The electrostatics problem with a dipole source: theoretical results and numerical approximation

Alberto Valli

Department of Mathematics, University of Trento, Italy

In collaboration with:

Ana Alonso Rodríguez, University of Trento, Italy

Jessika Camaño, University of Concepcion, Chile

Rodolfo Rodríguez, University of Concepcion, Chile.

Outline

EEG and MEG

Electroencephalography (EEG) and magnetoencephalography (MEG) are two non-invasive techniques used to localize electric activity in the brain from measurements of external electromagnetic signals.

EEG measures the electric potential (on the scalp), while MEG measures the magnetic flux (closed but external to the head).

The electromagnetic activity of the brain is due to the movements of ions within activated regions of the cortex sheet, the so-called impressed currents (or primary currents). In addition, Ohmic currents are generated in the surrounding medium, the so-called return currents.

EEG and MEG (cont'd)

The measures of EEG and MEG correspond to both impressed and return currents, but the source of interest are the impressed currents, as they represent the area of neural activity associated to a sensory stimulus.

- First EEG in man: H. Berger (1924)
- First MEG in man: D. Cohen (late 1960s).

EEG and MEG (cont'd)

Source localization is an inverse problem: knowing the value of the magnetic field or of the electric field on the surface of the head (or, possibly, external to the head, but close to its surface), the aim is to determine the position and some physical characteristics of the current density that has given rise to that value.

Since the current distribution inside a conductor cannot be retrieved uniquely from knowledge of the electromagnetic field outside the conductor, the mathematical problem does not have a unique solution unless some additional conditions on the source model are assumed.

Dipolar source models

We focus here on the dipolar source model.

In this model the primary current distribution is represented as a point source located at $\mathbf{x}_0$ with moment $\mathbf{p}$, namely,

$$\mathbf{J}_e(\mathbf{x}) = \mathbf{p}\,\delta(\mathbf{x} - \mathbf{x}_0)\,,$$

where δ is the Dirac delta distribution.

The dipole is a convenient representation for a uni-directional impressed current due to the activation of a large number of cells (in real situations may indeed extend over several square centimeters of the cortex). More generally, it is assumed that a primary current source can be decomposed as the sum of (few) current dipoles.

Minimization algorithms

A preliminary step for the solution of the inverse problem is an efficient resolution of the forward problem.

In fact, the procedure is essentially the following: given a source $\mathbf{J}_e$, solve the forward problem, thus determining the electric and magnetic fields generated by $\mathbf{J}_e$, and then minimize in a suitable way the difference between the computed and the measured data.

The current density $\mathbf{J}_e^*$ which achieves the minimum is the source we are trying to determine.

The forward problem: static approximation

Let us focus now on the forward problem.

The static approximation of Maxwell equations reads

$$\begin{aligned} \mathbf{curl}\,\mathbf{H} &= \mathbf{J}_e + \boldsymbol{\sigma}\mathbf{E} \\ \operatorname{div}\mathbf{B} &= 0 \\ \mathbf{curl}\,\mathbf{E} &= \mathbf{0}\,, \end{aligned} \tag{1}$$

neglecting the displacement current and the electromagnetic diffusion.

- Note that in this way the electric field $\mathbf{E}$ can be determined independently from the magnetic field $\mathbf{H}$.

From Ohm law the total current density $\mathbf{J}$ is the sum of the impressed currents plus the return currents

$$\mathbf{J} = \mathbf{J}_e + \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e - \boldsymbol{\sigma}\,\mathbf{grad}\,U\,,$$

where U is the electric scalar potential.

The forward problem: static approximation (cont'd)

From the first equation in (1) it follows that

$$0 = \operatorname{div} \mathbf{J} = \operatorname{div} (\mathbf{J}_e - \boldsymbol{\sigma} \mathbf{grad}\, U) \,.$$

Hence U can be obtained by solving the Poisson equation with Neumann boundary condition

$$\begin{cases} \operatorname{div} (\boldsymbol{\sigma} \mathbf{grad}\, U) = \operatorname{div} \mathbf{J}_e & \text{in } \Omega_C \\ \boldsymbol{\sigma} \mathbf{grad}\, U \cdot \mathbf{n} = \mathbf{J}_e \cdot \mathbf{n} & \text{on } \partial\Omega_C \,, \end{cases} \tag{2}$$

where the boundary condition is a consequence of the fact that outside the head Ω_C the magnetic field is supposed to be curl-free (the source $\mathbf{J}_e$ is located inside the head, and the conductivity is vanishing outside the head, so that $\mathbf{J}_I = \mathbf{0}$ in Ω_I and consequently $\mathbf{J}_C \cdot \mathbf{n} = 0$ on $\partial\Omega_C$).

The potential equation with a dipolar source

For EEG this is the point: solving this elliptic problem gives the potential of the electric field, and the inverse problem of source localization can be dealt with.

When we deal with a dipolar source, the potential equation (2) reads

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, U) = \operatorname{div}(\mathbf{p}\, \delta_{\mathbf{x}_0}) & \text{in } \Omega_C \\ (\boldsymbol{\sigma} \,\mathbf{grad}\, U) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_C \,, \end{cases} \tag{3}$$

where $\mathbf{x}_0 \in \Omega_C$ and we have set, for simplicity, $\delta_{\mathbf{x}_0}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0)$. Clearly, the solution U is defined up to an additive constant.

- We assume the local Lipschitz regularity condition for the conductivity:

$$\text{there exists } r_0 > 0 \text{ such that } \boldsymbol{\sigma} \in W^{1,\infty}(B_{r_0}(\mathbf{x}_0)) \,. \tag{4}$$

Weak formulation of the potential equation

We want to give a weak formulation of problem (3).

Introduce the linear space

$$X_q := \{\varphi \in H^1(\Omega_C) \,|\, \varphi \in C^1(B_{r_*}(\mathbf{x}_0)), \mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\varphi) \in L^q(\Omega_C), \\ (\boldsymbol{\sigma}\,\mathbf{grad}\,\varphi)\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_C\}\,,$$

where $0 < r_* < r_0$ is a fixed number, q is a fixed number satisfying $3 < q < +\infty$, and p is its Hölder dual exponent defined by $\frac{1}{p} + \frac{1}{q} = 1$ (hence $1 < p < \frac{3}{2}$).

Multiplying the first equation in (3) by $\varphi \in X_q$, integrating in Ω_C and integrating by parts we readily find

Weak formulation of the potential equation (cont'd)

$$
\begin{aligned}
&\int_{\Omega_C} \mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\, U)\,\varphi \\
&\quad = -\int_{\Omega_C} (\boldsymbol{\sigma}\,\mathbf{grad}\, U)\cdot \mathbf{grad}\,\varphi + \int_{\partial\Omega_C} (\boldsymbol{\sigma}\,\mathbf{grad}\, U)\cdot \mathbf{n}\,\varphi \\
&\quad = \int_{\Omega_C} U\,\mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\varphi) - \int_{\partial\Omega_C} U\,(\boldsymbol{\sigma}\,\mathbf{grad}\,\phi)\cdot \mathbf{n} \\
&\qquad + \int_{\partial\Omega_C} (\boldsymbol{\sigma}\,\mathbf{grad}\, U)\cdot \mathbf{n}\,\varphi \\
&\quad = \int_{\Omega_C} U\,\mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\varphi)
\end{aligned}
$$

and

$$
\int_{\Omega_C} \mathrm{div}(\mathbf{p}\,\delta_{\mathbf{x}_0})\,\varphi = -\int_{\Omega_C} \mathbf{p}\cdot \mathbf{grad}\,\varphi\,\delta_{\mathbf{x}_0} = -\,\mathbf{p}\cdot \mathbf{grad}\,\varphi(\mathbf{x}_0)\,,
$$

having taken into account the boundary conditions satisfied by U and φ.

Note that, for duality, the term $\int_{\Omega_C} U\,\mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\varphi)$ has a meaning also for $U \in L^p(\Omega_C)$.

Weak formulation of the potential equation (cont'd)

We are now in a position to describe the weak formulation of (3) that we consider:

$$\begin{cases} \text{find } U \in L^p(\Omega_C) \ : \\ \int_{\Omega_C} U \,\mathrm{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, \varphi) = -\mathbf{p} \cdot \mathbf{grad}\, \varphi(\mathbf{x}_0) \quad \forall\, \varphi \in X_q \\ \int_{\Omega_C} U = 0\,. \end{cases} \tag{5}$$

The following theorem gives the proof of the existence of the solution U of problem (5).

Existence theorem for the potential equation

Theorem

There exists a solution U to (5).

Proof. We use an approximation argument. Let us denote by δ_k a sequence of functions such that $\delta_k \in C_0^\infty(B_{r_*}(\mathbf{x}_0))$, $\delta_k \geq 0$, $\int_{\Omega_C} \delta_k = 1$ and $\int_{\Omega_C} \delta_k \, \xi \to \xi(\mathbf{x}_0)$ for each $\xi \in C^0(B_{r_*}(\mathbf{x}_0))$. We consider the solution $U_k \in H^1(\Omega_C)$ of the Neumann problem

$$\begin{cases} \mathrm{div}(\boldsymbol{\sigma} \, \mathbf{grad} \, U_k) = \mathrm{div}(\mathbf{p} \, \delta_k) & \text{in } \Omega_C \\ (\boldsymbol{\sigma} \, \mathbf{grad} \, U_k) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_C \\ \int_{\Omega_C} U_k = 0 \, . \end{cases}$$

The existence and uniqueness of U_k is assured as $\int_{\Omega_C} \mathrm{div}(\mathbf{p} \, \delta_k) = \int_{\partial\Omega_C} \mathbf{p} \cdot \mathbf{n} \, \delta_k = 0$, hence the compatibility condition is satisfied.

Existence theorem for the potential equation (cont'd)

By integrating by parts we see that U_k satisfies

$$\int_{\Omega_C} U_k \, \mathrm{div}(\boldsymbol{\sigma} \, \mathbf{grad} \, \varphi) = - \int_{\Omega_C} \mathbf{p} \cdot \mathbf{grad} \, \varphi \, \delta_k \quad \forall \, \varphi \in X_q \, .$$

Take now $\psi \in L^q(\Omega_C)$: we want to find an uniform estimate of $\int_{\Omega_C} U_k \, \psi$. Consider the solution $\hat{\varphi}$ of the Neumann problem

$$\begin{cases} \mathrm{div}(\boldsymbol{\sigma} \, \mathbf{grad} \, \hat{\varphi}) = \psi - \frac{1}{\mathrm{meas}(\Omega_C)} \left(\int_{\Omega_C} \psi \right) & \text{in } \Omega_C \\ (\boldsymbol{\sigma} \, \mathbf{grad} \, \hat{\varphi}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_C \\ \int_{\Omega_C} \hat{\varphi} = 0 \, . \end{cases} \qquad (6)$$

Since $\int_{\Omega_C} \left[\psi - \frac{1}{\mathrm{meas}(\Omega_C)} \left(\int_{\Omega_C} \psi \right) \right] = 0$, we have a unique solution $\hat{\varphi} \in H^1(\Omega_C)$.

Existence theorem for the potential equation (cont'd)

On the other hand, we have $\left[\psi - \frac{1}{\mathrm{meas}(\Omega_C)}\left(\int_{\Omega_C}\psi\right)\right] \in L^q(\Omega_C)$ and the regularity results for elliptic problems yield $\hat{\varphi} \in W^{2,q}(B_{r_*}(\mathbf{x}_0))$. The Sobolev embedding theorem also gives $\hat{\varphi} \in C^1(\overline{B_{r_*}(\mathbf{x}_0)})$, hence $\hat{\varphi} \in X_q$. Moreover, $\|\hat{\varphi}\|_{C^1(\overline{B_{r_*}(\mathbf{x}_0)})} \leq c_0\|\psi\|_{L^q(\Omega_C)}$, where c_0 depends on $\boldsymbol{\sigma}$, q, r_*, but not on ψ.

We are now in a position to obtain the needed estimate. We have

$$\begin{aligned}
\left|\int_{\Omega_C} U_k\,\psi\right| &= \left|\int_{\Omega_C} U_k\left[\psi - \frac{1}{\mathrm{meas}(\Omega_C)}\left(\int_{\Omega_C}\psi\right)\right]\right| \\
&= \left|\int_{\Omega_C} U_k \mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\hat{\varphi})\right| = \left|-\int_{\Omega_C}\mathbf{p}\cdot\mathbf{grad}\,\hat{\varphi}\,\delta_k\right| \\
&\leq |\mathbf{p}|\,\|\mathbf{grad}\,\hat{\varphi}\|_{C^0(\overline{B_{r_*}(\mathbf{x}_0)})}\int_{\Omega_C}\delta_k \leq c_0\,|\mathbf{p}|\|\psi\|_{L^q(\Omega_C)}\ .
\end{aligned}$$

Existence theorem for the potential equation (cont'd)

In other words,

$$\|U_k\|_{L^p(\Omega_C)} := \sup_{\psi \in L^q(\Omega_C)} \frac{|\int_{\Omega_C} U_k \, \psi|}{\|\psi\|_{L^q(\Omega_C)}} \leq c_0 \, |\mathbf{p}| \, .$$

We can thus select a subsequence (still denoted by U_k) that converges weekly in $L^p(\Omega_C)$ to $U \in L^p(\Omega_C)$. In particular, for each $\varphi \in X_q$

$$\begin{aligned} &\int_{\Omega_C} U_k \, \mathrm{div}(\boldsymbol{\sigma} \, \mathbf{grad} \, \varphi) \to \int_{\Omega_C} U \, \mathrm{div}(\boldsymbol{\sigma} \, \mathbf{grad} \, \varphi) \, , \\ &- \int_{\Omega_C} \mathbf{p} \cdot \mathbf{grad} \, \varphi \, \delta_k \to - \mathbf{p} \cdot \mathbf{grad} \, \varphi(\mathbf{x}_0) \, . \end{aligned}$$

Finally,

$$0 = \int_{\Omega_C} U_k \to \int_{\Omega_C} U \, ,$$

and U is a solution to (5). □

Uniqueness theorem for the potential equation

Theorem

The solution U to (5) is unique.

Proof. Let U be any solution to (5). For each $\psi \in L^q(\Omega_C)$, consider the solution $\hat{\varphi}$ of (6). Using it in (5) we find

$$\begin{aligned} |\int_{\Omega_C} U\,\psi| &= |\int_{\Omega_C} U\big[\psi - \tfrac{1}{\mathrm{meas}(\Omega_C)}\big(\int_{\Omega_C}\psi\big)\big]| \\ &= |\int_{\Omega_C} U\,\mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\hat{\varphi})| = |-\mathbf{p}\cdot\mathbf{grad}\,\hat{\varphi}(\mathbf{x}_0)| \\ &\leq |\mathbf{p}|\,\|\mathbf{grad}\,\hat{\varphi}\|_{C^0(\overline{B_{r_*}(\mathbf{x}_0)})} \leq c_0\,|\mathbf{p}|\|\psi\|_{L^q(\Omega_C)}\ , \end{aligned}$$

hence $\|U\|_{L^p(\Omega_C)} \leq c_0\,|\mathbf{p}|$, and uniqueness follows. □

Uniqueness theorem for the potential equation (cont'd)

Remark.
Since for $3 < s < q$ one has $X_s \supset X_q$ and $L^r(\Omega_C) \subset L^p(\Omega_C)$ (here $\frac{1}{r} + \frac{1}{s} = 1$), from the uniqueness result it follows readily that the solution U to (5) is the same for all finite values $s, q > 3$.
Therefore we have solved the problem

$$\begin{aligned} &\text{find } U \in \textstyle\bigcap_{q>3} L^p(\Omega_C) \text{ with } \int_{\Omega_C} U = 0 \ : \\ &\int_{\Omega_C} U \,\mathrm{div}(\boldsymbol{\sigma}\, \mathbf{grad}\, \varphi) = -\mathbf{p} \cdot \mathbf{grad}\, \varphi(\mathbf{x}_0) \\ &\forall\, \varphi \in \bigcup_{q>3} X_q \,. \end{aligned} \tag{7}$$

Direct approximation approach

In principle, the numerical approximation of the potential equation can be based on both formulations (3) or (5).

In the former case, one requires the non-isotropic homogeneity condition: there exist $r_0 > 0$ and a constant symmetric and positive definite matrix $\boldsymbol{\sigma}_0$ such that

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}_0 \quad \text{for each } \mathbf{x} \in B_{r_0}(\mathbf{x}_0)\,. \tag{8}$$

Then the fundamental solution $K^\sharp$ satisfying

$$\mathrm{div}(\boldsymbol{\sigma}_0 \mathbf{grad}\, K^\sharp) = \mathrm{div}(\mathbf{p}\, \delta_{\mathbf{x}_0})\,,$$

is constructed. It is given by

$$K^\sharp(\mathbf{x}) = \frac{1}{4\pi\sqrt{\det \boldsymbol{\sigma}_0}} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}_0^{-1}(\mathbf{x}-\mathbf{x}_0)}{[\boldsymbol{\sigma}_0^{-1}(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)]^{3/2}}$$

(see Sauter and Schwab (2011)).

Direct approximation approach (cont'd)

With this at hand, one looks for $U = K^\sharp + \mathcal{U}$, $\mathcal{U}$ being a solution of

$$\begin{cases} \mathrm{div}\,(\boldsymbol{\sigma}\,\mathbf{grad}\,\mathcal{U}) = g & \text{in } \Omega_C \\ \boldsymbol{\sigma}\,\mathbf{grad}\,\mathcal{U} \cdot \mathbf{n} = -\boldsymbol{\sigma}\,\mathbf{grad}\,K^\sharp \cdot \mathbf{n} & \text{on } \partial\Omega_C\,, \end{cases} \tag{9}$$

where

$$g(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \in B_{r_0}(\mathbf{x}_0) \\ -\mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,K^\sharp)(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_C \setminus \overline{B_{r_0}(\mathbf{x}_0)}\,. \end{cases}$$

Note that the data of problem (9) are smooth, and that g vanishes in $B_{r_0}(\mathbf{x}_0)$ as a consequence of the non-isotropic homogeneity condition.

Direct approximation approach (cont'd)

The non-isotropic homogeneity condition is a stronger assumption than the local Lipschitz regularity condition (4), and this has an influence on the efficiency of numerical computations.
In particular, when considering a head model in which the conductivity is jumping (and this is indeed the realistic case, as the conductivity is quite different in the skull or in the brain: in the skull it is from ten to one hundred times smaller), the subtraction method has shown some instabilities when the position $\mathbf{x}_0$ of the dipole is quite close to the discontinuity surface.

Therefore, a direct finite element approach based on (5) could be suitable. The results we are going to present have been obtained in Alonso Rodríguez, Camaño, Rodríguez and V. (2013).

Finite element approximation of the potential equation

The simplest finite element approximation of (5) reads:

$$\begin{cases} \text{find } U_h \in L^1_h \text{ with } \int_{\Omega_C} U_h = 0 \ : \\ \quad \int_{\Omega_C} \boldsymbol{\sigma} \, \mathbf{grad}\, U_h \cdot \mathbf{grad}\, \varphi_h = \mathbf{p} \cdot \mathbf{grad}\, (\varphi_h|_{T_0})(\mathbf{x}_0) \\ \forall\, \varphi_h \in L^1_h \,, \end{cases} \tag{10}$$

where $\mathbf{x}_0 \in T_0$ (if $\mathbf{x}_0$ belongs to many elements, just choose one of them) and

$$L^1_h := \{\varphi_h \in C^0(\Omega_C) \,|\, \varphi_{h|K} \in \mathbb{P}_1 \ \forall \ K\} \,.$$

To find an a priori error estimate in $L^p(\Omega_C)$, with $1 < p < 3/2$, a duality argument is used, and the solution $\hat{\varphi} \in H^1(\Omega_C)$ of problem (6) comes into play.

Error estimate for the potential equation

For utilizing the duality argument, we need that $\hat{\varphi} \in W^{2,q}(\Omega_C)$ for a suitable $q > 3$ ($q > 2$ in the two-dimensional case). This is true under some assumptions.

- In the two-dimensional case, we require that $\boldsymbol{\sigma} \in C^1(\overline{\Omega_C})$ and that Ω_C is convex; then the regularization result is true for all q such that $2 < q < q_0$, for a suitable $q_0 > 2$ (for the Laplace operator, $q_0 = \frac{2}{2-\pi/\theta}$, $\theta > \frac{\pi}{2}$ being the largest inner angle of Ω_C).
- In the three-dimensional case, we require that $\boldsymbol{\sigma} = \sigma_0 I$ ($\sigma_0 > 0$ a constant and I the identity matrix) and that Ω_C is a parallepiped; then the regularization result is true for all $q > 3$.

Error estimate for the potential equation (cont'd)

Finally we have (in the three-dimensional case):

Theorem

Let $\mathcal{T}_h$ be a quasiuniform family of triangulations of Ω_C. Let U and U_h be the solutions to problems (5) and (10). Then there exists $h_0 > 0$ such that

$$\|U - U_h\|_{0,p,\Omega_C} \leq Ch^{3/p-2}$$

for all $0 < h < h_0$, with p such that $\frac{1}{p} + \frac{1}{q} = 1$, and $q > 3$ the exponent such that the solution $\hat{\varphi}$ of problem (6) belongs to $W^{2,q}(\Omega_C)$.

A posteriori error analysis

The variational structure of problems (5) and (10) permits to perform an a posteriori error analysis, and therefore to devise an adaptive mesh strategy.

Let $\mathcal{F}_{h,i}$ be the set of all the inner faces and $\mathcal{F}_{h,e}$ that of external faces of the mesh $\mathcal{T}_h$. Let $\mathcal{F}_h := \mathcal{F}_{h,i} \cup \mathcal{F}_{h,e}$. For all $T \in \mathcal{T}_h$ we define

$$\widehat{\varrho}_{T,p} := \Big(\tfrac{1}{2} \sum_{F \in \mathcal{F}(T) \cap \mathcal{F}_{h,i}} |F|^{(p+3)/2} |[\![\, \mathbf{grad}\, U_h \cdot \mathbf{n}_F \,]\!]|^p + \sum_{F \in \mathcal{F}(T) \cap \mathcal{F}_{h,e}} |F|^{(p+3)/2} |\mathbf{grad}\, U_h \cdot \mathbf{n}_F|^p \Big)^{1/p} ,$$

where $\mathcal{F}(T)$ is the set of faces of T, $|F|$ is the area of F and $[\![\, \mathbf{grad}\, U_h \cdot \mathbf{n}_F \,]\!]$ denotes the jump of $\mathbf{grad}\, U_h \cdot \mathbf{n}_F$ across the face F.

A posteriori error analysis (cont'd)

We define the local a posteriori error indicator $\widehat{\eta}_{T,p}$ for all $T \in \mathcal{T}_h$ by

$$\widehat{\eta}_{T,p} := \begin{cases} \left(h_0^{3-2p} + \widehat{\varrho}^{p}_{T_0,p} \right)^{1/p} & \text{if } T = T_0\,, \\ \widehat{\varrho}_{T,p} & \text{otherwise}\,, \end{cases}$$

where $h_0 := h_{T_0}$, and the global error estimator from these indicators as follows:

$$\widehat{\eta}_p := \left(\sum_{T \in \mathcal{T}_h} \widehat{\eta}^{p}_{T,p} \right)^{1/p} .$$

A posteriori error analysis (cont'd)

Set $\omega_T := \{T' \in \mathcal{T}_h \,|\, T' \cap T \neq \emptyset\}$. Then we have

Theorem

Let U and U_h be the solutions of (5) and (10), respectively. Then the following estimates hold true:

$$\|U - U_h\|_{0,p,\Omega} \leq C\widehat{\eta}_p$$

and

$$\widehat{\eta}_{T,p} \leq C\|U - U_h\|_{0,p,\omega_T}$$

for all $T \in \mathcal{T}_h$.