“Hybrid” finite element approximation of time-harmonic eddy current problems

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Time-harmonic eddy current problems

Maxwell equations + time-harmonic structure (for a given frequency $\omega \neq 0$) + low frequency lead to:

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}) = 0 & \text{in } \Omega^I \,, \end{cases} \tag{1}$$

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where $\mathbf{E}$ and $\mathbf{H}$ are the electric field and magnetic field, respectively, $\boldsymbol{\epsilon}$ is the electric permittivity, $\boldsymbol{\mu}$ is the magnetic permeability, $\boldsymbol{\sigma}$ is the conductivity, and $\mathbf{J}_e$ is the applied density current. Moreover, the domain Ω is split into two parts, the conductor Ω_C and the insulator Ω_I, where $\boldsymbol{\sigma} = \mathbf{0}$.

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[We will consider (2).]

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In general topology ("handles", "holes") one has to add

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where:

- Γ_j are the connected components of the interface Γ between the insulator Ω^I and the conductor Ω_C
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[These are orthogonality conditions to the space of harmonic fields $\mathbf{w}$ with $\boldsymbol{\epsilon}\mathbf{w}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{w}\times\mathbf{n} = \mathbf{0}$ on Γ .]

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The unknowns of the problem are $\mathbf{H}_{|\Omega^C}$, $\mathbf{H}_{|\Omega^I}$, $\mathbf{E}_{|\Omega^C}$, $\mathbf{E}_{|\Omega^I}$, but one can consider reduced problems:

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$$\mathbf{E}_{|\Omega^C} = \boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_{|\Omega^C} - \mathbf{J}_{e|\Omega^C})$$

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General considerations (cont'd)

- $\mathbf{E}_{|\Omega^I}$ can be obtained from $\mathbf{H}_{|\Omega^I}$ and $\mathbf{E}_{|\Omega^C}$ by solving the electrostatic problem

$$\begin{cases} \mathbf{curl}\,\mathbf{E}_{|\Omega^I} = -i\omega\boldsymbol{\mu}\mathbf{H}_{|\Omega^I} & \text{in } \Omega^I \\ \operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}_{|\Omega^I}) = 0 & \text{in } \Omega^I \\ \mathbf{E}_{|\Omega^I} \times \mathbf{n} = \mathbf{E}_{|\Omega^C} \times \mathbf{n} & \text{on } \Gamma \\ \boldsymbol{\epsilon}\mathbf{E}_{|\Omega^I} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \int_{\Gamma_j} \boldsymbol{\epsilon}\mathbf{E}_{|\Omega^I} \cdot \mathbf{n} = 0 & \forall\, j = 1, \dots, p_\Gamma \\ \int_{\Sigma_k} \boldsymbol{\epsilon}\mathbf{E}_{|\Omega^I} \cdot \mathbf{n} = 0 & \forall\, k = 1, \dots, n_{\partial\Omega} \end{cases} \tag{4}$$

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Therefore, there are many possible formulations that can be proposed!

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- $(\mathbf{H}_{|\Omega^C}, \nabla\psi_{|\Omega^I})$ (scalar magnetic potential in Ω^I: need of cutting surfaces for non-bounding cycles on Γ)
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[similar to the approach via vector magnetic potential in Ω + scalar electric potential in Ω^C: nodal elements in Ω + piecewise linear elements in Ω^C]

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- $(\mathbf{H}_{|\Omega^C}, \mathbf{H}_{|\Omega^I})$ (saddle-point formulation: $\mathbf{E}_{|\Omega^I}$ is a Lagrange multiplier, and the divergence-free constraint reenters into play) [Alonso Rodríguez, Hiptmair, V., IMA J. Numer. Anal., 2004]
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- $(\mathbf{E}_{|\Omega^C}, \mathbf{E}_{|\Omega^I})$ (saddle-point formulation: a Lagrange multiplier is needed for the divergence-free constraint) [Alonso Rodríguez, V., ECCOMAS 2004]
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Let us see in more detail this last approach.

[The following results have been published on NMPDEs, 21 (2005), 742–763.]
The problem initially reads:

$$\left\{\begin{array}{l} \text{Find } (\mathbf{H}_{|\Omega^C}, \mathbf{E}_{|\Omega^I}) \in \boldsymbol{H}(\mathbf{curl};\Omega^C) \times \mathbf{Z}_I \text{ such that:} \\ \\ \int\limits_{\Omega^C} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H} \cdot \mathbf{curl}\,\overline{\mathbf{v}} + i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}}) + \int\limits_{\Gamma} \overline{\mathbf{v}} \times \mathbf{n} \cdot \mathbf{E} \\ \qquad\qquad = \int\limits_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_e \cdot \mathbf{curl}\,\overline{\mathbf{v}} \\ \int\limits_{\Gamma} \mathbf{H} \times \mathbf{n} \cdot \overline{\mathbf{z}} + i\omega^{-1} \int\limits_{\Omega^I} \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E} \cdot \mathbf{curl}\,\overline{\mathbf{z}} = \int\limits_{\Omega^I} \mathbf{J}_e \cdot \overline{\mathbf{z}} \\ \\ \forall\, (\mathbf{v}, \mathbf{z}) \in \boldsymbol{H}(\mathbf{curl};\Omega^C) \times \mathbf{Z}_I\ , \end{array}\right. \tag{5}$$

$(\mathbf{H}_{|\Omega^C}, \mathbf{E}_{|\Omega^I})$ saddle-point formulation

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where

$$\mathbf{Z}_I := \{\mathbf{z} \in \boldsymbol{H}(\mathbf{curl}; \Omega^I) \,|\, \mathbf{z} \text{ satisfies } (6)\} \,,$$

namely,

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[These are usually called gauge conditions.]

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We end up with the problem:

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$$
\begin{cases}
\text{Find } (\mathbf{H}_{|\Omega^C}, \mathbf{A}_{|\Omega^I}) \in \widetilde{\mathbf{X}}_C \times \widetilde{\mathbf{Z}}_I \text{ such that:} \\
\int\limits_{\Omega^C} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H} \cdot \mathbf{curl}\,\overline{\mathbf{v}} + i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}}) + \int\limits_{\Gamma} \overline{\mathbf{v}} \times \mathbf{n} \cdot \mathbf{A} \\
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where

$$\widetilde{\mathbf{X}}_C := \{\mathbf{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega^C) \,|\, \mathrm{div}_\Gamma(\mathbf{v} \times \mathbf{n}) = 0 \text{ on } \Gamma\}$$

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namely,

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- It is still a constrained problem!

$(\mathbf{H}_{|\Omega^C}, \mathbf{A}_{|\Omega^I})$ saddle-point formulation (cont.)

The final unconstrained problem is the following:
Find $(\mathbf{H}_{|\Omega^C}, \mathbf{A}_{|\Omega^I}, Q_{|\Gamma}, \phi_{|\Omega^I}) \in \mathcal{X}$ such that

$$\begin{cases} \int\limits_{\Omega^C} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}\cdot\mathbf{curl}\,\overline{\mathbf{v}} + i\omega\boldsymbol{\mu}\mathbf{H}\cdot\overline{\mathbf{v}}) + \int\limits_{\Gamma} \overline{\mathbf{v}}\times\mathbf{n}\cdot\mathbf{A} \\ \qquad\qquad - \int\limits_{\Gamma} \mathrm{div}_\Gamma(\overline{\mathbf{v}}\times\mathbf{n})\,Q = \int\limits_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_e\cdot\mathbf{curl}\,\overline{\mathbf{v}} \\ \int\limits_{\Gamma} \mathbf{H}\times\mathbf{n}\cdot\overline{\mathbf{z}} + i\omega^{-1}\int\limits_{\Omega^I} \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{A}\cdot\mathbf{curl}\,\overline{\mathbf{z}} \\ \qquad\qquad - \int\limits_{\Omega^I} \overline{\mathbf{z}}\cdot\mathbf{grad}\,\phi = \int\limits_{\Omega^I} \mathbf{J}_e\cdot\overline{\mathbf{z}} \\ \int\limits_{\Gamma} \mathrm{div}_\Gamma(\mathbf{H}\times\mathbf{n})\,\overline{P} = 0 \\ \int\limits_{\Omega^I} \mathbf{A}\cdot\mathbf{grad}\,\overline{\eta} = 0 \end{cases} \tag{9}$$

for all $(\mathbf{v}, \mathbf{z}, P, \eta) \in \mathcal{X}$,

$(\mathbf{H}_{|\Omega^C}, \mathbf{A}_{|\Omega^I})$ saddle-point formulation (cont.)

where

$$\mathcal{X} := \mathbf{X}_C^* \times \boldsymbol{H}(\mathbf{curl};\Omega^I) \times L^2(\Gamma) \times H^1(\Omega^I)$$

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- However, as we already remarked, the solution $\mathbf{A}_{|\Omega^I}$ is no more the electric field in Ω^I(the "physical" electric field does not satisfy (8)!). Indeed, it is a suitable magnetic vector potential: one can define the magnetic field in Ω^I by setting $\mathbf{curl}\,\mathbf{A}_{|\Omega^I} = -i\omega\boldsymbol{\mu}\mathbf{H}_{|\Omega^I}$, and then finding the electric field in Ω^I by solving the electrostatic problem (4).

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- Nédélec curl-conforming edge elements of the lowest order for $\mathbf{H}_{|\Omega^C}$
- Nédélec curl-conforming edge elements of the lowest order for $\mathbf{A}_{|\Omega^I}$
- piecewise constant elements for $Q_{|\Gamma}$
- piecewise linear continuous elements for $\phi_{|\Omega^I}$

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All the stability constants turn out to be independent on the mesh size h: the quasi-optimality of the algorithm then follows.