbol{\mu}\mathbf{V} \cdot \mathbf{grad}\,\varphi = -\int_\Omega G\,\varphi - \int_\Omega \boldsymbol{\mu}\mathbf{u}^* \cdot \mathbf{grad}\,\varphi + \int_{\partial\Omega} b\,\varphi \quad \forall\ \varphi \in H^1(\Omega)\,. \tag{27}$$

It is easily seen that $\mathbf{V}$ is indeed the unique solution of that problem: in fact, assuming $G = b = 0$, $\mathbf{u}^* = \mathbf{0}$, and taking φ such that $\mathbf{grad}\,\varphi = \mathbf{V}$, it follows at once $\int_\Omega \boldsymbol{\mu}\mathbf{V} \cdot \mathbf{V} = 0$, hence $\mathbf{V} = \mathbf{0}$.

Finite element approximation of the second case

The finite element approximation follows the same steps.

The first one is finding a finite element potential $\mathbf{u}_h^* \in N_h$ such that

$$\begin{cases} \mathbf{curl}\, \mathbf{u}_h^* = \mathbf{B}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}_h^* \cdot d\mathbf{s} = \beta_n & \forall\ n = 1, \ldots, g\,, \end{cases} \tag{28}$$

where $\mathbf{B}_h \in RT_h$ is the Raviart–Thomas interpolant $I_h^{RT}\mathbf{B}$ of $\mathbf{B}$ (we therefore assume that $\mathbf{B}$ is so regular that its interpolant $I_h^{RT}\mathbf{B}$ is defined; for instance, as already recalled, it is enough to assume $\mathbf{B} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$). The construction of $\mathbf{u}_h^*$ can be done as in (4).

Finite element approximation of the second case (cont'd)

The second step is related to the numerical approximation of problem (27). The natural choice for the finite element space is clearly the space $\mathcal{V}_{0,h}$ introduced in (10). The finite element approximation of (27) reads as follows:

$$
\begin{aligned}
&\mathbf{V}_h \in \mathcal{V}_{0,h} \ : \\
&\quad \int_\Omega \boldsymbol{\mu} \mathbf{V}_h \cdot \mathbf{grad}\, \varphi_h = \\
&\qquad\qquad - \int_\Omega G\, \varphi_h - \int_\Omega \boldsymbol{\mu} \mathbf{u}_h^* \cdot \mathbf{grad}\, \varphi_h + \int_{\partial\Omega} b\, \varphi_h \\
&\forall\ \varphi_h \in L_h^* \,,
\end{aligned}
\tag{29}
$$

where

$$
L_h^* = \mathrm{span}\{\psi_{h,i}\}_{i=1}^{n_v - 1} = \{\varphi_h \in L_h \,|\, \varphi_h(v_{n_v}) = 0\}\,. \tag{30}
$$

Finite element approximation of the second case (cont'd)

The corresponding algebraic problem is a square linear system of dimension $n_v - 1$, and it is uniquely solvable. In fact, since $\mathcal{V}_{0,h} = \mathbf{grad}\, L_h^*$, we can choose $\varphi_h^* \in L_h^*$ such that $\mathbf{grad}\, \varphi_h^* = \mathbf{V}_h$; from (29) we find at once $\mathbf{V}_h = \mathbf{0}$, provided that $G = b = 0$, $\mathbf{u}_h^* = \mathbf{0}$.

The convergence of this finite element scheme is easily shown by standard arguments.

Convergence of the approximation for the second case

Theorem B. Let $\mathbf{V} \in \mathcal{V}_0$ and $\mathbf{V}_h \in \mathcal{V}_{0,h}$ be the solutions of problem (27) and (29), respectively. Set $\mathbf{u} = \mathbf{V} + \mathbf{u}^*$ and $\mathbf{u}_h = \mathbf{V}_h + \mathbf{u}_h^*$, where $\mathbf{u}^* \in H(\mathbf{curl}\,;\Omega)$ and $\mathbf{u}_h^* \in N_h$ are solutions to problem (24) and (28), respectively. Assume that $\mathbf{u}$ and $\mathbf{B}$ are regular enough, so that the interpolants $I_h^N \mathbf{u}$ and $I_h^{RT} \mathbf{B}$ are defined. Then the following error estimate holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathbf{curl}\,;\Omega)} \leq c_0\big(\|\mathbf{u} - I_h^N \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{B} - I_h^{RT} \mathbf{B}\|_{L^2(\Omega)}\big)\,. \quad (31)$$

Convergence of the approximation for the second case (cont'd)

Proof. Since $L_h^* \subset H^1(\Omega)$, we can choose $\varphi = \varphi_h \in L_h^*$ in (27). By subtracting (29) from (27) we end up with

$$\int_\Omega \boldsymbol{\mu}[(\mathbf{V} + \mathbf{u}^*) - (\mathbf{V}_h + \mathbf{u}_h^*)] \cdot \mathbf{grad}\, \varphi_h = 0 \qquad \forall\ \varphi_h \in L_h^*\,,$$

namely, the consistency property

$$\int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{grad}\, \varphi_h = 0 \qquad \forall\ \varphi_h \in L_h^*\,. \tag{32}$$

Then, since $\mathcal{V}_{0,h} = \mathbf{grad}\, L_h^*$ and thus $\mathbf{V}_h = \mathbf{grad}\, \varphi_h^*$ for a suitable $\varphi_h^* \in L_h^*$, from (32) we find

Convergence of the approximation for the second case (cont'd)

$$
\begin{aligned}
c_1\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(\Omega)} &\leq \int_\Omega \boldsymbol{\mu}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{u}_h)\\
&= \int_\Omega \boldsymbol{\mu}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{V}_h-\mathbf{u}^*_h)\\
&= \int_\Omega \boldsymbol{\mu}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{grad}\,\varphi^*_h-\mathbf{u}^*_h)\\
&= \int_\Omega \boldsymbol{\mu}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{grad}\,\varphi_h-\mathbf{u}^*_h)\\
&\leq c_2\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}\|\mathbf{u}-\boldsymbol{\Psi}_h-\mathbf{u}^*_h\|_{L^2(\Omega)} \quad \forall\ \boldsymbol{\Psi}_h\in\mathcal{V}_{0,h}\,.
\end{aligned}
$$

We can choose $\boldsymbol{\Psi}_h=(I^N_h\mathbf{u}-\mathbf{u}^*_h)\in\mathcal{V}_{0,h}$; in fact, $\mathbf{curl}\,(I^N_h\mathbf{u})=I^{RT}_h(\mathbf{curl}\,\mathbf{u})=I^{RT}_h\mathbf{B}=\mathbf{B}_h$ and $\oint_{\sigma_n} I^N_h\mathbf{u}\cdot d\mathbf{s}=\oint_{\sigma_n}\mathbf{u}\cdot d\mathbf{s}=\beta_n$ for each $n=1,\dots,g$. Then we find at once $\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}\leq\frac{c_2}{c_1}\|\mathbf{u}-I^N_h\mathbf{u}\|_{L^2(\Omega)}$.
Moreover, $\mathbf{curl}\,(\mathbf{u}-\mathbf{u}_h)=\mathbf{B}-\mathbf{B}_h=\mathbf{B}-I^{RT}_h\mathbf{B}$. □

Convergence of the approximation for the second case (cont'd)

Note that a sufficient condition for defining the interpolants of $\mathbf{u}$ and $\mathbf{B} = \mathbf{curl}\,\mathbf{u}$ is that they both belong to $(H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$. Thus one has to assume that $\mathbf{B} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$; moreover, $\mathbf{u}$ belongs to $(H^{\frac{1}{2}+\delta}(\Omega))^3$ if, for instance, $\boldsymbol{\mu}$ is a scalar Lipschitz function in $\overline{\Omega}$ and $b \in H^{\gamma}(\Omega)$, $\gamma > 0$.

The algebraic problem for the second case

The solution $\mathbf{V}_h \in \mathcal{V}_{0,h}$ is given by $\mathbf{V}_h = \sum_{i=1}^{n_v-1} V_i \mathbf{grad}\, \psi_{h,i}$. Hence the finite dimensional problem (29) can be rewritten as

$$\begin{aligned} &\sum_{i=1}^{n_v-1} V_i \int_\Omega \boldsymbol{\mu}\, \mathbf{grad}\, \psi_{h,i} \cdot \mathbf{grad}\, \psi_{h,l} \\ &\quad = -\int_\Omega G\, \psi_{h,l} - \int_\Omega \boldsymbol{\mu}\, \mathbf{u}_h^* \cdot \mathbf{grad}\, \psi_{h,l} + \int_{\partial\Omega} b\, \psi_{h,l}\,, \end{aligned} \tag{33}$$

for each $l = 1, \ldots, n_v - 1$.
The matrix $\mathbf{K}^*$ with entries

$$K^*_{li} = \int_\Omega \boldsymbol{\mu}\, \mathbf{grad}\, \varphi_{h,i} \cdot \mathbf{grad}\, \varphi_{h,l}$$

is clearly symmetric and positive definite.

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The linking number

The linking number between $\widehat{\sigma}_j$ and another disjoint cycle σ is given by:

$$\ell_{\mathcal{K}}(\sigma, \widehat{\sigma}_j) = \frac{1}{4\pi} \oint_{\sigma} \left(\oint_{\widehat{\sigma}_j} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x \, .$$

- The linking number (introduced by Gauss...) is an integer that represents the number of times that each cycle winds around the other.

An explicit formula is available also for determining the basis elements $\mathbf{T}^{(j)}$: we have

$$\int_e \mathbf{T}^{(j)} \cdot \boldsymbol{\tau} = \ell_{\mathcal{K}}(D_e, \widehat{\sigma}_j) \qquad (34)$$

(where $\widehat{\sigma}_j$ has been chosen inside $\mathbb{R}^3 \setminus \overline{\Omega}$, namely, not intersecting D_e).