Coupling of eddy-current and circuit problems

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Time-harmonic eddy-current equations

Starting from Maxwell equations, assuming a sinusoidal dependence on time and disregarding displacement currents one obtains the so-called time-harmonic eddy-current problem

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{0} & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega\,. \end{cases} \tag{1}$$

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Here

- $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric fields, respectively
- $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the electric conductivity and the magnetic permeability, respectively
- $\omega \neq 0$ is the frequency.

Time-harmonic eddy-current equations (cont'd)

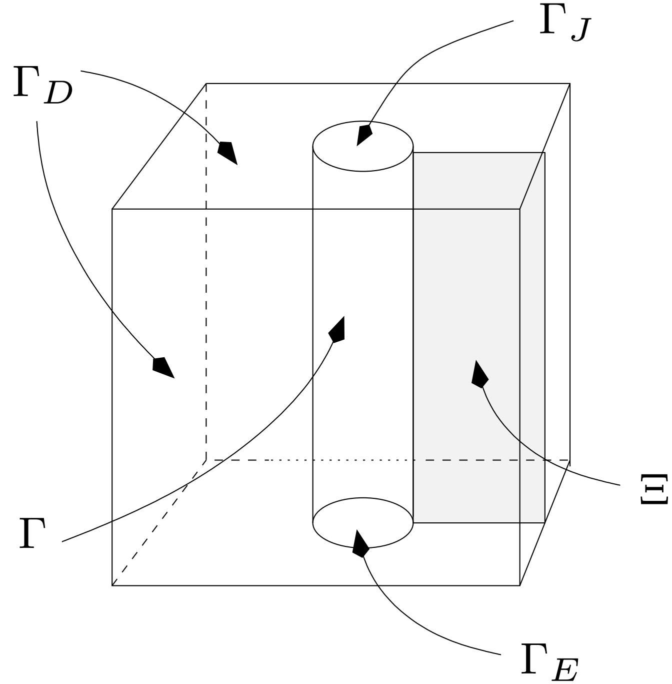
[As shown in the previous talk, in an insulator one has $\boldsymbol{\sigma} = \mathbf{0}$, therefore $\mathbf{E}$ is not uniquely determined in that region ($\mathbf{E} + \nabla\psi$ is still a solution).
Some additional conditions ("gauge" conditions) are thus necessary: as in the insulator Ω_I we have no charges, we impose

$$\mathrm{div}(\boldsymbol{\epsilon}\mathbf{E}) = 0 \qquad \text{in } \Omega_I \,, \tag{2}$$

where $\boldsymbol{\epsilon}$ is the electric permittivity.]

Geometry

The physical domain $\Omega \subset \mathbf{R}^3$ is a “box", and the conductor Ω_C is simply-connected with $\partial\Omega_C \cap \partial\Omega = \Gamma_E \cup \Gamma_J$, where Γ_E and Γ_J are connected and disjoint surfaces on $\partial\Omega$ (“electric ports"). Notation: $\Gamma = \overline{\Omega_C} \cap \overline{\Omega_I}$, $\partial\Omega = \Gamma_E \cup \Gamma_J \cup \Gamma_D$, $\partial\Omega_C = \Gamma_E \cup \Gamma_J \cup \Gamma$, $\partial\Omega_I = \Gamma_D \cup \Gamma$.



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- No-flux (Case C) [Bossavit, 2000]. One imposes $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\Gamma_E \cup \Gamma_J$, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on Γ_D.

Voltage and current intensity

When one wants to couple the eddy-current problem with a circuit problem, one has to consider, as the only external datum that determines the solution, a voltage V or a current intensity I_0.

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Question:

- how can we formulate the eddy-current problems when the excitation is given by a voltage or by a current intensity?

This is a delicate point, as eddy-current problems, for the two cases A and B, have a unique solution already before a voltage or a current intensity is assigned!

Poynting Theorem (energy balance)

In fact one has:

Uniqueness theorem. In the cases A and B for the solution of the eddy-current problem (1) the magnetic field $\mathbf{H}$ in Ω and the electric field $\mathbf{E}_C$ in Ω_C are uniquely determined. [Adding the "gauge" conditions, also the electric field $\mathbf{E}_I$ in Ω_I is uniquely determined.]

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Proof. Multiply the Faraday equation by $\overline{\mathbf{H}}$, integrate in Ω and integrate by parts: it holds

$$\begin{aligned} 0 &= \int_\Omega \mathbf{curl}\,\mathbf{E} \cdot \overline{\mathbf{H}} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} \\ &= \int_\Omega \mathbf{E} \cdot \mathbf{curl}\,\overline{\mathbf{H}} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}}\ . \end{aligned}$$

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Remembering that $\mathbf{curl}\,\mathbf{H}_I = \mathbf{0}$ in Ω_I and replacing $\mathbf{curl}\,\mathbf{H}_C$ with $\boldsymbol{\sigma}\mathbf{E}_C$, one has the Poynting Theorem (energy balance)

Poynting Theorem (energy balance) (cont'd)

$$\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{E}_C} \quad + \int_{\Omega} i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}} = - \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}}.$$

Poynting Theorem (energy balance) (cont'd)

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The term on $\partial\Omega$ is clearly vanishing in the cases A and B. $\square$

Poynting Theorem for the case C

In the case C, instead, since $\mathrm{div}_\tau(\mathbf{E}\times\mathbf{n}) = -i\omega\boldsymbol{\mu}\mathbf{H}\cdot\mathbf{n} = 0$ on $\partial\Omega$, one has

$$\mathbf{E}\times\mathbf{n} = \mathbf{grad}\,W\times\mathbf{n} \quad \text{on } \partial\Omega\,,$$

and therefore

$$\begin{aligned} -\int_{\partial\Omega}\mathbf{n}\times\mathbf{E}\cdot\overline{\mathbf{H}} &= -\int_{\partial\Omega}\overline{\mathbf{H}}\times\mathbf{n}\cdot\mathbf{grad}\,W \\ &= \int_{\partial\Omega}\mathrm{div}(\overline{\mathbf{H}}\times\mathbf{n})\,W \\ &= \int_{\partial\Omega}\mathbf{curl}\,\overline{\mathbf{H}}\cdot\mathbf{n}\,W = W_{|\Gamma_J}\int_{\Gamma_J}\mathbf{curl}\,\overline{\mathbf{H}_C}\cdot\mathbf{n}, \end{aligned}$$

as $\mathbf{curl}\,\mathbf{H}_I = \mathbf{0}$ in Ω_I, and we have denoted by $W_{|\Gamma_J}$ the (constant) value of the potential W on the electric port Γ_J (whereas $W_{|\Gamma_E} = 0$).

Poynting Theorem for the case C (cont'd)

- In this case a degree of freedom is indeed still free (either the voltage $W_{|\Gamma_J}$, that will be denoted by V, or else the current intensity $\int_{\Gamma_J} \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{n}$ in Ω_C, that will be denoted by I_0).

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We can thus conclude that the only meaningful boundary value problem is the one with assigned no-flux boundary conditions: the case C.

The case C: variational formulation

- How can we formulate the problem when the voltage or the current intensity are assigned?

[Alonso Rodríguez, Valli and Vázquez Hernández, 2009]

[Other approaches: Bíró, Preis, Buchgraber and Tičar, 2004; Bermúdez, Rodríguez and Salgado, 2005]

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This orthogonal decomposition result turns out to be useful: each vector function $\mathbf{v}_I$ can be decomposed as

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1}\,\mathbf{curl}\,\mathbf{q}_I + \mathbf{grad}\,\psi_I + \alpha\boldsymbol{\rho}_I\,,$$

where $\boldsymbol{\rho}_I$ is a harmonic field, namely, it belongs to the space

$$\mathcal{H}_{\mu_I}(\Omega_I) := \{\mathbf{v}_I \in (L^2(\Omega_I))^3 |\ \mathbf{curl}\,\mathbf{v}_I = \mathbf{0},\ \mathrm{div}(\boldsymbol{\mu_I}\mathbf{v}_I) = 0,\ \boldsymbol{\mu_I}\mathbf{v}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_I\}\,.$$

The case C: variational formulation (cont'd)

The harmonic field $\boldsymbol{\rho}_I$ is known from the data of the problem, and satisfies $\int_{\partial\Gamma_J} \boldsymbol{\rho}_I \cdot d\boldsymbol{\tau} = 1$; moreover, if the vector field $\mathbf{v}_I$ satisfies $\mathbf{curl}\,\mathbf{v}_I = \mathbf{0}$, it follows $\mathbf{q}_I = \mathbf{0}$ and therefore $\alpha = \int_{\partial\Gamma_J} \mathbf{v}_I \cdot d\boldsymbol{\tau}$.

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In particular, setting $\mathbf{H}_I = \mathbf{grad}\,\psi_I + \alpha_I \boldsymbol{\rho}_I$, from the Stokes Theorem one has

$$I_0 = \int_{\Gamma_J} \mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{n}_C = \int_{\partial\Gamma_J} \mathbf{H}_C \cdot d\boldsymbol{\tau} = \int_{\partial\Gamma_J} \mathbf{H}_I \cdot d\boldsymbol{\tau} = \alpha_I \,,$$

hence

$$\mathbf{H}_I = \mathbf{grad}\,\psi_I + I_0 \boldsymbol{\rho}_I \,. \qquad (3)$$

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We want to provide a "coupled" variational formulation, in terms of $\mathbf{E}_C$ in Ω_C and of $\mathbf{H}_I$ in Ω_I.

The case C: variational formulation (cont'd)

Inserting the Faraday equation into the Ampère equation in Ω_C we find

$$\int_{\Omega_C} \boldsymbol{\mu}_C^{-1} \,\mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{w}_C} \\ -i\omega \int_{\Gamma} \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \mathbf{H}_I = 0 \,. \tag{4}$$

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Instead, the Faraday equation in Ω_I gives

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{grad}\, \overline{\varphi_I} + \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} = 0 \tag{5}$$

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and

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I + \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I = V\,. \tag{6}$$

The case C: variational formulation (cont'd)

Here we have to note that

$$\int_{\Gamma_D} \mathbf{E}_I \times \mathbf{n}_I \cdot \boldsymbol{\rho}_I = \int_{\Gamma_D} \mathbf{grad}\, W \times \mathbf{n}_I \cdot \boldsymbol{\rho}_I$$
$$= \int_{\Gamma_D} \mathrm{div}_\tau(\boldsymbol{\rho}_I \times \mathbf{n}_I) W + V \int_{\partial\Gamma_J} \boldsymbol{\rho}_I \cdot d\boldsymbol{\tau} = V \ .$$

The case C: variational formulation (cont'd)

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Using (3) in (4), (5) and (6) one has

$$\int_{\Omega_C} \boldsymbol{\mu}_C^{-1}\, \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{w}_C}$$
$$-i\omega \int_\Gamma \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \mathbf{grad}\, \psi_I - i\omega I_0 \int_\Gamma \overline{\mathbf{w}_C} \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I = 0 \qquad (7)$$

$$-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathbf{grad}\, \overline{\varphi_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I\, \mathbf{grad}\, \psi_I \cdot \mathbf{grad}\, \overline{\varphi_I} = 0 \quad (8)$$

$$-i\omega \overline{Q} \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I + \omega^2 I_0 \overline{Q} \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_I \cdot \boldsymbol{\rho}_I = -i\omega V \overline{Q}\ . \quad (9)$$

The case C: existence and uniqueness

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Both problems are well-posed, namely, they have a unique solution, since the associated sesquilinear form is coercive (thus one can apply the Lax–Milgram Lemma).

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Moreover, it is simple to propose an approximation method based on finite elements, of "edge" type for $\mathbf{E}_C$ in Ω_C and of (scalar) nodal type for ψ_I in Ω_I. Convergence is assured by the Céa Lemma.

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Moreover, it is simple to propose an approximation method based on finite elements, of "edge" type for $\mathbf{E}_C$ in Ω_C and of (scalar) nodal type for ψ_I in Ω_I. Convergence is assured by the Céa Lemma. [However, an efficient implementation demands to replace the harmonic field $\boldsymbol{\rho}_I$ with an easily computable function.]

Physical interpretation

Note: the physical interpretation of equation (9) is that

$$-\int_{\gamma} \mathbf{E}_C \cdot d\mathbf{r} + i\omega \int_{\Xi} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n}_{\Xi} = V \ ,$$

where $\gamma = \partial\Xi \cap \Gamma$ is oriented from Γ_J to Γ_E, and $\mathbf{n}_{\Xi}$ is directed in such a way that γ is clockwise oriented with respect to it.

In other words, if it is possible to determine the electric field $\mathbf{E}_I$ in Ω_I satisfying the Faraday equation, it follows that

$$\int_{\gamma_*} \mathbf{E}_I \cdot d\mathbf{r} = V \ ,$$

where $\gamma_* = \partial\Xi \cap \Gamma_D$ is oriented from Γ_E to Γ_J: hence (9) is indeed determining the voltage drop between the electric ports.

Physical interpretation (cont'd)

This explains from another point of view why, when the source is a voltage drop or a current intensity, it is not possible to assume the electric boundary conditions $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.

Physical interpretation (cont'd)

This explains from another point of view why, when the source is a voltage drop or a current intensity, it is not possible to assume the electric boundary conditions $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.
In fact, in that case one would have

$$\int_{\gamma_*} \mathbf{E}_I \cdot d\mathbf{r} = 0 \,,$$

hence from (9)

$$\begin{aligned} i\omega \int_\Xi \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n}_\Xi &= V + \int_\gamma \mathbf{E}_C \cdot d\mathbf{r} = V + \int_{\gamma \cup \gamma_*} \mathbf{E} \cdot d\mathbf{r} \\ &= V + \int_{\partial\Xi} \mathbf{E} \cdot d\mathbf{r} \,, \end{aligned}$$

with $\partial\Xi$ clockwise oriented with respect $\mathbf{n}_\Xi$: due to the term V the Faraday equation would be violated on Ξ!

Numerical results for the Case C

Coming back to the case C and to its variational formulation (7), (8), (9), we use edge finite elements of the lowest degree ($\mathbf{a} + \mathbf{b} \times \mathbf{x}$ in each element) for approximating $\mathbf{E}_C$, and scalar piecewise-linear elements for approximating ψ_I.

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The problem description is the following: the conductor Ω_C and the whole domain Ω are two coaxial cylinders of radius R_C and R_D, respectively, and height L. Assuming that $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are scalar constants, the exact solution for an assigned current intensity I_0 is known (through suitable Bessel functions), and also the basis function $\boldsymbol{\rho}_I$ is known, thus from (9) one easily computes the voltage V, too.

Numerical results for the Case C (cont'd)

We have the following data:

$$
\begin{aligned}
R_C &= 0.25 \text{ m} \\
R_D &= 0.5 \text{ m} \\
L &= 0.25 \text{ m} \\
\sigma &= 151565.8 \text{ S/m} \\
\mu &= 4\pi \times 10^{-7} \text{ H/m} \\
\omega &= 2\pi \times 50 \text{ rad/s}
\end{aligned}
$$

and

$$I_0 = 10^4 \text{ A} \quad \text{or} \quad V = 0.08979 + 0.14680i$$

[the voltage corresponds to the current intensity $I_0 = 10^4$ A].

Numerical results for the Case C (cont'd)

The relative errors (for $\mathbf{E}_C$ in $H(\mathbf{curl};\Omega_C)$ and for $\mathbf{H}_I$ in $L^2(\Omega_I)$) with respect to the number of degrees of freedom are given by:

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Elements	DoF	e_E	e_H	e_V
2304	1684	0.2341	0.1693	0.0312
18432	11240	0.1132	0.0847	0.0089
62208	35580	0.0750	0.0567	0.0048
147456	81616	0.0561	0.0425	0.0018

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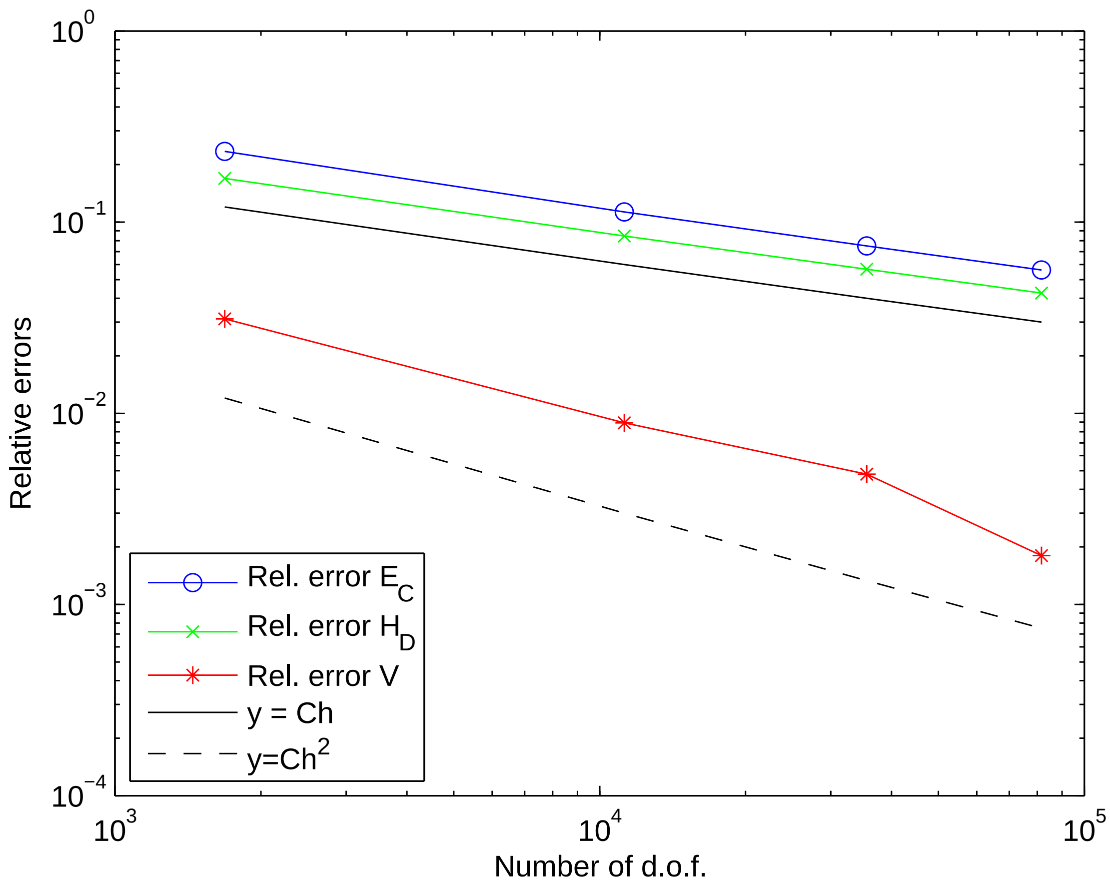
18432	11241	0.1132	0.0847	0.0085
62208	35581	0.0750	0.0566	0.0041
147456	81617	0.0561	0.0425	0.0024

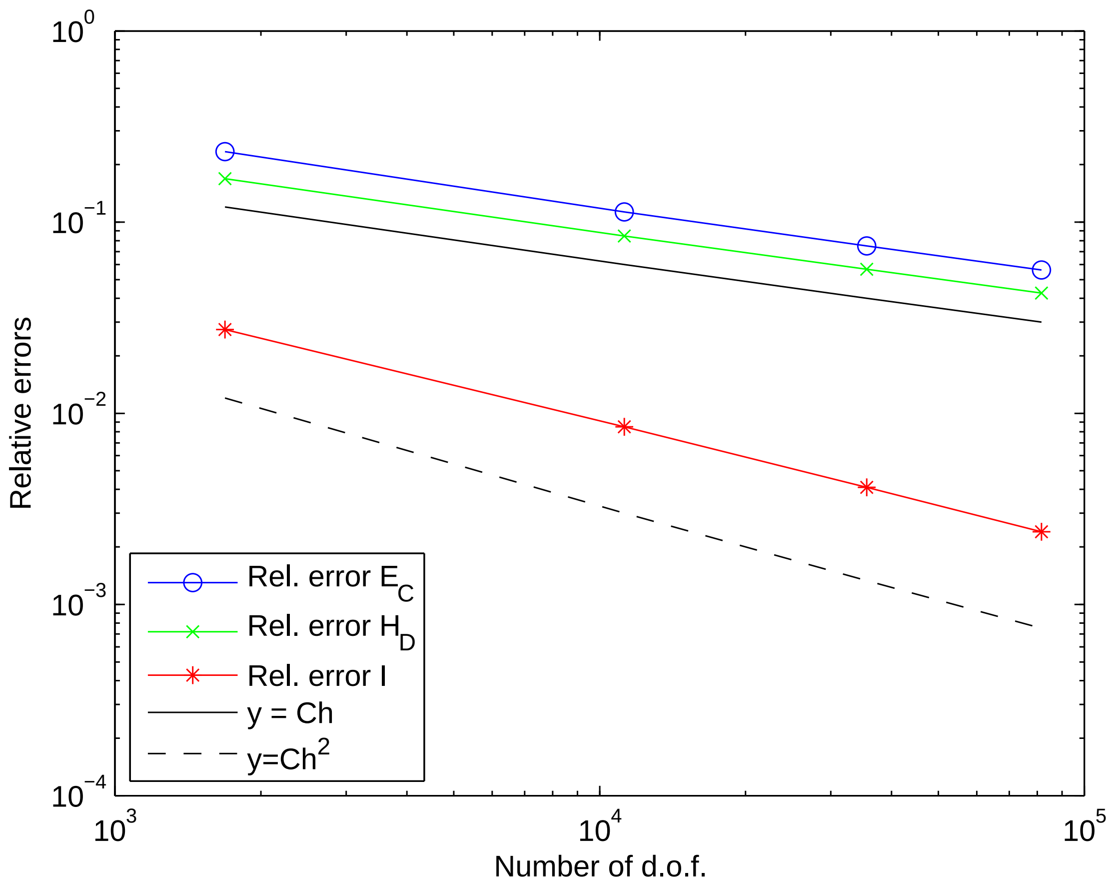
Numerical results for the Case C (cont'd)

On a graph: for assigned current intensity



Numerical results for the Case C (cont'd)

for assigned voltage



Numerical results for the Case C (cont'd)

A more realistic problem, considered by Bermúdez, Rodríguez and Salgado, 2005, is that of a cylindrical electric furnace with three electrodes ELSA [dimensions: furnace height 2 m; furnace diameter 8.88 m; electrode height 1.25 m; electrode diameter 1 m; distance of the center of the electrode from the wall 3 m].

Numerical results for the Case C (cont'd)

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The three electrodes ELSA are constituted by a graphite core of 0.4 m of diameter, and by an outer part of Söderberg paste. The electric current enters the electrodes through horizontal copper bars of rectangular section (0.07 m $\times$ 0.25 m), connecting the top of the electrode with the external boundary.

Numerical results for the Case C (cont'd)

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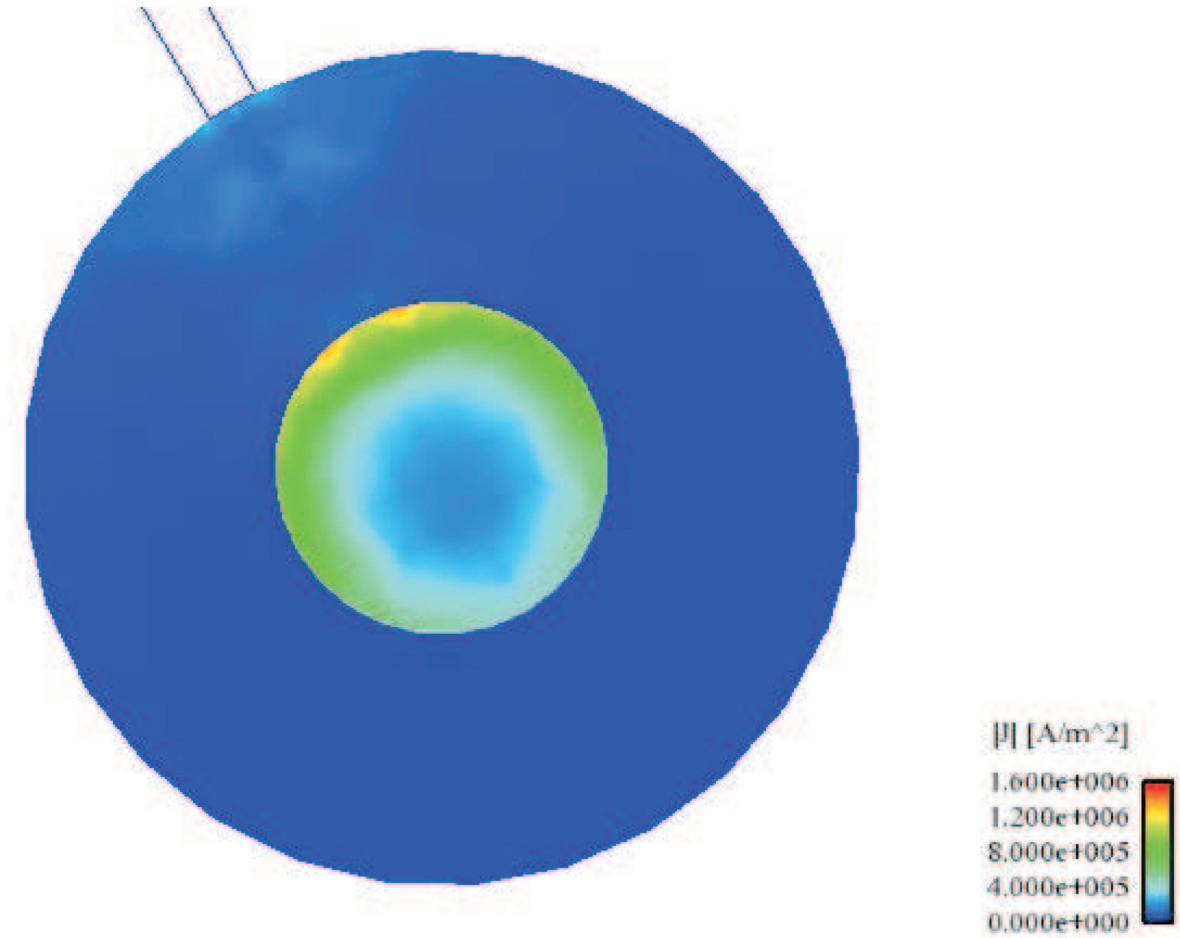
Data: $\sigma = 10^6$ S/m for graphite, $\sigma = 10^4$ S/m for Söderberg paste, $\sigma = 5 \times 10^6$ S/m for copper, $\mu = 4\pi \times 10^{-7}$ H/m, $\omega = 2\pi \times 50$ rad/s, $I_0 = 7 \times 10^4$ A for each electrode.

Numerical results for the Case C (cont'd)

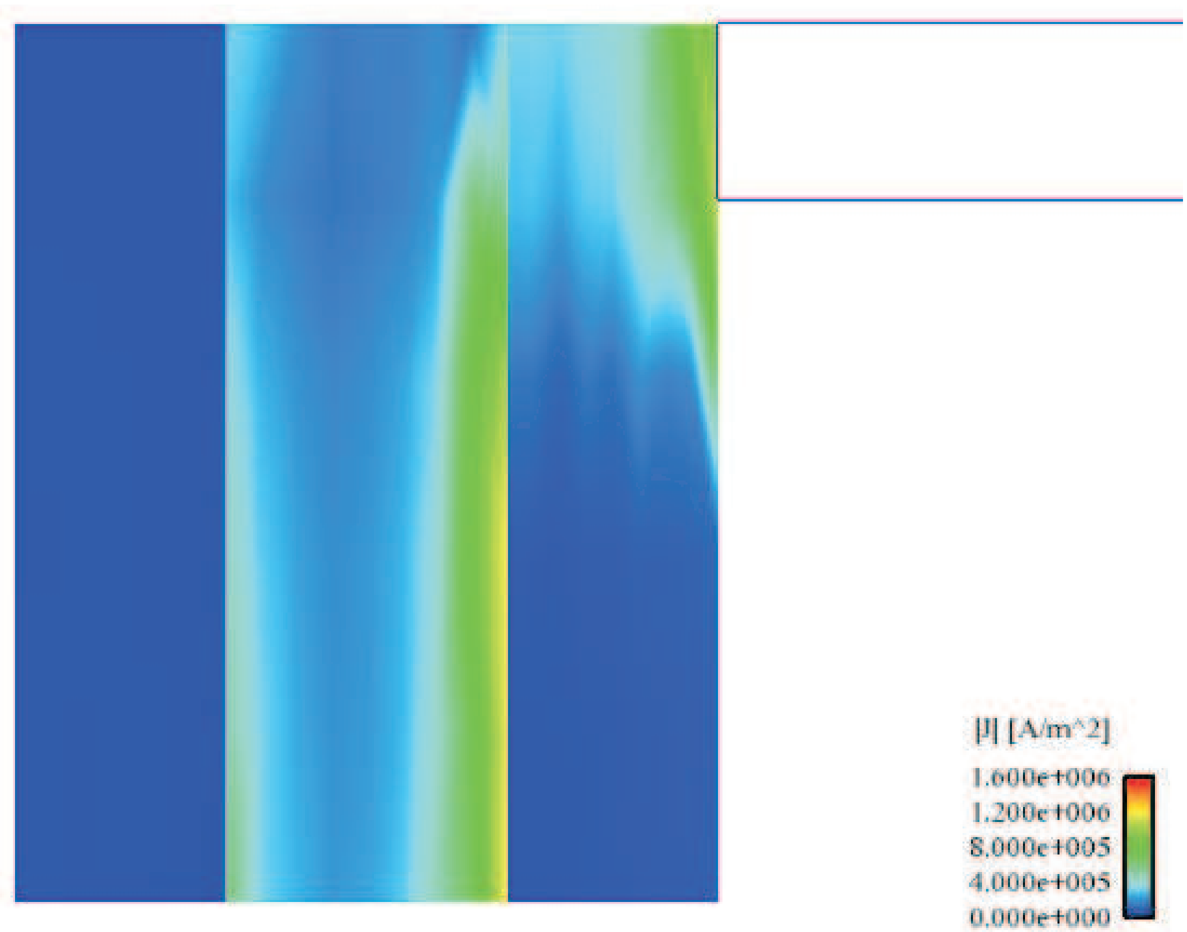
The value of the magnetic "potential" in the insulator: the magnetic field is the gradient of the represented function (not taking into account the jump surfaces).

Numerical results for the Case C (cont'd)



The magnitude of the current density $\sigma \mathbf{E}_C$ on a horizontal section of one electrode.

Numerical results for the Case C (cont'd)



The magnitude of the current density $\sigma \mathbf{E}_C$ on a vertical section of one electrode.

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