Eddy current problems in the time-harmonic regime

ALBERTO VALLI

Department of Mathematics, University of Trento

Maxwell equations

The complete Maxwell system of electromagnetism reads

$$
\begin{cases}
\dfrac{\partial \mathcal{D}}{\partial t} + \mathcal{J} = \text{curl}\, \mathcal{H} & \text{Maxwell–Ampère equation} \\
\dfrac{\partial \mathcal{B}}{\partial t} + \text{curl}\, \mathcal{E} = 0 & \text{Faraday equation} \\
\text{div}\, \mathcal{D} = \rho & \text{Gauss electrical equation} \\
\text{div}\, \mathcal{B} = 0 & \text{Gauss magnetic equation}\ .
\end{cases}
\tag{1}
$$

- $\mathcal{H}$ and $\mathcal{E}$ are the magnetic and electric fields, respectively
- $\mathcal{B}$ and $\mathcal{D}$ are the magnetic and electric inductions, respectively
- $\mathcal{J}$ and ρ are the (surface) electric current density and (volume) electric charge density, respectively.

Maxwell equations (cont'd)

These fields are related through some constitutive equations: it is usually assumed a linear dependence like

$$\mathcal{D} = \boldsymbol{\varepsilon}\mathcal{E} \;\;,\;\; \mathcal{B} = \boldsymbol{\mu}\mathcal{H} \;\;,\;\; \mathcal{J} = \boldsymbol{\sigma}\mathcal{E} + \mathcal{J}_e \;,$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are the electric permittivity and magnetic permeability, respectively, and $\boldsymbol{\sigma}$ is the electric conductivity.

In general, $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are not constant, but are symmetric and uniformly positive definite matrices (with entries that are bounded functions of the space variable $\mathbf{x}$). Clearly, the conductivity $\boldsymbol{\sigma}$ is only present in conductors, and is identically vanishing in any insulator.

- $\mathcal{J}_e$ is the applied electric current density.

Eddy current equations

As observed in experiments and stated by the Faraday law, a time-variation of the magnetic field generates an electric field. Therefore, in each conductor a current density $\mathbf{J}_{\text{eddy}} = \boldsymbol{\sigma}\mathbf{E}$ arises; this term expresses the presence in conducting media of the so-called eddy currents.

This phenomenon, and the related heating of the conductor, was observed and studied by the French physicist L. Foucault in the mid of the nineteenth century, and in fact the generated currents are also known as Foucault currents.

Eddy current equations (cont'd)

In many real-life applications, the time of propagation of the electromagnetic waves is very small with respect to some characteristic time scale, or, equivalently, their wave length is much larger than the diameter of the physical domain.

Therefore one can think that the speed of propagation is infinite, and take into account only the diffusion of the electromagnetic fields, neglecting electromagnetic waves.

Rephrasing this concept, one can also say that, when considering time-dependent problems in electromagnetism, one can distinguish between "fast" varying fields and "slowly" varying fields. In the latter case, one is led to simplify the set of equations, neglecting time derivatives, or, depending on the specific situation at hand, one time derivative, either $\frac{\partial \mathcal{D}}{\partial t}$ or $\frac{\partial \mathcal{B}}{\partial t}$.

Eddy current equations (cont’d)

Typically, problems of this type are peculiar of electrical engineering, where low frequencies are involved, but not of electronic engineering, where the frequency ranges in much larger bands.

We focus on the case in which the displacement current term $\frac{\partial \mathcal{D}}{\partial t}$ can be disregarded, while the time-variation of the magnetic induction is still important, as well as the related presence of eddy currents in the conductors.

Eddy current equations (cont'd)

A thumb rule for deciding wheter $\frac{\partial \mathcal{D}}{\partial t}$ can be dropped is the following: if L is a typical length in Ω (say, its diameter), and we choose ω^{-1} as a typical time, it is possibile to disregard the displacement current term provided that

$$|\mathcal{D}||\omega| \ll |\mathcal{H}|L^{-1} \ , \ \ |\mathcal{D}||\omega| \ll |\boldsymbol{\sigma}\mathcal{E}| \, .$$

Using the Faraday equation, we can write $\mathcal{E}$ is terms of $\mathcal{H}$, finding

$$|\mathcal{E}|L^{-1} \approx |\omega||\boldsymbol{\mu}\mathcal{H}| \, .$$

Eddy current equations (cont'd)

Hence, recalling that $\mathcal{D} = \boldsymbol{\varepsilon}\mathcal{E}$ and putting everything together, one should have

$$\mu_{\max}\varepsilon_{\max}\omega^2 L^2 \ll 1 \ \ , \ \ \sigma_{\min}^{-1}\varepsilon_{\max}|\omega| \ll 1 \ ,$$

where $\mu_{\max}$ and $\varepsilon_{\max}$ are uniform upper bounds in Ω for the maximum eigenvalues of $\boldsymbol{\mu}(\mathbf{x})$ and $\boldsymbol{\varepsilon}(\mathbf{x})$, respectively, and $\sigma_{\min}$ denotes a uniform lower bound in Ω_C for the minimum eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$.

Since the magnitude of the velocity of the electromagnetic wave can be estimated by $(\mu_{\max}\varepsilon_{\max})^{-1/2}$, the first relation is requiring that the wave length is large compared to L.

Eddy current equations (cont'd)

Let us also note that for industrial electrical applications some typical values of the parameters involved are $\mu_0 = 4\pi \times 10^{-7}$ H/m, $\varepsilon_0 = 8.9 \times 10^{-12}$ F/m, $\sigma_{\rm copper} = 5.7 \times 10^7$ S/m, $\omega = 2\pi \times 50$ rad/s (power frequency of 50 Hz), hence in that case

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}|\omega|} \approx 10^6 \, \text{m} \ \ , \ \ \sigma^{-1}_{\rm copper}\varepsilon_0|\omega| \approx 4.9 \times 10^{-17} \, ,$$

and dropping the displacement current term looks appropriate.

Though less apparent, the same is true for a typical conductivity in physiological problem, say, $\sigma_{\rm tissue} \approx 10^{-1}$ S/m, for which $\sigma^{-1}_{\rm tissue}\varepsilon_0|\omega| \approx 2.8 \times 10^{-8}$.

Time-harmonic eddy current equations

Starting from the Maxwell equations, assuming a sinusoidal dependence on time and disregarding the displacement current term $\frac{\partial \mathcal{D}}{\partial t}$ one obtains the so-called time-harmonic eddy current equations

$$\begin{cases} \text{curl}\, \mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \text{curl}\, \mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega\,. \end{cases} \quad (2)$$

Here

- $\omega \neq 0$ is the (angular) frequency.

As a consequence one has $\text{div}(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω, and the electric charge in conductors is defined by $\rho = \text{div}(\boldsymbol{\varepsilon}\mathbf{E})$.

Time-harmonic eddy current equations (cont'd)

Since in an insulator one has $\boldsymbol{\sigma} = \mathbf{0}$, it follows that $\mathbf{E}$ is not uniquely determined in that region ($\mathbf{E} + \nabla\psi$ is still a solution).
Some additional conditions ("gauge" conditions) are thus necessary: the most natural idea is to impose the conditions satisfied by the solution $\mathbf{E}^{\varepsilon}$ of the Maxwell equations.
As in the insulator Ω_I we have no charges, the first additional condition is

$$\text{div}(\boldsymbol{\varepsilon}_I \mathbf{E}_I) = 0 \qquad \text{in } \Omega_I \tag{3}$$

($\mathbf{E}_I$ means $\mathbf{E}_{|\Omega_I}$, and similarly for other quantities).

Topological gauge conditions for the electric field

Other gauge conditions are related to the topology of the insulator Ω_I. Denoting by Ω_C the conductor (strictly contained in the physical domain Ω, and surrounded by the insulator Ω_I) and by $\Gamma := \overline{\Omega_C} \cap \overline{\Omega_I}$, let us define

$$\begin{aligned}\mathcal{H}_I := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \text{curl}\,\mathbf{G}_I = \mathbf{0}, \text{div}(\boldsymbol{\varepsilon}_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \mathit{BC}_E(\mathbf{G}_I) = 0 \text{ on } \partial\Omega\}\,,\end{aligned}$$

where BC_E denotes the boundary condition imposed on $\mathbf{E}_I$. The topological gauge conditions can be written as

$$\boldsymbol{\varepsilon}_I \mathbf{E}_I \perp \mathcal{H}_I\,. \tag{4}$$

Topological gauge conditions for the electric field (cont'd)

Thus these conditions are assuring that, if in addition one has curl $\mathbf{E}_I = \mathbf{0}$ in Ω_I, div$(\boldsymbol{\varepsilon}_I \mathbf{E}_I) = 0$ in Ω_I, $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on Γ and $BC_E(\mathbf{E}_I) = 0$ on $\partial\Omega$, then it follows $\mathbf{E}_I = \mathbf{0}$ in Ω_I.

- It can be shown that the orthogonality condition $\boldsymbol{\varepsilon}_I \mathbf{E}_I \perp \mathcal{H}_I$ is equivalent to impose that the flux of $\boldsymbol{\varepsilon}_I \mathbf{E}_I$ is vanishing on a suitable set of surfaces.
  [This set depends on the choice of the boundary condition for $\mathbf{E}_I$; for instance, for $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ the surfaces are the connected components of $\partial\Omega \cup \Gamma$.]

The spaces of harmonic fields

Let us consider a couple of questions.

- If a vector field satisfies curl $\mathbf{v} = \mathbf{0}$ and div $\mathbf{v} = 0$ in a domain, together with the boundary conditions $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on a part of the boundary and $\mathbf{v} \cdot \mathbf{n} = 0$ on the other part, is it non-trivial, namely, not vanishing everywhere in the domain? [A field like that is called harmonic field.]
- If that is the case, do harmonic fields appear in electromagnetism?

Both questions have an affermative answer.

The spaces of harmonic fields (cont'd)

Let us start from the first question.

If the domain $\mathcal{O}$ is homeomorphic to a three-dimensional ball, a curl-free vector field $\mathbf{v}$ must be a gradient of a scalar function ψ, that must be harmonic due to the constraint on the divergence.

If the boundary condition is $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{O}$, which in this case is a connected surface, then it follows $\psi = \text{const.}$ on $\partial\mathcal{O}$, and therefore $\psi = \text{const.}$ in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

The spaces of harmonic fields (cont'd)

If the boundary condition is $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\mathcal{O}$, then ψ satisfies a homogeneous Neumann boundary condition and thus $\psi =$ const. in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

The same result follows if the boundary conditions are $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_N, and Γ_D is a connected surface: in fact, we still have $\psi =$ const. on Γ_D and grad $\psi \cdot \mathbf{n} = 0$ on Γ_N, hence ψ satisfies a mixed boundary value problem and we obtain $\psi =$ const. in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

The spaces of harmonic fields (cont'd)

However, the problem is different in a more general geometry.

In fact, take the magnetic field generated in the vacuum by a current of constant intensity I^0 passing along the x_3-axis: as it is well-known, for $x_1^2 + x_2^2 > 0$ it is given by

$$\mathbf{H}(x_1, x_2, x_3) = \frac{I^0}{2\pi} \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0 \right) \ .$$

The spaces of harmonic fields (cont'd)

It is easily checked that, as Maxwell equations require, $\mathrm{curl}\, \mathbf{H} = \mathbf{0}$ and $\mathrm{div}\, \mathbf{H} = 0$.

Let us consider now the torus $\mathcal{T}$ obtained by rotating around the x_3-axis the disk of centre $(a, 0, 0)$ and radius b, with $0 < b < a$. One sees at once that $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}$; hence we have found a non-trivial harmonic field $\mathbf{H}$ in $\mathcal{T}$ satisfying $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}$.

The spaces of harmonic fields (cont'd)

On the other hand, consider now the electric field generated in the vacuum by a pointwise charge ρ_0 placed at the origin. For $\mathbf{x} \neq \mathbf{0}$ it is given by

$$\mathbf{E}(x_1, x_2, x_3) = \frac{\rho_0}{4\pi\varepsilon_0} \frac{\mathbf{x}}{|\mathbf{x}|^3} ,$$

where ε_0 is the electric permittivity of the vacuum.

It satisfies $\operatorname{div} \mathbf{E} = 0$ and $\operatorname{curl} \mathbf{E} = \mathbf{0}$, and moreover $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on the boundary of $\mathcal{C} := B_{R_2} \setminus \overline{B_{R_1}}$, where $0 < R_1 < R_2$ and $B_R := \{\mathbf{x} \in^3 \mid |\mathbf{x}| < R\}$ is the ball of centre $\mathbf{0}$ and radius R. We have thus found a non-trivial harmonic field $\mathbf{E}$ in $\mathcal{C}$ satisfying $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{C}$.

The spaces of harmonic fields (cont'd)

These two examples show that the geometry of the domain and the type of boundary conditions play an essential role when considering harmonic fields.

What are the relevant differences between the set $\mathcal{O}$, homeomorphic to a ball, and the sets $\mathcal{T}$ and $\mathcal{C}$?

For the former, the point is that in $\mathcal{T}$ we have a non-bounding cycle, namely, a cycle that is not the boundary of a surface contained in $\mathcal{T}$ (take for instance the circle of centre $\mathbf{0}$ and radius a in the (x_1, x_2)-plane).

In the latter case, the boundary of $\mathcal{C}$ is not connected.

The spaces of harmonic fields (cont'd)

Four types of spaces of harmonic fields are coming into play.

- For the electric field

$$\mathcal{H}_I^{(A)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathrm{curl}\, \mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\varepsilon}_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,$$

$$\mathcal{H}_I^{(B)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathrm{curl}\, \mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\varepsilon}_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \boldsymbol{\varepsilon}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,,$$

The spaces of harmonic fields (cont'd)

- For the magnetic field

$$\mathcal{H}_I^{(C)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathrm{curl}\,\mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0$$
$$\boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma, \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,$$

$$\mathcal{H}_I^{(D)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathrm{curl}\,\mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0$$
$$\boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma, \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$$

All are finite dimensional! Their dimension is a topological invariant (precisely,... see below!).

The basis functions of the spaces of harmonic fields

Let us make precise which are the basis functions of $\mathcal{H}_I^{(D)}$ and $\mathcal{H}_I^{(C)}$.

For $\mathcal{H}_I^{(D)}$ one has first to introduce the "cutting" surfaces $\Xi_\alpha^* \subset \Omega_I$, $\alpha = 1, \ldots, n_{\Omega_I}$, with $\partial\Xi_\alpha^* \subset \partial\Omega \cup \Gamma$, such that every curl-free vector field in Ω_I has a global potential in $\Omega_I \setminus \cup_\alpha \Xi_\alpha^*$.

The number n_{Ω_I} is the number of (independent) non-bounding cycles in Ω_I, namely, the first Betti number of Ω_I, or, equivalently, the dimension of the first homology space of Ω_I.

These surfaces "cuts" the non-bounding cycles in Ω_I.

The basis functions of the spaces of harmonic fields (cont'd)

The basis functions $\boldsymbol{\rho}^*_{\alpha,I}$ are the $(L^2(\Omega_I))^3$-extension of grad $p^*_{\alpha,I}$, where $p^*_{\alpha,I}$ is the solution to

$$
\begin{cases}
\operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} p^*_{\alpha,I}) = 0 & \text{in } \Omega_I \setminus \Xi^*_\alpha \\
\boldsymbol{\mu}_I \operatorname{grad} p^*_{\alpha,I} \cdot \mathbf{n}_I = 0 & \text{on } (\partial\Omega \cup \Gamma) \setminus \partial\Xi^*_\alpha \\
\left[\boldsymbol{\mu}_I \operatorname{grad} p^*_{\alpha,I} \cdot \mathbf{n}_{\Xi^*}\right]_{\Xi^*_\alpha} = 0 & \\
\left[p^*_{\alpha,I}\right]_{\Xi^*_\alpha} = 1\ , &
\end{cases}
$$

having denoted by $[\,\cdot\,]_{\Xi^*_\alpha}$ the jump across the surface Ξ^*_α and by $\mathbf{n}_{\Xi^*}$ the unit normal vector on Ξ^*_α.

The basis functions of the spaces of harmonic fields (cont'd)

The basis functions for $\mathcal{H}_I^{(C)}$ can be defined as follows.

First of all we have $\text{grad}\, z_{r,I}$, the solutions to

$$\begin{cases} \text{div}(\boldsymbol{\mu}_I \,\text{grad}\, z_{r,I}) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \,\text{grad}\, z_{r,I} \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\ z_{r,I} = 0 & \text{on } \partial\Omega \setminus (\partial\Omega)_r \\ z_{r,I} = 1 & \text{on } (\partial\Omega)_r \ , \end{cases}$$

where $r = 1, \ldots, p_{\partial\Omega}$, and $p_{\partial\Omega} + 1$ is the number of connected components of $\partial\Omega$.

The basis functions of the spaces of harmonic fields (cont'd)

To complete the construction of the basis functions we have to proceed further.
For that, as in the preceding case, let us recall that in Ω_I there exist a set of "cutting" surfaces Ξ_l, with $\partial\Xi_l \subset \Gamma$, such that every curl-free vector field in Ω_I with vanishing tangential component on $\partial\Omega$ has a global potential in $\Omega_I \setminus \cup_l \Xi_l$.

These surfaces "cuts" the $\partial\Omega$-independent non-bounding cycles in Ω_I (whose number is denoted by n_Γ).

The basis functions of the spaces of harmonic fields (cont'd)

Then introduce the functions $p_{l,I}$, defined in $\Omega_I \setminus \Xi_l$ and solutions to

$$\begin{cases} \text{div}(\boldsymbol{\mu}_I \,\text{grad}\, p_{l,I}) = 0 & \text{in } \Omega_I \setminus \Xi_l \\ \boldsymbol{\mu}_I \,\text{grad}\, p_{l,I} \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \setminus \partial\Xi_l \\ p_{l,I} = 0 & \text{on } \partial\Omega \\ \left[\boldsymbol{\mu}_I \,\text{grad}\, p_{l,I} \cdot \mathbf{n}_\Xi\right]_{\Xi_l} = 0 & \\ \left[p_{l,I}\right]_{\Xi_l} = 1 \,, & \end{cases}$$

having denoted by $[\,\cdot\,]_{\Xi_l}$ the jump across the surface Ξ_l and by $\mathbf{n}_\Xi$ the unit normal vector on Ξ_l.

The other basis functions $\boldsymbol{\rho}_{l,I}$ are the $(L^2(\Omega_I))^3$-extension of $\text{grad}\, p_{l,I}$ (computed in $\Omega_I \setminus \Xi_l$).

Boundary conditions

We will distinguish among two types of boundary conditions.

- Electric. One imposes $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. [As a consequence, one also has $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$.]
- Magnetic. One imposes $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

The notation $\mathit{BC}_E(\mathbf{E}_I)$ on $\partial\Omega$ therefore refers to $\mathbf{E}_I \times \mathbf{n}$ for the electric boundary condition, and to $\boldsymbol{\varepsilon}_I\mathbf{E}_I \cdot \mathbf{n}$ for the magnetic boundary conditions.

[A third type of boundary conditions can be considered:

- No-flux [Bossavit (2000)]. One imposes $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ and $\boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

We will not dwell on these boundary conditions in the sequel.]

$\mathbf{E}$ and $\mathbf{H}$ formulations

As for the Maxwell equations, the problem can be formulated in terms of $\mathbf{E}$ or $\mathbf{H}$ only.

- $\mathbf{E}$ formulation

$$\begin{cases} \text{curl}(\boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{E}) + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{J}_e & \text{in } \Omega \\ \text{div}(\boldsymbol{\varepsilon}_I\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ BC_E(\mathbf{E}_I) = 0 & \text{on } \partial\Omega \\ \boldsymbol{\varepsilon}_I\mathbf{E}_I \perp \mathcal{H}_I \end{cases} \quad (5)$$

[where the condition $\boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{E}\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$ has to be dropped if considering the electric boundary condition].

$\mathbf{E}$ and $\mathbf{H}$ formulations (cont'd)

Once the electric field $\mathbf{E}$ is available, one sets

$$\mathbf{H} = i\omega^{-1}\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{E} \qquad \text{in } \Omega\,,$$

and the complete solution is recovered.

E and H formulations (cont'd)

- **H formulation**

$$
\begin{cases}
\text{curl}(\boldsymbol{\sigma}^{-1}\,\text{curl}\,\mathbf{H}_C) + i\omega\boldsymbol{\mu}_C\mathbf{H}_C & \\
\qquad\qquad = \text{curl}(\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}) & \text{in } \Omega_C \\
\text{curl}\,\mathbf{H}_I = \mathbf{J}_{e,I} & \text{in } \Omega_I \\
\text{div}(\boldsymbol{\mu}\mathbf{H}) = 0 & \text{in } \Omega \\
BC_H(\mathbf{H}_I) = 0 & \text{on } \partial\Omega \\
\mathbf{H}_I \times \mathbf{n}_I + \mathbf{H}_C \times \mathbf{n}_C = \mathbf{0} & \text{on } \Gamma \\
TOP(\mathbf{H}) = 0\ , &
\end{cases}
\tag{6}
$$

where $BC_H(\mathbf{H}_I)$ means $\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n}$ for the electric boundary condition, and $\mathbf{H}_I \times \mathbf{n}$ for the magnetic boundary conditions, and $TOP(\mathbf{H}) = 0$ is a set of topological conditions that have to be satisfied by the magnetic field $\mathbf{H}$.

$\mathbf{E}$ and $\mathbf{H}$ formulations (cont'd)

Having determined $\mathbf{H}$, the electric field is obtained by setting

$$\mathbf{E}_C = \boldsymbol{\sigma}^{-1}(\text{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C}) \qquad \text{in } \Omega_C \,,$$

and solving the problem

$$\begin{cases} \text{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}_I\mathbf{H}_I & \text{in } \Omega_I \\ \text{div}(\boldsymbol{\varepsilon}_I\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ BC_E(\mathbf{E}_I) = 0 & \text{on } \partial\Omega \\ \mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C & \text{on } \Gamma \\ \boldsymbol{\varepsilon}_I\mathbf{E}_I \perp \mathcal{H}_I \,. & \end{cases}$$

This last problem is not always solvable, but needs that some compatibility conditions on the data are satisfied.

Topological conditions on the magnetic field

Besides the conditions $\text{div}(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω and $\boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$ (if $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$), that are clearly satisfied, it is important to underline that the other needed compatibility conditions are the topological conditions $\mathit{TOP}(\mathbf{H}) = 0$.

Let us make clear their structure. For the sake of definiteness, let us focus on the electric boundary condition. We need to consider again the (finite dimensional) space

$$\mathcal{H}_I^{(D)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \text{curl}\, \mathbf{G}_I = \mathbf{0}, \text{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0 \\ \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \cup \Gamma\}\,,$$

and its basis functions $\boldsymbol{\rho}^*_{\alpha,I}$, $\alpha = 1, \ldots, n_{\Omega_I}$ [let us recall that n_{Ω_I} is the first Betti number of Ω_I, or, equivalently, the number of (independent) non-bounding cycles in Ω_I].

Topological conditions on the magnetic field (cont'd)

The topological conditions *TOP*$(\mathbf{H}) = 0$ mean that

$$\begin{aligned}\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}^*_{\alpha,I} \\ + \int_\Gamma [\boldsymbol{\sigma}^{-1}(\operatorname{curl} \mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot \boldsymbol{\rho}^*_{\alpha,I} = 0\end{aligned} \tag{7}$$

for each $\alpha = 1, \ldots, n_{\Omega_I}$.

Note that one has $n_{\Omega_I} \geq 1$ if the conductor Ω_C is not simply-connected, and therefore in that case these conditions have to be taken into account.

- It can be proved that the topological conditions *TOP*$(\mathbf{H}) = 0$ are equivalent to the integral form of the Faraday equation on each surface that "cuts" a non-bounding cycle [Seifert surface].

Don't forget the Faraday equation!

Instead of proving this statement, let us change our point of view and show that, if *TOP*$(\mathbf{H})$ are not imposed, the Faraday equation is not completely solved.

Since we have imposed the Faraday equation in Ω_C and the electric field $\mathbf{E}_I$ will be determined by solving the Faraday equation in Ω_I (with $\mathbf{H}_I$ already known), it really seems that everything is all right...

But, as already remarked, finding $\mathbf{E}_I$ is possible only if some compatibility conditions are satisfied!

Thus let us see in more detail: the Faraday equation relates the flux of the magnetic induction through a surface with the line integral of the electric field on the boundary of that surface.

Don't forget the Faraday equation! (cont'd)

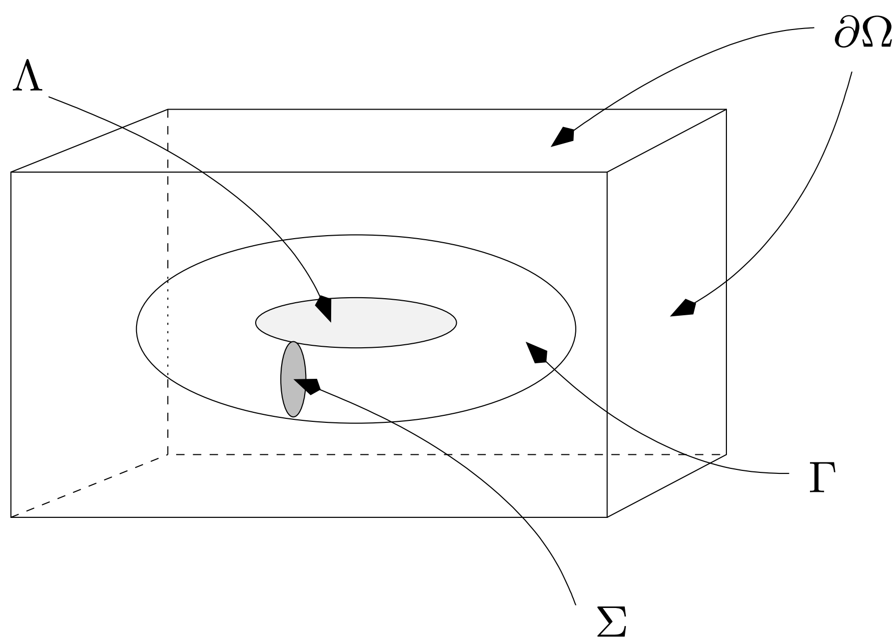
Since we know the magnetic field in the whole Ω, surfaces can stay everywhere in Ω; but, before determining $\mathbf{E}_I$, we know the electric field only in Ω_C, therefore the boundary of the surface must stay in $\overline{\Omega_C}$.

On the other hand, the Faraday equation (in differential form) is satisfied in Ω_C, therefore for a surface contained in Ω_C everything is all right.

Thus we must verify if there are surfaces in Ω_I with boundary on Γ, and moreover such that this boundary is not the boundary of a surface in Ω_C [if this is not the case, the Divergence Theorem says that again everything is all right, as the magnetic induction is divergence free in Ω...].

Don't forget the Faraday equation! (cont'd)

- Conclusion: the Faraday equation has not been imposed on the "cutting" surface Λ! [The non-bounding cycle is the boundary of the surface Σ.]



Weak formulations

Let us come back to our eddy current problems.

Looking at the $\mathbf{E}$-formulation (5) and the $\mathbf{H}$-formulation (6) one sees that they have not a simple structure, and that a degeneration occurs where $\boldsymbol{\sigma}$ is vanishing (namely, in the insulator Ω_I).

The constraints on the divergence should balance in some way the degeneration of the operator: but it does not look so trivial to take into account this fact.

However, passing to weak formulations permits to show the well-posedness of eddy current problems.

Weak H-formulation

First of all, under the necessary conditions

$$\begin{aligned} &\text{div}\, \mathbf{J}_{e,I} = 0 \quad \text{in } \Omega_I \\ &\mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \\ &\mathbf{J}_{e,I} \perp \mathcal{H}_I \,, \end{aligned}$$

it can be shown that there exists a vector field $\mathbf{H}_e \in H(\text{curl}; \Omega)$ satisfying

$$\begin{cases} \text{curl}\, \mathbf{H}_{e,I} = \mathbf{J}_{e,I} & \text{in } \Omega_I \\ BC_H(\mathbf{H}_{e,I}) = 0 & \text{on } \partial\Omega \end{cases}$$

[the boundary conditions for $\mathbf{J}_{e,I}$ and $\mathbf{H}_{e,I}$ have to be dropped if considering the electric boundary condition].

Weak H-formulation (cont'd)

Setting

$$V := \{\mathbf{v} \in H(\text{curl};\Omega) \mid \text{curl}\,\mathbf{v}_I = \mathbf{0} \text{ in } \Omega_I, \mathbf{v}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$$

[the boundary condition has to be dropped if considering the electric boundary condition], multiplying the Faraday equation by $\overline{\mathbf{v}}$, with $\mathbf{v} \in V$, integrating in Ω and integrating by parts one finds

$$\int_{\Omega_C} \mathbf{E}_C \cdot \text{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega_I} \mathbf{E}_I \cdot \text{curl}\,\overline{\mathbf{v}_I} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{v}} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} = 0 \ ,$$

thus

$$\int_{\Omega_C} \mathbf{E}_C \cdot \text{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} = 0 \ ,$$

as $\text{curl}\,\mathbf{v}_I = \mathbf{0}$ in Ω_I.

Weak H-formulation (cont'd)

Using the Ampère equation in Ω_C for expressing $\mathbf{E}_C$, we end up with the following problem

Find $(\mathbf{H} - \mathbf{H}_e) \in V$:

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \,\mathrm{curl}\, \mathbf{H}_C \cdot \mathrm{curl}\, \overline{\mathbf{v}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{v}} = \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \mathrm{curl}\, \overline{\mathbf{v}_C} \qquad (8)$$

for each $\mathbf{v} \in V$.

This formulation is well-posed via the Lax–Milgram lemma, as the sesquilinear form

$$a(\mathbf{u}, \mathbf{v}) := \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \,\mathrm{curl}\, \mathbf{u}_C \cdot \mathrm{curl}\, \overline{\mathbf{v}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{u} \cdot \overline{\mathbf{v}}$$

is clearly continuous and coercive in V.

Weak E-formulation

For deriving the weak $\mathbf{E}$-formulation one starts from the Ampère equation: multiplying by $\overline{\mathbf{z}}$, integrating in Ω and integrating by parts one easily sees that

$$\int_\Omega \mathbf{H} \cdot \text{curl}\, \overline{\mathbf{z}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{H} \cdot \overline{\mathbf{z}} - \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}}$$

for all $\mathbf{z} \in H(\text{curl}; \Omega)$.

The boundary term disappears if $\mathbf{H}$ satisfies the magnetic boundary condition, or if $\mathbf{z}$ satisfies the electric boundary condition.

Set

$$Z := \{\mathbf{z} \in H(\text{curl}; \Omega) \mid \text{div}(\boldsymbol{\varepsilon}_I \mathbf{z}_I) = 0 \text{ in } \Omega_I, \\ BC_E(\mathbf{z}_I) = 0,\ \boldsymbol{\varepsilon}_I \mathbf{z}_I \perp \mathcal{H}_I\} .$$

Weak E-formulation (cont'd)

Expressing $\mathbf{H}$ through the Faraday equation, the weak $\mathbf{E}$-formulation finally reads

Find $\mathbf{E} \in Z$:

$$\int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{E} \cdot \operatorname{curl} \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} = -i\omega \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}} \quad (9)$$

for each $\mathbf{z} \in Z$.

Though less straightforward, it can be proved that the sesquilinear form

$$a_e(\mathbf{w}, \mathbf{z}) \quad := \int_\Omega \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{w} \cdot \operatorname{curl} \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{z}_C}$$

is continuous and coercive in Z, and well-posedness of the weak $\mathbf{E}$-formulation follows from Lax–Milgram lemma.

From the weak to the strong formulations

Since we have proved well-posedness for the weak problems (8) and (9), in order to prove that the eddy current problem is completely solved it is necessary to show that (5) or (6) are satisfied.

The easiest case is the proof that (5) holds. For that, it is enough to choose suitable test functions $\mathbf{v}$ in (8).

For the sake of definiteness, let us consider the electric boundary case.

From the weak to the strong formulations (cont'd)

- Taking as test function $\mathbf{v} = \operatorname{grad}\varphi$ it follows $\operatorname{div}(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω and $\boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$.
- Taking as test function $\mathbf{v}$ with compact support in Ω_C one finds $\operatorname{curl}(\boldsymbol{\sigma}^{-1} \operatorname{curl}\mathbf{H}_C) + i\omega\boldsymbol{\mu}_C\mathbf{H}_C = \operatorname{curl}(\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C})$ in Ω_C.
- Taking as test function $\mathbf{v}$ such that $\mathbf{v}_I = \boldsymbol{\rho}^*_{\alpha,I}$ in Ω_I gives

$$
\begin{aligned}
\int_{\Omega_I} i\omega\boldsymbol{\mu}_I\mathbf{H}_I \cdot \boldsymbol{\rho}^*_{\alpha,I} &= -\int_{\Omega_C} i\omega\boldsymbol{\mu}_C\mathbf{H}_C \cdot \overline{\mathbf{v}_C} \\
&\quad + \int_{\Omega_C} \boldsymbol{\sigma}^{-1}(\mathbf{J}_{e,C} - \operatorname{curl}\mathbf{H}_C) \cdot \operatorname{curl}\overline{\mathbf{v}_C} \\
&= \int_\Gamma \boldsymbol{\sigma}^{-1}(\mathbf{J}_{e,C} - \operatorname{curl}\mathbf{H}_C) \cdot (\mathbf{n}_C \times \overline{\mathbf{v}_C}) \\
&= \int_\Gamma \boldsymbol{\sigma}^{-1}(\mathbf{J}_{e,C} - \operatorname{curl}\mathbf{H}_C) \cdot (\mathbf{n}_C \times \boldsymbol{\rho}^*_{\alpha,I})\,,
\end{aligned}
$$

namely, $\mathit{TOP}(\mathbf{H}) = 0$.

Numerical approximation

Both problems (8) and (9) contain a differential constraint: the former on the curl, the latter on the divergence.

- Numerical approximation needs some care!

Possible ways of attack:

- saddle-point formulations [Lagrange multipliers]
- a scalar potential for $\mathbf{H}_I - \mathbf{H}_{e,I}$
- a vector potential for $\varepsilon_I \mathbf{E}_I$.

Numerical approximation (cont'd)

The first choice has been considered by Alonso Rodríguez, Hiptmair and V. (2004) (for the magnetic field) and by Alonso Rodríguez and V. (2004) (for the electric field); hybrid formulations in terms of $(\mathbf{H}_C, \mathbf{E}_I)$ or $(\mathbf{E}_C, \mathbf{H}_I)$ have been also proposed and analyzed (Alonso Rodríguez, Hiptmair and V. (2004, 2005)).

The second possibility will be described here below.

To our knowledge, the third choice has not been completely exploited. A possible modification is to look for a vector potential for $\mu\mathbf{H}$: this (classical) approach will be illustrated in the following.

Scalar potential formulation

Again, for the sake of definiteness let us consider the electric boundary condition.

The starting point is to consider $\mathbf{H}_e \in H(\text{curl}; \Omega)$ satisfying

$$\text{curl}\, \mathbf{H}_{e,I} = \mathbf{J}_{e,I} \qquad \text{in } \Omega_I \, .$$

Then the main step is to use the orthogonal decomposition

$$\mathbf{H}_I - \mathbf{H}_{e,I} = \text{grad}\, \psi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \ , \tag{10}$$

where $\psi_I^* \in H^1(\Omega_I)/\mathbb{C}$ and $\eta_{I,\alpha}^* \in \mathbb{C}$ (the two terms of the decomposition are orthogonal, with respect to the scalar product $(\mathbf{u}_I, \mathbf{v}_I)_{\mu_I,\Omega_I} := \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{u}_I \cdot \mathbf{v}_I$).

Orthogonal decompositions

There are infinitely many of these decomposition results...

Let us recall the two that are interesting for the magnetic field:

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1} \, \text{curl} \, \mathbf{Q}_I^* + \text{grad} \, \chi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \theta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^*$$

and

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1} \, \text{curl} \, \mathbf{Q}_I + \text{grad} \, \chi_I + \sum_{r=1}^{p_{\partial\Omega}} a_{I,r} \, \text{grad} \, z_{r,I} + \sum_{l=1}^{n_\Gamma} b_{I,l} \boldsymbol{\rho}_{l,I} \; .$$

Orthogonal decompositions (cont'd)

Let us explain the first decomposition.

The vector function $\mathbf{Q}_I^*$ is the solution to

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{Q}_I^*) = \operatorname{curl} \mathbf{v}_I & \text{in } \Omega_I \\ \operatorname{div} \mathbf{Q}_I^* = 0 & \text{in } \Omega_I \\ \mathbf{Q}_I^* \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \cup \partial\Omega \\ \mathbf{Q}_I^* \perp \mathcal{H}_{I,\varepsilon_0}^{(A)} & \end{cases}$$

[$\mathcal{H}_{I,\varepsilon_0}^{(A)}$ denotes $\mathcal{H}_I^{(A)}$ for $\boldsymbol{\varepsilon}_I = \varepsilon_0$, a positive constant].
The scalar function χ_I^* is the solution to the elliptic Neumann boundary value problem

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}_I \operatorname{grad} \chi_I^*) = \operatorname{div}(\boldsymbol{\mu}_I \mathbf{v}_I) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \operatorname{grad} \chi_I^* \cdot \mathbf{n}_I = \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I & \text{on } \Gamma \cup \partial\Omega\,. \end{cases}$$

Orthogonal decompositions (cont'd)

Finally the vector $\theta^*_{I,\alpha}$ is the solution of the linear system

$$\sum_{\alpha=1}^{n_{\Omega_I}} A^*_{\beta\alpha}\theta^*_{I,\alpha} = \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \boldsymbol{\rho}^*_{\beta,I} \ ,$$

where

$$A^*_{\beta\alpha} := \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}^*_{\alpha,I} \cdot \boldsymbol{\rho}^*_{\beta,I} \ ,$$

and the harmonic vector fields $\boldsymbol{\rho}^*_{\alpha,I}$ are the basis functions of the space $\mathcal{H}_I^{(D)}$.

Orthogonal decompositions (cont'd)

Let us explain the second decomposition.

The vector function $\mathbf{Q}_I$ is the solution to

$$\begin{cases} \text{curl}(\boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{Q}_I) = \text{curl}\,\mathbf{v}_I & \text{in } \Omega_I \\ \text{div}\,\mathbf{Q}_I = 0 & \text{in } \Omega_I \\ \mathbf{Q}_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\ \mathbf{Q}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ (\boldsymbol{\mu}_I^{-1}\,\text{curl}\,\mathbf{Q}_I) \times \mathbf{n} = \mathbf{v}_I \times \mathbf{n} & \text{on } \partial\Omega \\ \mathbf{Q}_I \perp \mathcal{H}^{(B)}_{I,\varepsilon_0} & \end{cases}$$

[$\mathcal{H}^{(B)}_{I,\varepsilon_0}$ denotes $\mathcal{H}^{(B)}_I$ for $\boldsymbol{\varepsilon}_I = \varepsilon_0$, a positive constant].

Orthogonal decompositions (cont'd)

The scalar function χ_I is the solution to the elliptic mixed boundary value problem

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I \,\mathrm{grad}\, \chi_I) = \mathrm{div}(\boldsymbol{\mu}_I \mathbf{v}_I) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \,\mathrm{grad}\, \chi_I \cdot \mathbf{n}_I = \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathbf{n}_I & \text{on } \Gamma \\ \chi_I = 0 & \text{on } \partial\Omega \,. \end{cases}$$

Finally the vector $(a_{I,r}, b_{I,l})$ is the solution of the linear system

$$A \begin{pmatrix} a_{I,r} \\ b_{I,l} \end{pmatrix} = \begin{pmatrix} \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \mathrm{grad}\, z_{s,I} \\ \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \boldsymbol{\rho}_{m,I} \end{pmatrix} \,,$$

Orthogonal decompositions (cont'd)

where $A := \begin{pmatrix} D & B \\ B^T & C \end{pmatrix}$ with

$$
\begin{aligned}
D_{sr} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \, \mathrm{grad}\, z_{r,I} \cdot \mathrm{grad}\, z_{s,I} \\
B_{sl} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_{l,I} \cdot \mathrm{grad}\, z_{s,I} \\
C_{ml} &:= \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_{l,I} \cdot \boldsymbol{\rho}_{m,I} \ ,
\end{aligned}
$$

and the harmonic vector fields $\mathrm{grad}\, z_{r,I}$ and $\boldsymbol{\rho}_{l,I}$ are the basis functions of the space $\mathcal{H}_I^{(C)}$.

Scalar potential formulation (cont'd)

Coming back to the scalar potential formulation, in (8) each test function $\mathbf{v} \in V$ can be thus written as

$$\mathbf{v}_I = \text{grad}\, \chi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \theta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \,. \tag{11}$$

Inserting (10) and (11) in (8) and using orthogonality one easily finds, for the unknowns $\mathbf{Z}_C := \mathbf{H}_C - \mathbf{H}_{e,C}$, ψ_I^*, $\eta_{I,\alpha}^*$,

$$\begin{aligned} &\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \text{curl}\, \mathbf{Z}_C \cdot \text{curl}\, \overline{\mathbf{v}_C} + \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{Z}_C \cdot \overline{\mathbf{v}_C} \\ &\qquad + \int_{\Omega_I} i\omega \boldsymbol{\mu}_I\, \text{grad}\, \psi_I^* \cdot \text{grad}\, \overline{\chi_I^*} + i\omega [A^* \boldsymbol{\eta}_I^*, \boldsymbol{\theta}_I^*] \\ &= - \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \text{curl}\, \mathbf{H}_{e,C} \cdot \text{curl}\, \overline{\mathbf{v}_C} - \int_{\Omega_C} i\omega \boldsymbol{\mu}_C \mathbf{H}_{e,C} \cdot \overline{\mathbf{v}_C} \\ &\qquad - \int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot (\text{grad}\, \overline{\chi_I^*} + \textstyle\sum_{\alpha=1}^{n_{\Omega_I}} \overline{\theta_{I,\alpha}^*} \boldsymbol{\rho}_{\alpha,I}^*) \\ &\qquad + \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \text{curl}\, \overline{\mathbf{v}_C} \,, \end{aligned} \tag{12}$$

Scalar potential formulation (cont'd)

where we recall that the matrix A^* is defined by

$$A^*_{\alpha\beta} := \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}^*_{\alpha,I} \cdot \boldsymbol{\rho}^*_{\beta,I} \,,$$

and is symmetric and positive definite (the fields $\boldsymbol{\rho}^*_{\alpha,I}$ form a basis for the space $\mathcal{H}^*_I$).
Clearly, the solutions $\mathbf{Z}_C$, ψ^*_I and $\boldsymbol{\eta}^*_I$ have to satisfy on Γ the matching condition

$$\mathbf{Z}_C \times \mathbf{n}_C + \text{grad}\, \psi^*_I \times \mathbf{n}_I + \sum_{\alpha=1}^{n_{\Omega_I}} \eta^*_{I,\alpha} \boldsymbol{\rho}^*_{\alpha,I} \times \mathbf{n}_I = \mathbf{0}\,.$$

The same holds for the test functions $\mathbf{v}_C$, χ^*_I and $\boldsymbol{\theta}^*_I$.

Scalar potential formulation (cont'd)

The left hand side in (12) is a continuous and coercive sesquilinear form, therefore the problem is well-posed.

The numerical approximation is standard:

- (vector) edge finite elements in Ω_C
- (scalar) nodal finite elements in Ω_I.

In addition, one looks for

- other n_{Ω_I} degrees of freedom (expressing the line integrals of $\mathbf{H}_I - \mathbf{H}_{e,I}$ along the non-bounding cycles contained in $\overline{\Omega_I}$).

Convergence is assured by Céa lemma.

[Bermúdez, Rodríguez and Salgado (2002), Alonso Rodríguez, Fernandes and V. (2003).]

Scalar potential formulation (cont'd)

Some remarks about implementation issues:

- The matching condition on the interface Γ is easily imposed by eliminating the degrees of freedom of $\mathbf{v}_{C,h}$ associated to the edges and faces on Γ in terms of those of grad $\chi^*_{I,h} + \sum_{\alpha=1}^{n_{\Omega_I}} \theta^*_{I,\alpha} \boldsymbol{\rho}^*_{\alpha,I}$.
- The construction of the fields $\boldsymbol{\rho}^*_{\alpha,I}$ (or of a suitable approximation of them) is not needed.
  It is enough to construct n_{Ω_I} interpolants $\boldsymbol{\lambda}^*_{\alpha}$, each one jumping by 1 on a "cutting" surface (and continuous across all the others).
  One looses (in part) orthogonality properties, but everything works well.

Scalar potential formulation (cont'd)

- For the electric boundary condition, the construction of the vector $\mathbf{H}_{e,I}$ can be done through the Biot–Savart formula

$$\begin{aligned} \mathbf{H}_{e,I}(\mathbf{x}) &:= \operatorname{curl}\left(\int_{\Omega_I} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \mathbf{J}_{e,I}(\mathbf{y})\, d\mathbf{y} \right) \\ &= \int_{\Omega_I} \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \times \mathbf{J}_{e,I}(\mathbf{y})\, d\mathbf{y} \end{aligned}$$

[at least for $\mathbf{J}_{e,I} \cdot \mathbf{n} = 0$ on $\partial\Omega \cup \Gamma$; if this is not satisfied, one has to extend $\mathbf{J}_{e,I}$ on a set larger than Ω_I, in such a way that $\mathbf{J}_{e,I}$ is tangential on the boundary of this set].

Scalar potential formulation (cont'd)

- When considering the magnetic boundary condition, it must be noted that the Biot–Savart formula gives a vector field $\mathbf{H}_{e,I}$ that does not satisfy the boundary condition $\mathbf{H}_{e,I} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.
  Then, a couple of procedures can be adopted:
  - construct $\mathbf{H}_{e,I}$ (or a suitable approximation of it) by means of a different approach, in such a way that $\mathbf{H}_{e,I} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, and decompose $\mathbf{H}_I - \mathbf{H}_{e,I}$ as a sum of orthogonal terms, each one with vanishing tangential value on $\partial\Omega$
  - use again the Biot–Savart formula, and decompose $\mathbf{H}_I - \mathbf{H}_{e,I}$ as in the case of the electric boundary condition.

Scalar potential formulation (cont'd)

Let us illustrate this second approach: we again write

$$\mathbf{Z}_I = \mathbf{H}_I - \mathbf{H}_{e,I} = \operatorname{grad} \psi_I^* + \sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \ ,$$

but now we have to consider a non-homogeneous boundary value problem (on $\partial\Omega$ we have $\mathbf{Z}_I \times \mathbf{n} \neq \mathbf{0}$).

The problem reads as follows: one looks for $\mathbf{Z}_C$, ψ_I^*, $\boldsymbol{\eta}_I^*$ such that

Scalar potential formulation (cont'd)

$$
\begin{aligned}
&\text{grad}\,\psi_I^* \times \mathbf{n} + \textstyle\sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^*\, \boldsymbol{\rho}_{\alpha,I}^* \times \mathbf{n} = -\mathbf{H}_{e,I} \times \mathbf{n} \text{ on } \partial\Omega \\
&\textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \text{curl}\,\mathbf{Z}_C \cdot \text{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega_C} i\omega\boldsymbol{\mu}_C \mathbf{Z}_C \cdot \overline{\mathbf{v}_C} \\
&\qquad + \textstyle\int_{\Omega_I} i\omega\boldsymbol{\mu}_I\, \text{grad}\,\psi_I^* \cdot \text{grad}\,\overline{\chi_I^*} + i\omega[A^*\boldsymbol{\eta}_I^*, \boldsymbol{\theta}_I^*] \\
&\quad = -\textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\, \text{curl}\,\mathbf{H}_{e,C} \cdot \text{curl}\,\overline{\mathbf{v}_C} - \int_{\Omega_C} i\omega\boldsymbol{\mu}_C \mathbf{H}_{e,C} \cdot \overline{\mathbf{v}_C} \\
&\qquad - \textstyle\int_{\Omega_I} i\omega\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot (\text{grad}\,\overline{\chi_I^*} + \sum_{\alpha=1}^{n_{\Omega_I}} \overline{\theta_{I,\alpha}^*} \boldsymbol{\rho}_{\alpha,I}^*) \\
&\qquad + \textstyle\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \text{curl}\,\overline{\mathbf{v}_C}\,,
\end{aligned}
\tag{13}
$$

where the test functions have to satisfy

$$
\text{grad}\,\chi_I^* \times \mathbf{n} + \sum_{\alpha=1}^{n_{\Omega_I}} \theta_{I,\alpha}^*\, \boldsymbol{\rho}_{\alpha,I}^* \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\,,
$$

Scalar potential formulation (cont'd)

and moreover the matching condition on Γ

$$\mathbf{Z}_C \times \mathbf{n}_C + \text{grad}\, \psi_I^* \times \mathbf{n}_I + \sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \times \mathbf{n}_I = \mathbf{0}$$

is still imposed (also for $\mathbf{v}_C$, χ_I^*, $\boldsymbol{\theta}_I^*$).

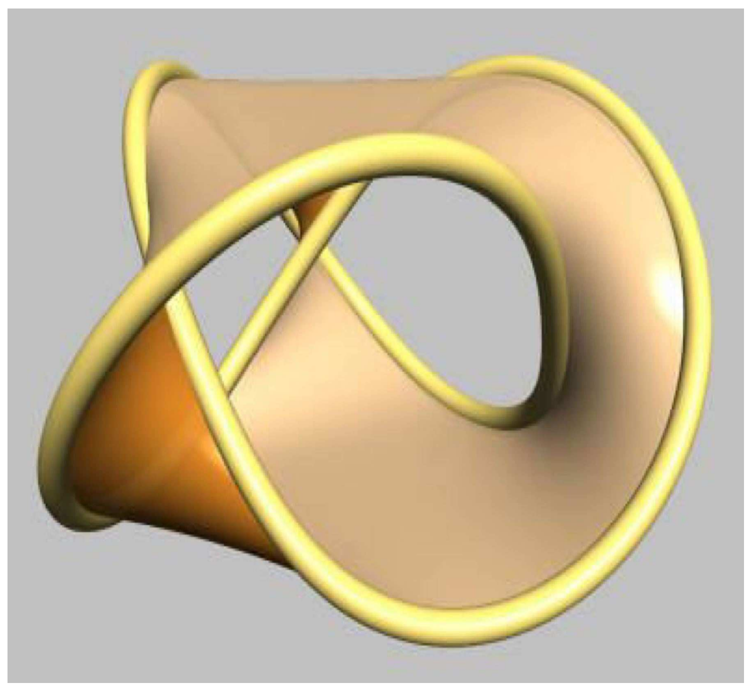
At the finite dimensional level the constraint on $\partial\Omega$ can be imposed by means of a Lagrange multiplier [Bermúdez, Rodríguez and Salgado (2002)].

Scalar potential formulation (cont'd)

- For implementation it is necessary to determine the "cutting" surfaces of the non-bounding cycles (their knowledge is necessary for constructing the basis functions $\boldsymbol{\rho}^*_{\alpha,I}$ or the interpolants $\boldsymbol{\lambda}^*_{\alpha}$).
  This can be easy in many situations, but for a general topological domain it can be computationally expensive: here below you see the "cutting" surface when Ω_C is the trifoil knot (thanks to J.J. van Wijk).

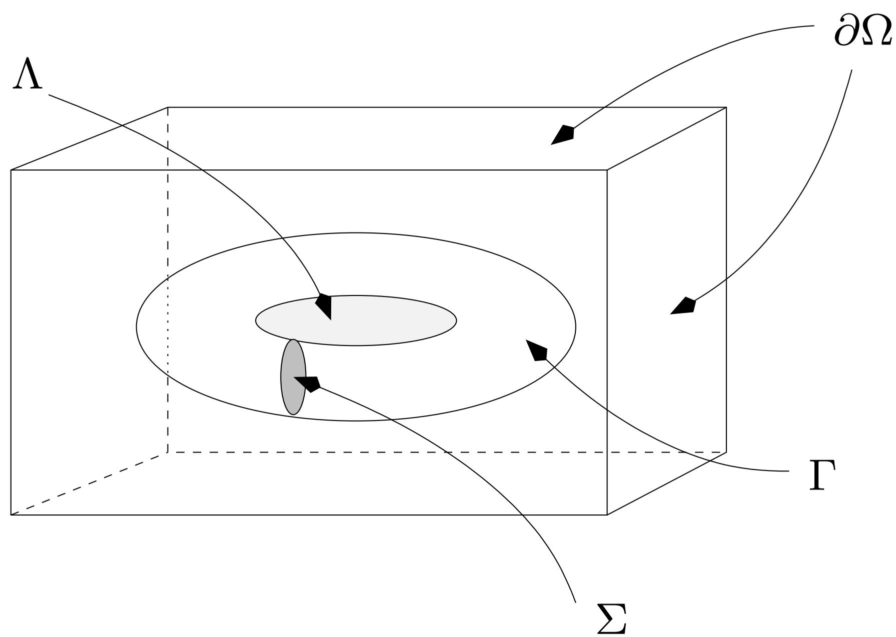
Scalar potential formulation (cont'd)

- For implementation it is necessary to determine the "cutting" surfaces of the non-bounding cycles (their knowledge is necessary for constructing the basis functions $\boldsymbol{\rho}^*_{\alpha,I}$ or the interpolants $\boldsymbol{\lambda}^*_\alpha$).
  This can be easy in many situations, but for a general topological domain it can be computationally expensive: here below you see the "cutting" surface when Ω_C is the trifoil knot (thanks to J.J. van Wijk).



Scalar potential formulation (cont'd)

Instead, if Ω_C is a torus, we have the "cutting" surface Λ:



Some algorithms have been proposed to the aim of constructing "cutting" surfaces: see Kotiuga (1987, 1988, 1989), Leonard and Rodger (1989) and the book by Gross and Kotiuga (2004).

Scalar potential formulation (cont'd)

- A formulation in terms of $\mathbf{E}_C$, ψ_I^* and $\boldsymbol{\eta}_I^*$ is also possible.

  From the Ampère equation in Ω_C, multiplying by $\overline{\mathbf{z}_C}$, integrating in Ω_C and integrating by parts one finds

$$\begin{aligned}\int_{\Omega_C} \mathbf{H}_C \cdot \operatorname{curl} \overline{\mathbf{z}_C} + \int_\Gamma \mathbf{n}_C \times \mathbf{H}_C \cdot \overline{\mathbf{z}_C} - \int_{\Omega_C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\ = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \,.\end{aligned}$$

  Using the Faraday equation for expressing $\mathbf{H}_C$ and recalling that $\mathbf{n}_C \times \mathbf{H}_C = \mathbf{n}_C \times \mathbf{H}_I$ on Γ, it holds

$$\begin{aligned}\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{E}_C \cdot \operatorname{curl} \overline{\mathbf{z}_C} + i\omega\boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{z}_C}) \\ + i\omega \int_\Gamma \mathbf{H}_I \times \mathbf{n}_C \cdot \overline{\mathbf{z}_C} = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \,.\end{aligned}$$

Scalar potential formulation (cont'd)

On the other hand, multiplying the Faraday equation in Ω_I by a test function $\overline{\mathbf{v}_I}$ such that $\text{curl}\,\mathbf{v}_I = \mathbf{0}$ in Ω_I and recalling that $\mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C$ on Γ, by integration by parts one has

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \overline{\mathbf{v}_I} = -\int_{\Omega_I} \text{curl}\,\mathbf{E}_I \cdot \overline{\mathbf{v}_I} = -\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \overline{\mathbf{v}_I}\,.$$

Setting

$$V_I(\mathbf{G}) := \{\mathbf{v}_I \in H(\text{curl};\Omega_I) \mid \text{curl}\,\mathbf{v}_I = \mathbf{G} \text{ in } \Omega_I\}\,,$$

we are thus looking for $\mathbf{E}_C \in H(\text{curl};\Omega_C)$ and $\mathbf{H}_I \in V_I(\mathbf{J}_{e,I})$ such that

Scalar potential formulation (cont'd)

$$\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \,\mathrm{curl}\, \mathbf{E}_C \cdot \mathrm{curl}\, \overline{\mathbf{z}_C} + i\omega\boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{z}_C})$$
$$-i\omega \int_\Gamma \overline{\mathbf{z}_C} \times \mathbf{n}_C \cdot \mathbf{H}_I = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \qquad (14)$$
$$-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \overline{\mathbf{v}_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \overline{\mathbf{v}_I} = 0\,,$$

where $\mathbf{z}_C \in H(\mathrm{curl}; \Omega_C)$ and $\mathbf{v}_I \in V_I(\mathbf{0})$.

Using in (14) the orthogonal decompositions of $\mathbf{H}_I - \mathbf{H}_{e,I}$ and $\mathbf{v}_I$ one finds

$$\mathcal{K}((\mathbf{E}_C, \psi_I^*, \boldsymbol{\eta}_I^*), (\mathbf{z}_C, \chi_I^*, \boldsymbol{\theta}_I^*))$$
$$= -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} + i\omega \int_\Gamma \mathbf{H}_{e,I} \cdot \overline{\mathbf{z}_C} \times \mathbf{n}_C \qquad (15)$$
$$-\omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot (\mathrm{grad}\, \overline{\chi_I^*} + \textstyle\sum_{\alpha=1}^{n_{\Omega_I}} \overline{\theta_{I,\alpha}^*} \boldsymbol{\rho}_{\alpha,I}^*)\,,$$

Scalar potential formulation (cont'd)

where the sesquilinear form $\mathcal{K}(\cdot,\cdot)$, that can be proved to be continuous and coercive, is given by

$$\begin{aligned}
&\mathcal{K}((\mathbf{E}_C,\psi_I^*,\boldsymbol{\eta}_I^*),(\mathbf{z}_C,\chi_I^*,\boldsymbol{\theta}_I^*)) \\
&\quad := \int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\,\mathrm{curl}\,\mathbf{E}_C\cdot\mathrm{curl}\,\overline{\mathbf{z}_C} + i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{z}_C}) \\
&\qquad -i\omega\int_\Gamma(\mathrm{grad}\,\psi_I^* + \textstyle\sum_{\alpha=1}^{n_{\Omega_I}}\eta_{I,\alpha}^*\boldsymbol{\rho}_{\alpha,I}^*)\cdot\overline{\mathbf{z}_C}\times\mathbf{n}_C \\
&\qquad -i\omega\int_\Gamma(\mathrm{grad}\,\overline{\chi_I^*} + \textstyle\sum_{\alpha=1}^{n_{\Omega_I}}\overline{\theta_{I,\alpha}^*}\boldsymbol{\rho}_{\alpha,I}^*)\cdot\mathbf{E}_C\times\mathbf{n}_C \\
&\qquad +\omega^2\int_{\Omega_I}\boldsymbol{\mu}_I\,\mathrm{grad}\,\psi_I^*\cdot\mathrm{grad}\,\overline{\chi_I^*} \\
&\qquad +\omega^2[A^*\boldsymbol{\eta}_I^*,\boldsymbol{\theta}_I^*]\ .
\end{aligned}$$

Note that the interaction between $\mathbf{E}_C$ and $\mathbf{H}_I$ is driven in a weak way by boundary integrals, and no strong matching conditon on Γ has to be imposed: non-matching meshes can be employed!

Scalar potential formulation (cont'd)

- Domain decomposition approaches can be devised. Let us specify it for the formulation in terms of $\mathbf{E}_C$, ψ_I^* and $\boldsymbol{\eta}_I^*$.

  Given $\mathbf{e}_\Gamma^{\mathrm{old}}$ on Γ, find the solutions to

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I \,\mathrm{grad}\, \psi_I^*) = -\,\mathrm{div}(\boldsymbol{\mu}_I \mathbf{H}_{e,I}) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \,\mathrm{grad}\, \psi_I^* \cdot \mathbf{n}_I = -i\omega^{-1}\,\mathrm{div}_\tau\, \mathbf{e}_\Gamma^{\mathrm{old}} & \\ \qquad\qquad\qquad\qquad -\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \mathbf{n}_I & \text{on } \Gamma \\ \boldsymbol{\mu}_I \,\mathrm{grad}\, \psi_I^* \cdot \mathbf{n} = -\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \mathbf{n} & \text{on } \partial\Omega \end{cases} \qquad (16)$$

$$\begin{aligned}(A^* \boldsymbol{\eta}_I^*)_\beta = i\omega^{-1} \int_\Gamma \mathbf{e}_\Gamma^{\mathrm{old}} \cdot \boldsymbol{\rho}_{\beta,I}^* - \int_{\Omega_I} \boldsymbol{\mu}_I \,\mathrm{grad}\, \psi_I^* \cdot \boldsymbol{\rho}_{\beta,I}^* \\ - \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \boldsymbol{\rho}_{\beta,I}^* \quad \forall\, \beta = 1, \dots, n_{\Omega_I}\end{aligned} \qquad (17)$$

Scalar potential formulation (cont'd)

$$\begin{cases} \operatorname{curl}(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{E}_C) + i\omega\boldsymbol{\sigma}\mathbf{E}_C = -i\omega\mathbf{J}_{e,C} & \text{in } \Omega_C \\ (\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{E}_C) \times \mathbf{n}_C = i\omega \operatorname{grad} \psi_I^* \times \mathbf{n}_I & \\ \qquad + i\omega \sum_{\alpha=1}^{n_{\Omega_I}} \eta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \times \mathbf{n}_I + i\omega\mathbf{H}_{e,I} \times \mathbf{n}_I & \text{on } \Gamma\,, \end{cases} \tag{18}$$

finally set

$$\mathbf{e}_\Gamma^{\text{new}} = (1-\delta)\mathbf{e}_\Gamma^{\text{old}} + \delta\,\mathbf{E}_C \times \mathbf{n}_C \quad \text{on } \Gamma \tag{19}$$

and iterate until convergence ($\delta > 0$ is an acceleration parameter). At convergence one has $\mathbf{e}_\Gamma^\infty = \mathbf{E}_C \times \mathbf{n}_C$ on Γ, the right tangential value of the electric field on Γ.

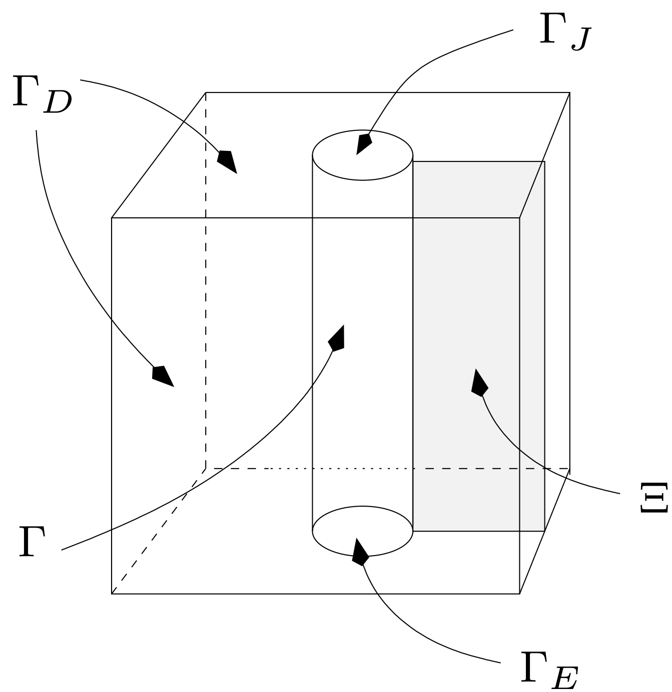
This iteration-by-subdomain procedure has shown good convergence properties (convergence rate independent of the mesh size [Alonso and V. (1997)]).

Pros and cons

- *Pros*:
  - few degrees of freedom;
  - "positive definite" algebraic problem.
- *Cons*:
  - need of computing in advance a vector potential of the current density;
  - some difficulties coming from the topology of the computational domain, in particular of the conductor [construction of the "cutting" surfaces];
  - cancellation errors?

Voltage or current excitation

In a geometrical situation like the following



we can study the eddy current problem under voltage or current intensity excitation.
[Alonso Rodríguez, V. and Vázquez Hernández, (2009); also Bíró, Preis, Buchgraber and Tičar (2004), Bermúdez, Rodríguez and Salgado (2005).]

Voltage or current excitation (cont'd)

It is assumed that $\mathbf{J}_e = \mathbf{0}$, and the boundary conditions must be $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\Gamma_E \cup \Gamma_J$, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ and $\boldsymbol{\epsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on Γ_D [for other types of boundary conditions the problem has no solution].

Proof. Multiply the Faraday equation by $\overline{\mathbf{H}}$, integrate in Ω and integrate by parts: it holds

$$\begin{aligned} 0 &= \int_\Omega \mathrm{curl}\, \mathbf{E} \cdot \overline{\mathbf{H}} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} \\ &= \int_\Omega \mathbf{E} \cdot \mathrm{curl}\, \overline{\mathbf{H}} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{H}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}}\ . \end{aligned}$$

Remembering that $\mathrm{curl}\, \mathbf{H}_I = \mathbf{0}$ in Ω_I and replacing $\mathrm{curl}\, \mathbf{H}_C$ with $\boldsymbol{\sigma}\mathbf{E}_C$, one has the Poynting Theorem (energy balance)

Voltage or current excitation (cont'd)

$$\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{E}_C} \;\; + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{H}} = - \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}}.$$

The term on $\partial\Omega$ is clearly vanishing for the electric and the magnetic boundary conditions (or for a mixed electric–magnetic boundary condition). □

For the proposed boundary conditions, instead, since $\mathrm{div}_\tau(\mathbf{E} \times \mathbf{n}) = -i\omega\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$, one has

$$\mathbf{E} \times \mathbf{n} = \mathrm{grad}\, W \times \mathbf{n} \;\; \text{on } \partial\Omega \; ,$$

and therefore

$$\begin{aligned} - \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{H}} \;\; &= - \int_{\partial\Omega} \overline{\mathbf{H}} \times \mathbf{n} \cdot \mathrm{grad}\, W \\ &= \int_{\partial\Omega} \mathrm{div}_\tau(\overline{\mathbf{H}} \times \mathbf{n})\, W \\ &= \int_{\partial\Omega} \mathrm{curl}\, \overline{\mathbf{H}} \cdot \mathbf{n}\, W = W_{|\Gamma_J} \int_{\Gamma_J} \mathrm{curl}\, \overline{\mathbf{H}_C} \cdot \mathbf{n}, \end{aligned}$$

Voltage or current excitation (cont'd)

as $\operatorname{curl} \mathbf{H}_I = \mathbf{0}$ in Ω_I, and we have denoted by $W_{|\Gamma_J}$ the (constant) value of the potential W on the electric port Γ_J (whereas $W_{|\Gamma_E} = 0$).

- In this case a degree of freedom is indeed still free (either the voltage $W_{|\Gamma_J}$, that will be denoted by V, or else the current intensity $\int_{\Gamma_J} \operatorname{curl} \mathbf{H}_C \cdot \mathbf{n}$ in Ω_C, that will be denoted by I_0).

Voltage or current excitation (cont'd)

For formulating the voltage or current excitation problem we come back to the usual orthogonal decomposition result

$$\mathbf{v}_I = \text{grad}\, \chi_I^* + Q \boldsymbol{\rho}_I^* \,, \tag{20}$$

valid for a vector field $\mathbf{v}_I$ satisfying $\text{curl}\, \mathbf{v}_I = \mathbf{0}$. The harmonic field $\boldsymbol{\rho}_I^*$ can be chosen such that $\int_{\partial\Gamma_J} \boldsymbol{\rho}_I^* \cdot d\boldsymbol{\tau} = 1$; therefore $Q = \int_{\partial\Gamma_J} \mathbf{v}_I \cdot d\boldsymbol{\tau}$.

In particular, from the Stokes Theorem one has

$$I_0 = \int_{\Gamma_J} \text{curl}\, \mathbf{H}_C \cdot \mathbf{n}_C = \int_{\partial\Gamma_J} \mathbf{H}_C \cdot d\boldsymbol{\tau} = \int_{\partial\Gamma_J} \mathbf{H}_I \cdot d\boldsymbol{\tau} \,,$$

hence

$$\mathbf{H}_I = \text{grad}\, \psi_I^* + I_0 \boldsymbol{\rho}_I^* \,. \tag{21}$$

Voltage or current excitation (cont'd)

We can provide a "coupled" variational formulation, in terms of $\mathbf{E}_C$ in Ω_C and of $\mathbf{H}_I$ in Ω_I.
Proceeding as done before for the formulation in terms of $\mathbf{E}_C$, ψ_I^* and $\boldsymbol{\eta}_I^*$, we find

$$\begin{gathered}\int_{\Omega_C} \boldsymbol{\mu}_C^{-1} \,\mathrm{curl}\, \mathbf{E}_C \cdot \mathrm{curl}\, \overline{\mathbf{z}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\ -i\omega \int_\Gamma \overline{\mathbf{z}_C} \times \mathbf{n}_C \cdot \mathbf{H}_I = 0\end{gathered} \tag{22}$$

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathrm{grad}\, \overline{\chi_I^*} + \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathrm{grad}\, \overline{\chi_I^*} = 0 \tag{23}$$

and

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}_I^* + \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I^* = V \ , \tag{24}$$

Voltage or current excitation (cont'd)

as

$$\begin{aligned}\int_{\Gamma_D} \mathbf{E}_I \times \mathbf{n}_I \cdot \boldsymbol{\rho}_I^* &= \int_{\Gamma_D} \operatorname{grad} W \times \mathbf{n}_I \cdot \boldsymbol{\rho}_I^* \\ &= \int_{\Gamma_D} \operatorname{div}_\tau(\boldsymbol{\rho}_I^* \times \mathbf{n}_I)\, W + V \int_{\partial\Gamma_J} \boldsymbol{\rho}_I^* \cdot d\boldsymbol{\tau} \\ &= \int_{\Gamma_D} \operatorname{curl} \boldsymbol{\rho}_I^* \cdot \mathbf{n}_I\, W + V = V \,.\end{aligned}$$

Using (21) in (22), (23) and (24) one has

$$\begin{aligned}&\int_{\Omega_C} \boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{E}_C \cdot \operatorname{curl} \overline{\mathbf{z}_C} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\ &\quad -i\omega \int_\Gamma \overline{\mathbf{z}_C} \times \mathbf{n}_C \cdot \operatorname{grad} \psi_I^* - i\omega {\color{red}I_0} \int_\Gamma \overline{\mathbf{z}_C} \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I^* = 0\end{aligned} \tag{25}$$

$$-i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \operatorname{grad} \overline{\chi_I^*} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \operatorname{grad} \psi_I^* \cdot \operatorname{grad} \overline{\chi_I^*} = 0 \tag{26}$$

$$-i\omega\overline{Q} \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\rho}_I^* + \omega^2 {\color{red}I_0} \overline{Q} \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}_I^* \cdot \boldsymbol{\rho}_I^* = -i\omega {\color{red}V} \overline{Q} \,. \tag{27}$$

Voltage or current excitation (cont'd)

- If V is given, one solves (25), (26), (27) and determines $\mathbf{E}_C$, ψ_I^* and I_0 (hence $\mathbf{H}_C$ and $\mathbf{H}_I$).
- If I_0 is given, one solves (25), (26) and determines $\mathbf{E}_C$ and ψ_I^* (hence $\mathbf{H}_C$ and $\mathbf{H}_I$); then from (27) one can also compute V.

Both problems are well-posed, namely, they have a unique solution, since the associated sesquilinear form is coercive (thus one can apply the Lax–Milgram Lemma).

As before, it is simple to propose an approximation method based on finite elements, of "edge" type for $\mathbf{E}_C$ in Ω_C and of (scalar) nodal type for ψ_I^* in Ω_I. Convergence is assured by the Céa Lemma.

Voltage or current excitation (cont'd)

Note: the physical interpretation of equation (27) is that

$$-\int_{\gamma} \mathbf{E}_C \cdot d\boldsymbol{\tau} + i\omega \int_{\Xi} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n}_{\Xi} = V \,,$$

where $\gamma = \partial\Xi \cap \Gamma$ is oriented from Γ_J to Γ_E, and $\mathbf{n}_{\Xi}$ is directed in such a way that γ is clockwise oriented with respect to it.
In other words, if it is possible to determine the electric field $\mathbf{E}_I$ in Ω_I satisfying the Faraday equation, it follows that

$$\int_{\gamma_*} \mathbf{E}_I \cdot d\boldsymbol{\tau} = V \,,$$

where $\gamma_* = \partial\Xi \cap \Gamma_D$ is oriented from Γ_E to Γ_J: hence (27) is indeed determining the voltage drop between the electric ports.

Voltage or current excitation (cont'd)

This explains from another point of view why, when the source is a voltage drop or a current intensity, it is not possible to assume the electric boundary conditions $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.
In fact, in that case one would have

$$\int_{\gamma_*} \mathbf{E}_I \cdot d\boldsymbol{\tau} = 0 \,,$$

hence from (24)

$$\begin{aligned} i\omega \int_\Xi \boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n}_\Xi &= V + \int_\gamma \mathbf{E}_C \cdot d\boldsymbol{\tau} = V + \int_{\gamma \cup \gamma_*} \mathbf{E} \cdot d\boldsymbol{\tau} \\ &= V + \int_{\partial\Xi} \mathbf{E} \cdot d\boldsymbol{\tau} \,, \end{aligned}$$

with $\partial\Xi$ clockwise oriented with respect $\mathbf{n}_\Xi$: due to the term V the Faraday equation would be violated on Ξ!

Numerical results for voltage or current intensity excitation

We use edge finite elements of the lowest degree ($\mathbf{a} + \mathbf{b} \times \mathbf{x}$ in each element) for approximating $\mathbf{E}_C$, and scalar piecewise-linear elements for approximating ψ_I^*.

The problem description is the following: the conductor Ω_C and the whole domain Ω are two coaxial cylinders of radius R_C and R_D, respectively, and height L. Assuming that $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are scalar constants, the exact solution for an assigned current intensity I_0 is known (through suitable Bessel functions), and also the basis function $\boldsymbol{\rho}_I^*$ is known, thus from (9) one easily computes the voltage V, too.

Numerical results for voltage or current intensity excitation (cont'd)

We have the following data:

$$\begin{aligned} R_C &= 0.25 \text{ m} \\ R_D &= 0.5 \text{ m} \\ L &= 0.25 \text{ m} \\ \sigma &= 151565.8 \text{ S/m} \\ \mu &= 4\pi \times 10^{-7} \text{ H/m} \\ \omega &= 2\pi \times 50 \text{ rad/s} \end{aligned}$$

and

$$I_0 = 10^4 \text{ A} \quad \text{or} \quad V = 0.08979 + 0.14680i$$

[the voltage corresponds to the current intensity $I_0 = 10^4$ A].

Numerical results for voltage or current intensity excitation (cont'd)

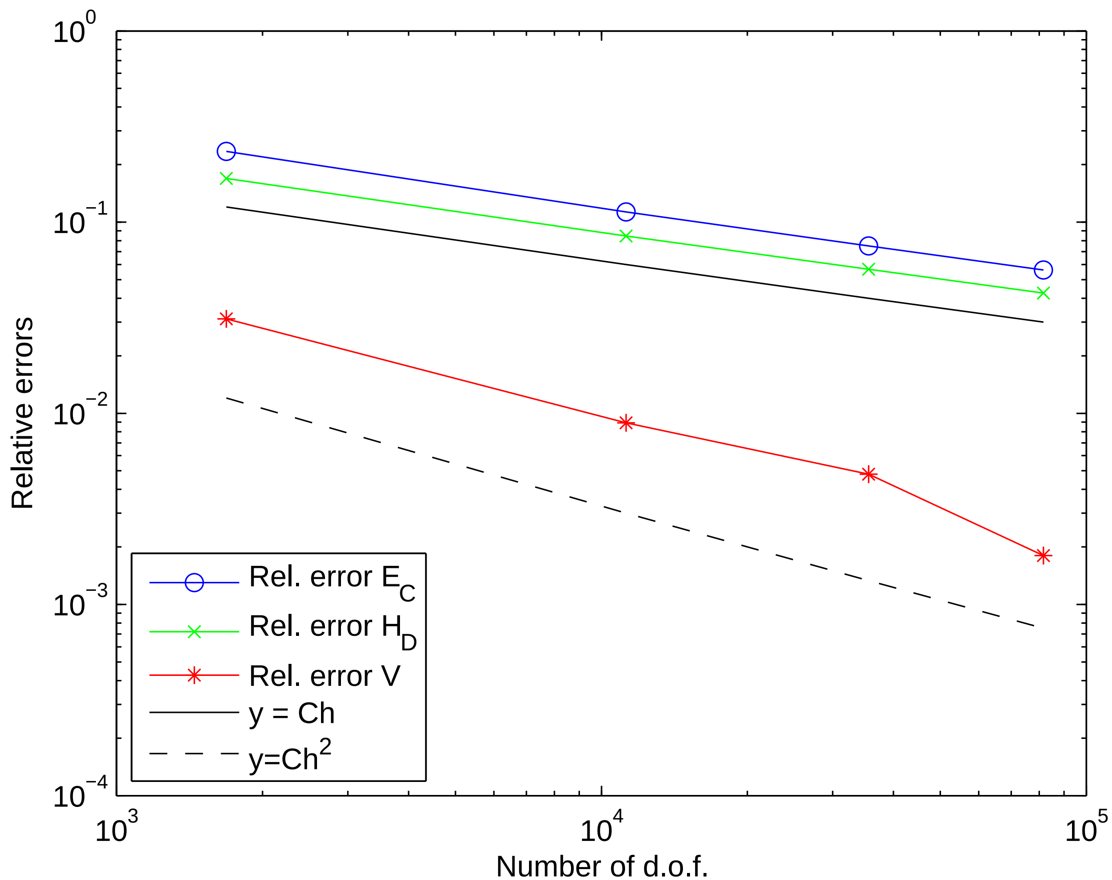
The relative errors (for $\mathbf{E}_C$ in $H(\text{curl};\Omega_C)$ and for $\mathbf{H}_I$ in $L^2(\Omega_I)$) with respect to the number of degrees of freedom are given by:

Elements	DoF	e_E	e_H	e_V
2304	1684	0.2341	0.1693	0.0312
18432	11240	0.1132	0.0847	0.0089
62208	35580	0.0750	0.0567	0.0048
147456	81616	0.0561	0.0425	0.0018

Elements	DoF	e_E	e_H	e_{I_0}
2304	1685	0.2336	0.1685	0.0274
18432	11241	0.1132	0.0847	0.0085
62208	35581	0.0750	0.0566	0.0041
147456	81617	0.0561	0.0425	0.0024

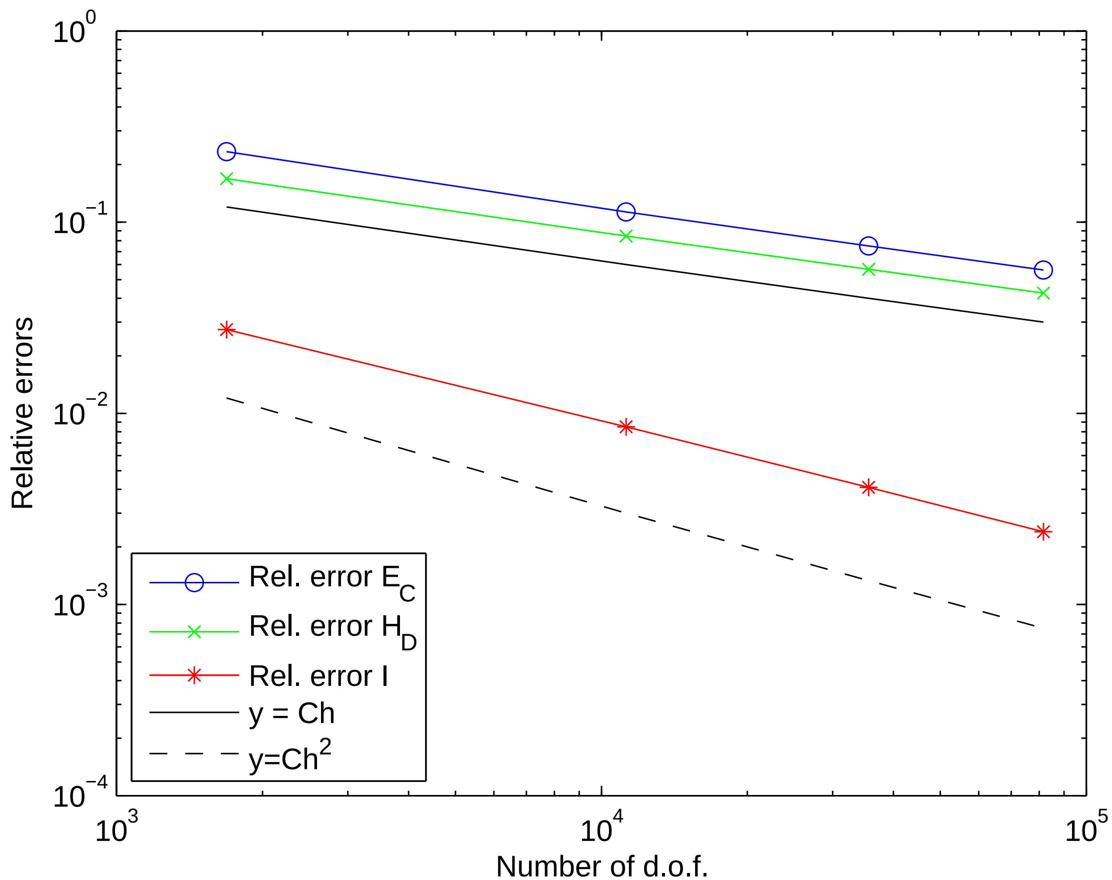
Numerical results for voltage or current intensity excitation (cont'd)

On a graph: for assigned current intensity



Numerical results for voltage or current intensity excitation (cont'd)

for assigned voltage



Numerical results for voltage or current intensity excitation (cont'd)

A more realistic problem, considered by Bermúdez, Rodríguez and Salgado (2005), is that of a cylindrical electric furnace with three electrodes ELSA [dimensions: furnace height 2 m; furnace diameter 8.88 m; electrode height 1.25 m; electrode diameter 1 m; distance of the center of the electrode from the wall 3 m].

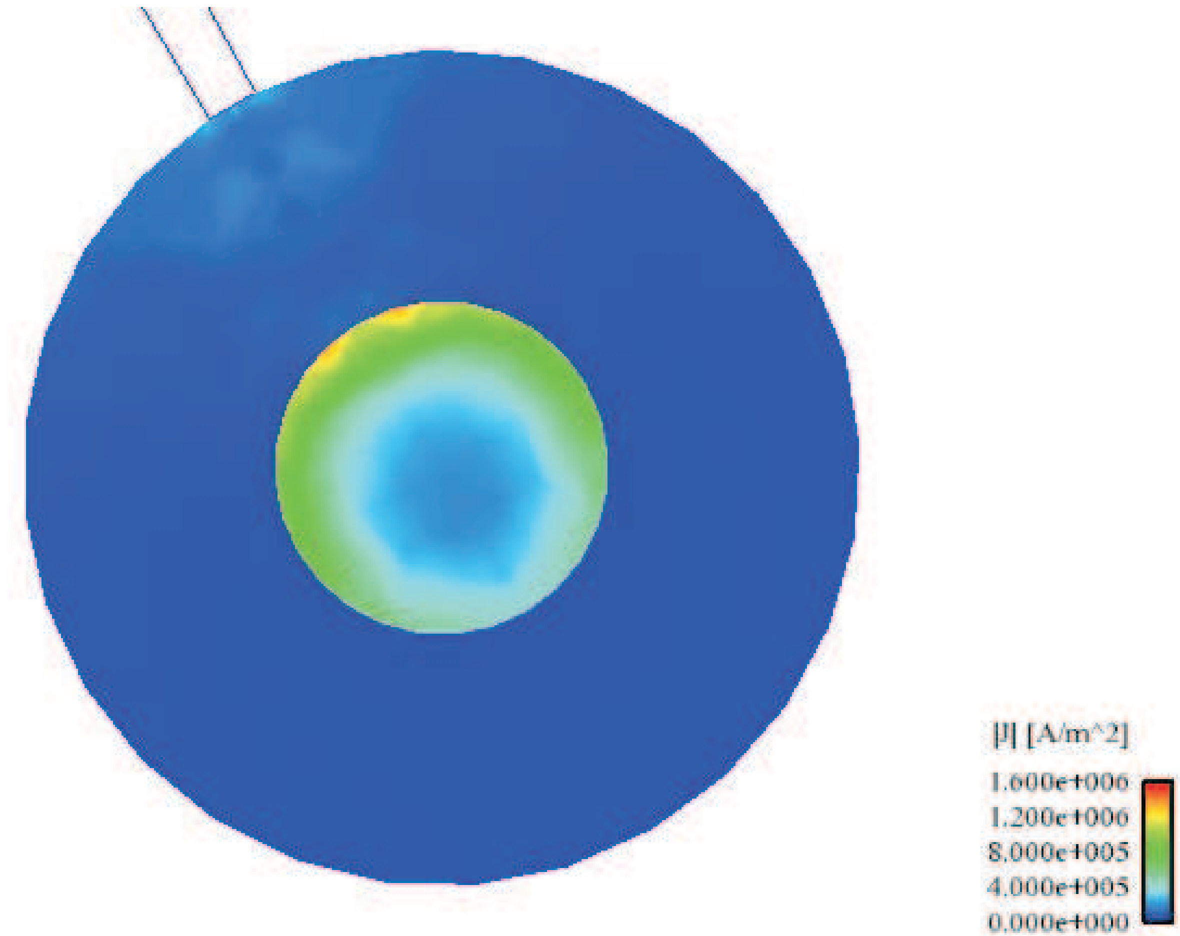
The three electrodes ELSA are constituted by a graphite core of 0.4 m of diameter, and by an outer part of Söderberg paste. The electric current enters the electrodes through horizontal copper bars of rectangular section (0.07 m $\times$ 0.25 m), connecting the top of the electrode with the external boundary.

Data: $\sigma = 10^6$ S/m for graphite, $\sigma = 10^4$ S/m for Söderberg paste, $\sigma = 5 \times 10^6$ S/m for copper, $\mu = 4\pi \times 10^{-7}$ H/m, $\omega = 2\pi \times 50$ rad/s, $I_0 = 7 \times 10^4$ A for each electrode.

Numerical results for voltage or current intensity excitation (cont'd)

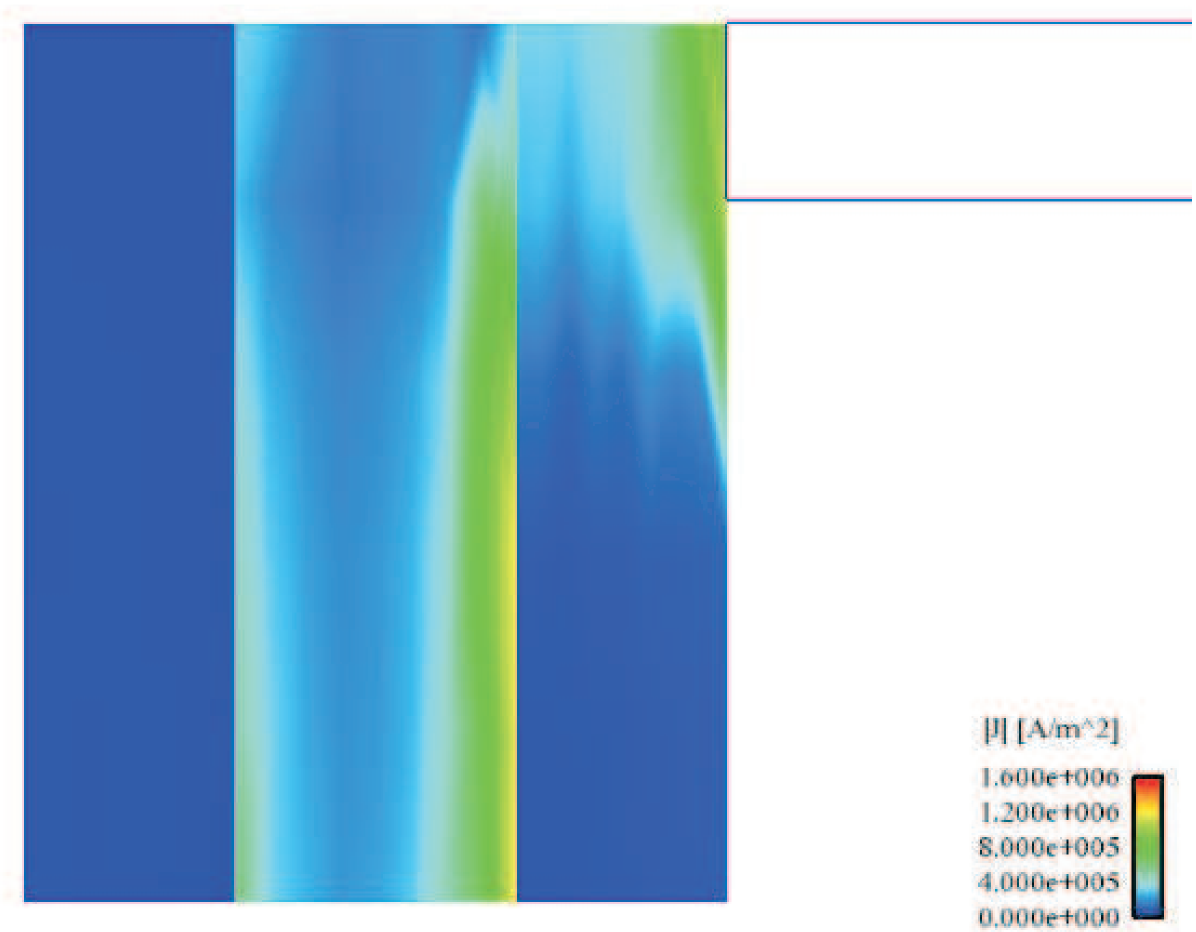
The value of the magnetic "potential" in the insulator: the magnetic field is the gradient of the represented function (not taking into account the jump surfaces).

Numerical results for voltage or current intensity excitation (cont'd)



The magnitude of the current density $\sigma \mathbf{E}_C$ on a horizontal section of one electrode.

Numerical results for voltage or current intensity excitation (cont'd)



The magnitude of the current density $\sigma \mathbf{E}_C$ on a vertical section of one electrode.

Vector potential formulation

Again, for the sake of definiteness let us consider the electric boundary condition.

Motivated by the fact that the magnetic induction $\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$ is divergence-free in Ω, a classical approach to the Maxwell equations and to eddy current problems is that based on the introduction of a vector magnetic potential $\mathbf{A}$ such that curl $\mathbf{A} = \boldsymbol{\mu}\mathbf{H}$. Often, this is also accompanied by the use of a scalar electric potential V_C in the conductor Ω_C, satisfying $i\omega\mathbf{A}_C + \text{grad}\, V_C = -\mathbf{E}_C$.

This approach opens the problem of determining correct gauge conditions assuring the uniqueness of $\mathbf{A}$ and V_C (these conditions can be necessary when considering numerical approximation, in order to avoid that the discrete problem becomes singular).

Vector potential formulation (cont'd)

Let us describe the problem: one looks for a magnetic vector potential $\mathbf{A}$ and a scalar electric potential V_C such that

$$\mathbf{E}_C = -i\omega\mathbf{A}_C - \mathrm{grad}\, V_C \ \ , \ \ \boldsymbol{\mu}\mathbf{H} = \mathrm{curl}\, \mathbf{A} \ . \tag{28}$$

We see at once that $\mathrm{curl}\, \mathbf{E}_C = -i\omega\, \mathrm{curl}\, \mathbf{A}_C = -i\omega\boldsymbol{\mu}_C\mathbf{H}_C$, thus the Faraday equation in Ω_C is satisfied. Moreover, $\boldsymbol{\mu}\mathbf{H}$ is equal to $\mathrm{curl}\, \mathbf{A}$ in Ω, therefore it is a solenoidal vector field in Ω.

The boundary condition $\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$ is satisfied provided that we require $\mathbf{A}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, as this gives $0 = \mathrm{div}_\tau(\mathbf{A}_I \times \mathbf{n}) = \mathrm{curl}\, \mathbf{A}_I \cdot \mathbf{n} = \boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n}$.

Also the topological conditions (7) are satisfied: in fact,

Vector potential formulation (cont'd)

$$\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}^*_{\alpha,I} = \int_{\Omega_I} i\omega\, \mathsf{curl}\, \mathbf{A}_I \cdot \boldsymbol{\rho}^*_{\alpha,I}$$
$$= i\omega \int_\Gamma (\mathbf{n}_I \times \mathbf{A}_I) \cdot \boldsymbol{\rho}^*_{\alpha,I} = i\omega \int_\Gamma (\mathbf{A}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}^*_{\alpha,I}$$
$$= -\int_\Gamma (\mathbf{E}_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}^*_{\alpha,I} - \int_\Gamma (\mathsf{grad}\, V_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}^*_{\alpha,I} \ .$$

Moreover,

$$\int_\Gamma (\mathsf{grad}\, V_C \times \mathbf{n}_C) \cdot \boldsymbol{\rho}^*_{\alpha,I}$$
$$= \int_\Gamma (\boldsymbol{\rho}^*_{\alpha,I} \times \mathbf{n}_I) \cdot \mathsf{grad}\, V_C$$
$$= -\int_\Gamma \mathsf{div}_\tau (\boldsymbol{\rho}^*_{\alpha,I} \times \mathbf{n}_I)\, V_C$$
$$= -\int_\Gamma \mathsf{curl}\, \boldsymbol{\rho}^*_{\alpha,I} \cdot \mathbf{n}_I\, V_C = 0 \ .$$

Assuming that the Ampère equation is satisfied in Ω_C (so that $\mathbf{E}_C = \boldsymbol{\sigma}^{-1}(\mathsf{curl}\, \mathbf{H}_C - \mathbf{J}_{e,C})$), we have thus we proved that the topological conditions (7) hold.

Vector potential formulation (cont'd)

In conclusion, we have only to require that the Ampère equation is satisfied in Ω.

Concerning the gauge conditions, the most frequently used is the Coulomb gauge

$$\mathrm{div}\,\mathbf{A} = 0 \qquad \text{in } \Omega\,. \tag{29}$$

In a general geometrical situation, this can be not enough for determining a unique vector potential $\mathbf{A}$ in Ω. In fact, there exist non-trivial irrotational, solenoidal vector fields with vanishing tangential component, namely, the elements of the space of harmonic fields

$$\begin{aligned}\mathcal{H}(e;\Omega) := \{\mathbf{w} \in (L^2(\Omega))^3 \mid \; & \mathrm{curl}\,\mathbf{w} = \mathbf{0}, \mathrm{div}\,\mathbf{w} = 0, \\ & \mathbf{w} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,\end{aligned}$$

Vector potential formulation (cont'd)

whose dimension is given by the number of connected components of $\partial\Omega$ minus 1 (say, as stated before, $p_{\partial\Omega}$). Imposing orthogonality, namely, $\mathbf{A}\perp\mathcal{H}(e;\Omega)$, turns out to be equivalent to require

$$\int_{(\partial\Omega)_r} \mathbf{A}\cdot\mathbf{n} = 0 \qquad \forall\, r = 1,\ldots,p_{\partial\Omega}\,. \tag{30}$$

In conclusion, we are left with the problem

$$\begin{cases} \mathrm{curl}(\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{A}) + i\omega\boldsymbol{\sigma}\mathbf{A} & \\ \qquad\qquad\qquad +\boldsymbol{\sigma}\,\mathrm{grad}\,V_C = \mathbf{J}_e & \text{in } \Omega \\ \mathrm{div}\,\mathbf{A} = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{A}\cdot\mathbf{n} = 0 & \forall\, r = 1,\ldots,p_{\partial\Omega} \\ \mathbf{A}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases} \tag{31}$$

Vector potential formulation (cont'd)

[Clearly, V_C is determined up to an additive constant in each connected component $\Omega_{C,j}$ of Ω_C, $j = 1, \ldots, p_\Gamma + 1$.]

The solenoidal constraint can be imposed by adding of a penalization term. Introducing the constant $\mu_* > 0$, representing a suitable average in Ω of the entries of the matrix $\boldsymbol{\mu}$, the Coulomb gauge condition $\operatorname{div} \mathbf{A} = 0$ in Ω can be incorporated in the Ampère equation, which becomes

$$\begin{aligned}\operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A}) - \mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{A} + i\omega\boldsymbol{\sigma}\mathbf{A} + \boldsymbol{\sigma} \operatorname{grad} V_C \\ = \mathbf{J}_e \qquad \text{in } \Omega\,.\end{aligned}$$

A boundary condition for $\operatorname{div} \mathbf{A}$ is now necessary, and we impose

$$\operatorname{div} \mathbf{A} = 0 \qquad \text{on } \partial\Omega\,.$$

Vector potential formulation (cont'd)

Moreover one adds the two equations

$$\begin{aligned}
&\text{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\text{grad}\,V_C) = \text{div}\,\mathbf{J}_{e,C} && \text{in } \Omega_C \\
&(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\text{grad}\,V_C)\cdot\mathbf{n}_C = \mathbf{J}_{e,C}\cdot\mathbf{n}_C + \mathbf{J}_{e,I}\cdot\mathbf{n}_I && \text{on } \Gamma\ ,
\end{aligned}$$

that are necessary as, due to the modification in the Ampère equation, it is no more assured that the electric field $\mathbf{E}_C = -i\omega\mathbf{A}_C - \text{grad}\,V_C$ satisfies the necessary conditions

$$\begin{aligned}
&\text{div}(\boldsymbol{\sigma}\mathbf{E}_C) = -\,\text{div}\,\mathbf{J}_{e,C} && \text{in } \Omega_C \\
&\boldsymbol{\sigma}\mathbf{E}_C\cdot\mathbf{n}_C = -\mathbf{J}_{e,C}\cdot\mathbf{n}_C - \mathbf{J}_{e,I}\cdot\mathbf{n}_I && \text{on } \Gamma\ .
\end{aligned}$$

Vector potential formulation (cont'd)

The complete $(\mathbf{A}, V_C)$ formulation is therefore

$$\left\{\begin{array}{ll}
\text{curl}(\boldsymbol{\mu}^{-1}\,\text{curl}\,\mathbf{A}) - \mu_*^{-1}\,\text{grad}\,\text{div}\,\mathbf{A} & \\
\qquad\qquad +i\omega\boldsymbol{\sigma}\mathbf{A} + \boldsymbol{\sigma}\,\text{grad}\,V_C = \mathbf{J}_e & \text{in } \Omega \\
\text{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\text{grad}\,V_C) = \text{div}\,\mathbf{J}_{e,C} & \text{in } \Omega_C \\
(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\text{grad}\,V_C)\cdot\mathbf{n}_C & \\
\qquad\qquad = \mathbf{J}_{e,C}\cdot\mathbf{n}_C + \mathbf{J}_{e,I}\cdot\mathbf{n}_I & \text{on } \Gamma \\
\int_{(\partial\Omega)_r} \mathbf{A}\cdot\mathbf{n} = 0 & \forall\, r = 1, \ldots, p_{\partial\Omega} \\
\text{div}\,\mathbf{A} = 0 & \text{on } \partial\Omega \\
\mathbf{A}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\ .
\end{array}\right. \qquad (32)$$

[For the magnetic boundary conditions see Bíró and V. (2007).]

Vector potential formulation (cont'd)

It is important to show that any solution to (32) satisfies div $\mathbf{A} = 0$ in Ω. In fact, taking the divergence of $(32)_1$ and using $(32)_2$ we have $-\Delta \operatorname{div} \mathbf{A}_C = 0$ in Ω_C. Moreover, since div $\mathbf{J}_{e,I} = 0$ in Ω_I, one also obtains $-\Delta \operatorname{div} \mathbf{A}_I = 0$ in Ω_I. On the other hand, using $(32)_3$, on the interface Γ we have

$$\begin{aligned} &-\mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{A}_C \cdot \mathbf{n}_C \\ &\qquad = -\mathbf{J}_{e,I} \cdot \mathbf{n}_I - \operatorname{curl}\big(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C\big) \cdot \mathbf{n}_C \\ &\qquad = -\mathbf{J}_{e,I} \cdot \mathbf{n}_I - \operatorname{div}_\tau\big[\big(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C\big) \times \mathbf{n}_C\big] \ , \end{aligned}$$

and also

$$\begin{aligned} &-\mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{A}_I \cdot \mathbf{n}_I \\ &\qquad = \mathbf{J}_{e,I} \cdot \mathbf{n}_I - \operatorname{curl}\big(\boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{A}_I\big) \cdot \mathbf{n}_I \\ &\qquad = \mathbf{J}_{e,I} \cdot \mathbf{n}_I - \operatorname{div}_\tau\big[\big(\boldsymbol{\mu}_I^{-1} \operatorname{curl} \mathbf{A}_I\big) \times \mathbf{n}_I\big] \ . \end{aligned}$$

Vector potential formulation (cont'd)

Moreover, a solution to $(32)_1$ satisfies on the interface Γ

$$\mathbf{n}_C \times (\boldsymbol{\mu}_C^{-1} \,\mathrm{curl}\, \mathbf{A}_C) - \mu_*^{-1} \,\mathrm{div}\, \mathbf{A}_C \,\mathbf{n}_C$$
$$+ \mathbf{n}_I \times (\boldsymbol{\mu}_I^{-1} \,\mathrm{curl}\, \mathbf{A}_I) - \mu_*^{-1} \,\mathrm{div}\, \mathbf{A}_I \,\mathbf{n}_I = \mathbf{0} \ ,$$

therefore, due to orthogonality,

$$\mathbf{n}_C \times (\boldsymbol{\mu}_C^{-1} \,\mathrm{curl}\, \mathbf{A}_C) + \mathbf{n}_I \times (\boldsymbol{\mu}_I^{-1} \,\mathrm{curl}\, \mathbf{A}_I) = \mathbf{0} \ , \ \ \mathrm{div}\, \mathbf{A}_C = \mathrm{div}\, \mathbf{A}_I \ .$$

Hence we have obtained

$$\mathrm{grad}\,\mathrm{div}\, \mathbf{A}_C \cdot \mathbf{n}_C + \mathrm{grad}\,\mathrm{div}\, \mathbf{A}_I \cdot \mathbf{n}_I = 0 \quad \text{on } \Gamma \ ,$$

and this last condition, together with the matching of $\mathrm{div}\, \mathbf{A}$ on Γ, furnishes that $\mathrm{div}\, \mathbf{A}$ is a harmonic function in the whole Ω. Since it vanishes on $\partial\Omega$, it vanishes in Ω.

Vector potential weak formulation

We are now interested in finding a weak formulation of (32).

First of all, multiplying $(32)_1$ by $\overline{\mathbf{w}}$ with $\mathbf{w} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and integrating in Ω, we obtain by integration by parts

$$\begin{aligned} &\int_\Omega (\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \overline{\mathbf{w}} + \mu_*^{-1} \operatorname{div} \mathbf{A} \ \operatorname{div} \overline{\mathbf{w}}) \\ &\qquad + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma} \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\ &= \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \ , \end{aligned}$$

having used $(32)_5$.

Let us now multiply $(32)_2$ by $i\omega^{-1}\overline{Q_C}$ and integrate in Ω_C: by integration by parts and using $(32)_3$ we find

$$\begin{aligned} &\int_{\Omega_C} (-\boldsymbol{\sigma} \mathbf{A}_C \cdot \operatorname{grad} \overline{Q_C} + i\omega^{-1} \boldsymbol{\sigma} \operatorname{grad} V_C \cdot \operatorname{grad} \overline{Q_C}) \\ &\qquad = i\omega^{-1} \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \operatorname{grad} \overline{Q_C} + i\omega^{-1} \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{Q_C} \ . \end{aligned}$$

Vector potential weak formulation (cont'd)

Introducing the sesquilinear form

$$
\begin{aligned}
&\mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w}, Q_C)] \\
&\quad := \int_\Omega (\boldsymbol{\mu}^{-1} \,\text{curl}\, \mathbf{A} \cdot \text{curl}\, \overline{\mathbf{w}} + \mu_*^{-1} \,\text{div}\, \mathbf{A} \ \text{div}\, \overline{\mathbf{w}}) \\
&\qquad + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma} \,\text{grad}\, V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad - \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{A}_C \cdot \text{grad}\, \overline{Q_C} \\
&\qquad + i\omega^{-1} \int_{\Omega_C} \boldsymbol{\sigma} \,\text{grad}\, V_C \cdot \text{grad}\, \overline{Q_C} \ ,
\end{aligned}
\tag{33}
$$

we have finally rewritten (32) as

Find $(\mathbf{A}, V_C) \in W_\sharp \times H^1_\sharp(\Omega_C)$ such that

$$
\begin{aligned}
&\mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w}, Q_C)] = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} \\
&\quad + i\omega^{-1} \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\, \overline{Q_C} + i\omega^{-1} \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{Q_C}
\end{aligned}
\tag{34}
$$

for all $(\mathbf{w}, Q_C) \in W_\sharp \times H^1_\sharp(\Omega_C)$,

Vector potential weak formulation (cont'd)

where

$$W_\sharp := \{\mathbf{w} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega) \mid \int_{(\partial\Omega)_r} \mathbf{w}\cdot\mathbf{n} = 0 \ \forall\, r = 1,\dots,p_{\partial\Omega}\}\ ,$$

and

$$H^1_\sharp(\Omega_C) := \prod_{j=1}^{p_\Gamma+1} H^1(\Omega_{C,j})/\mathbb{C}\ .$$

[$\Omega_{C,j}$ are the connected components of Ω_C.]

- The sesquilinear form $\mathcal{A}[\cdot,\cdot]$ is continuous and coercive [we will see this result later on...], therefore existence and uniqueness of the solution is assured by the Lax–Milgram lemma.

Vector potential: from the weak to the strong formulation

To complete the argument, it is necessary to show that a solution of the weak problem is in fact a solution of the eddy current problem.

- This is not a trivial fact, as the functional spaces $W_\sharp$ and $H^1_\sharp(\Omega_C)$ contain some constraints.

The first step is to show that (34) is satisfied for any $\mathbf{w} \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$, $Q_C \in H^1(\Omega_C)$.
First note that (34) does not change if we add to Q_C a (different) constant in $\Omega_{C,j}$. In fact, the necessary conditions on $\mathbf{J}_{e,I}$ are $\text{div}\, \mathbf{J}_{e,I} = 0$ in Ω_I and $\mathbf{J}_{e,I} \perp \mathcal{H}_I$, and the latter can be rewritten as $\int_{\Gamma_j} \mathbf{J}_{e,I} \cdot \mathbf{n}_I = 0$ for each $j = 1, \dots, p_\Gamma + 1$ and $\int_{(\partial\Omega)_r} \mathbf{J}_{e,I} \cdot \mathbf{n} = 0$ for each $r = 1, \dots, p_{\partial\Omega}$. Hence a solution $(\mathbf{A}, V_C)$ of (34) satisfies it also for each $Q_C \in H^1(\Omega_C)$.

Vector potential: from the weak to the strong formulation (cont'd)

Taking $\mathbf{w} = \mathbf{0}$, a first general result is that any solution to (34) satisfies

$$\begin{cases} \mathrm{div}(i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathrm{grad}\,V_C) = \mathrm{div}\,\mathbf{J}_{e,C} & \text{in } \Omega_C \\ (i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\mathrm{grad}\,V_C)\cdot\mathbf{n}_C = \mathbf{J}_{e,C}\cdot\mathbf{n}_C + \mathbf{J}_{e,I}\cdot\mathbf{n}_I & \text{on } \Gamma\,. \end{cases}$$

Therefore, setting

$$\mathbf{J} := \begin{cases} -i\omega\boldsymbol{\sigma}\mathbf{A}_C - \boldsymbol{\sigma}\,\mathrm{grad}\,V_C + \mathbf{J}_{e,C} & \text{in } \Omega_C \\ \mathbf{J}_{e,I} & \text{in } \Omega_I\,, \end{cases}$$

we have proved that $\mathrm{div}\,\mathbf{J} = 0$ in Ω.

Vector potential: from the weak to the strong formulation (cont'd)

For any $\mathbf{w} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ we can define by $\mathbf{w}_e$ the harmonic field in $\mathcal{H}(e;\Omega)$ satisfying $\int_{(\partial\Omega)_r} \mathbf{w}_e \cdot \mathbf{n} = \int_{(\partial\Omega)_r} \mathbf{w} \cdot \mathbf{n}$ for all $r = 1, \ldots, p_{\partial\Omega}$. Clearly, the difference $\mathbf{w} - \mathbf{w}_e$ belongs to $W_\sharp$. Hence

$$
\begin{aligned}
&\mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w}, Q_C)] \\
&\quad = \mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w} - \mathbf{w}_e, Q_C)] + \mathcal{A}[(\mathbf{A}, V_C), (\mathbf{w}_e, 0)] \\
&\quad = \int_\Omega \mathbf{J}_e \cdot (\overline{\mathbf{w}} - \overline{\mathbf{w}_e}) + i\omega^{-1} \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\, \overline{Q_C} \\
&\qquad + i\omega^{-1} \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{Q_C} \\
&\qquad + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C + \boldsymbol{\sigma}\, \text{grad}\, V_C) \cdot \overline{\mathbf{w}_{e,C}} \\
&\quad = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{w}} + i\omega^{-1} \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \text{grad}\, \overline{Q_C} \\
&\qquad + i\omega^{-1} \int_\Gamma \mathbf{J}_{e,I} \cdot \mathbf{n}_I \, \overline{Q_C} - \int_\Omega \mathbf{J} \cdot \overline{\mathbf{w}_e} \; .
\end{aligned}
$$

Vector potential: from the weak to the strong formulation (cont'd)

Therefore, the only result that remains to be proved is

$$\int_\Omega \mathbf{J} \cdot \overline{\mathbf{w}_e} = 0 \ .$$

The basis functions of $\mathcal{H}(e;\Omega)$ are given by grad w_r^*, $r = 1, \dots, p_{\partial\Omega}$, where w_r^* is the (real-valued) solution to

$$\begin{cases} \Delta w_r^* = 0 & \text{in } \Omega \\ w_r^* = 0 & \text{on } (\partial\Omega) \setminus (\partial\Omega)_r \\ w_r^* = 1 & \text{on } (\partial\Omega)_r \ , \end{cases}$$

and we have

$$\begin{aligned} \int_\Omega \mathbf{J} \cdot \operatorname{grad} w_r^* &= -\int_\Omega \operatorname{div} \mathbf{J} \, w_r^* + \int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n} \, w_r^* \\ &= \int_{(\partial\Omega)_r} \mathbf{J} \cdot \mathbf{n} = \int_{(\partial\Omega)_r} \mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \, . \end{aligned}$$

Vector potential: from the weak to the strong formulation (cont'd)

Taking now in (34) a test function $\mathbf{w} \in (C_0^\infty(\Omega))^3$, by integration by parts we find at once that

$$\begin{aligned} \mathrm{curl}(\boldsymbol{\mu}^{-1}\, \mathrm{curl}\, \mathbf{A}) - \mu_*^{-1}\, \mathrm{grad}\, \mathrm{div}\, \mathbf{A} \\ + i\omega\boldsymbol{\sigma}\mathbf{A} + \boldsymbol{\sigma}\, \mathrm{grad}\, V_C = \mathbf{J}_e \quad \text{in } \Omega\,. \end{aligned}$$

Repeating the same argument for $\mathbf{w} \in H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div};\Omega)$ gives $\mathrm{div}\, \mathbf{A} = 0$ on $\partial\Omega$, and therefore a weak solution $(\mathbf{A}, V_C)$ to (34) is a solution to the strong problem (32).

Vector potential formulation: existence and uniqueness

The proof of existence and uniqueness derives from the Lax–Milgram lemma.
We have only to check that the sesquilinear form $\mathcal{A}[\cdot,\cdot]$ is coercive in $W_\sharp \times H^1_\sharp(\Omega_C)$, namely, that there exists a constant $\kappa_0 > 0$ such that for each $(\mathbf{w}, Q_C) \in W_\sharp \times H^1(\Omega_C)$ with $\int_{\Omega_{C,j}} Q_{C|\Omega_j} = 0$, $j = 1, \ldots, p_\Gamma + 1$, it holds

$$\begin{aligned} &|\mathcal{A}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)]| \\ &\qquad \geq \kappa_0 \Big(\int_\Omega (|\mathbf{w}|^2 + |\,\mathrm{curl}\,\mathbf{w}|^2 + |\,\mathrm{div}\,\mathbf{w}|^2) \\ &\qquad\qquad + \int_{\Omega_C} (|Q_C|^2 + |\,\mathrm{grad}\,Q_C|^2) \Big) \,. \end{aligned} \tag{35}$$

Vector potential formulation: existence and uniqueness (cont'd)

First of all, we can easily obtain

$$\mathcal{A}[(\mathbf{w}, Q_C), (\mathbf{w}, Q_C)]$$
$$= \int_\Omega (\boldsymbol{\mu}^{-1} \,\mathsf{curl}\, \mathbf{w} \cdot \mathsf{curl}\, \overline{\mathbf{w}} + \mu_*^{-1} |\,\mathsf{div}\, \mathbf{w}|^2)$$
$$+ i\omega^{-1} \int_{\Omega_C} \boldsymbol{\sigma} (i\omega \mathbf{w}_C + \mathsf{grad}\, Q_C) \cdot (-i\omega \overline{\mathbf{w}_C} + \mathsf{grad}\, \overline{Q_C}) \; .$$

Then, observe that, given a couple of real numbers a and b, for each $0 < \delta < 1$ it holds

$$|2ab| \leq \delta a^2 + \delta^{-1} b^2 \, .$$

Vector potential formulation: existence and uniqueness (cont'd)

Hence one has

$$
\begin{aligned}
|\omega|^{-1} \int_{\Omega_C} \boldsymbol{\sigma}(i\omega \mathbf{w}_C + \operatorname{grad} Q_C) \cdot (-i\omega \overline{\mathbf{w}_C} + \operatorname{grad} \overline{Q_C}) \\
\geq |\omega|^{-1}\sigma_{\min} \int_{\Omega_C} [|\operatorname{grad} Q_C|^2 + \omega^2 |\mathbf{w}_C|^2 \\
+2\,\mathrm{Re}(i\omega \mathbf{w}_C \cdot \operatorname{grad} \overline{Q_C})] \\
\geq |\omega|^{-1}\sigma_{\min}(1-\delta) \int_{\Omega_C} |\operatorname{grad} Q_C|^2 \\
-|\omega|\sigma_{\min}(1-\delta)\delta^{-1} \int_{\Omega_C} |\mathbf{w}_C|^2 \,,
\end{aligned}
$$

where $\sigma_{\min}$ is an uniform lower bound in Ω_C of the minimum eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$.

Vector potential formulation: existence and uniqueness (cont'd)

The Poincaré inequality gives that

$$\begin{aligned}\int_{\Omega_C} |\,\mathrm{grad}\, Q_C|^2 &= \sum_{j=1}^{p_\Gamma+1} \int_{\Omega_{C,j}} |\,\mathrm{grad}\, Q_{C|\Omega_{C,j}}|^2 \\ &\geq K_1 \sum_{j=1}^{p_\Gamma+1} \int_{\Omega_{C,j}} (|\,\mathrm{grad}\, Q_{C|\Omega_{C,j}}|^2 + |Q_{C|\Omega_{C,j}}|^2) \\ &= K_1 \int_{\Omega_C} (|\,\mathrm{grad}\, Q_C|^2 + |Q_C|^2)\end{aligned}$$

[recall that $\int_{\Omega_{C,j}} Q_{C|\Omega_{C,j}} = 0$, $j = 1, \ldots, p_\Gamma + 1$].
Moreover, the Poincaré-like inequality yields

$$\begin{aligned}\int_\Omega (\boldsymbol{\mu}^{-1}\,\mathrm{curl}\,\mathbf{w} \cdot \mathrm{curl}\,\overline{\mathbf{w}} + \mu_*^{-1} |\,\mathrm{div}\,\mathbf{w}|^2) &\\ \geq \int_\Omega (\mu_{\max}^{-1} |\,\mathrm{curl}\,\mathbf{w}|^2 + \mu_*^{-1} |\,\mathrm{div}\,\mathbf{w}|^2) &\\ \geq K_2 \int_\Omega (|\,\mathrm{curl}\,\mathbf{w}|^2 + |\,\mathrm{div}\,\mathbf{w}|^2 + |\mathbf{w}|^2)\ ,&\end{aligned}$$

Vector potential formulation: existence and uniqueness (cont'd)

where $\mu_{\max}$ is a uniform upper bound in Ω of the maximum eigenvalues of $\boldsymbol{\mu}(\mathbf{x})$ [recall that, for a divergence-free vector field, the conditions $\int_{(\partial\Omega)_r} \mathbf{w} \cdot \mathbf{n} = 0$ for all $r = 1, \ldots, p_{\partial\Omega}$ are equivalent to the orthogonality to $\mathcal{H}(e;\Omega)$].
Choosing $(1-\delta)$ so small that $\sigma_{\min}|\omega|(1-\delta) < K_2\delta$, we find at once (35).

Vector potential formulation: numerical approximation

- Numerical approximation is performed by means of nodal finite elements, for all the components of $\mathbf{A}$ and for V_C.

Via Céa lemma we have

$$\Big(\int_\Omega (|\mathbf{A} - \mathbf{A}_h|^2 + |\,\mathrm{curl}(\mathbf{A} - \mathbf{A}_h)|^2 + |\,\mathrm{div}(\mathbf{A} - \mathbf{A}_h)|^2) + \int_{\Omega_C} |\,\mathrm{grad}(V_C - V_{C,h})|^2 \Big)^{1/2}$$
$$\leq C_0 \Big(\int_\Omega (|\mathbf{A} - \mathbf{w}_h|^2 + |\,\mathrm{curl}(\mathbf{A} - \mathbf{w}_h)|^2 + |\,\mathrm{div}(\mathbf{A} - \mathbf{w}_h)|^2) + \int_{\Omega_C} |\,\mathrm{grad}(V_C - Q_{C,h})|^2 \Big)^{1/2} ,$$

for each choice of $\mathbf{w}_h$ and $Q_{C,h}$ (the former satisfying the constraints $\int_{(\partial\Omega)_r} \mathbf{w}_h \cdot \mathbf{n} = 0$ for all $r = 1, \ldots, p_{\partial\Omega}$).

Vector potential formulation: numerical approximation(cont'd)

- It is not possible to choose $\mathbf{w}_h = \mathbf{I}_h \mathbf{A}$, the interpolant of the solution $\mathbf{A}$, as the constraints $\int_{(\partial\Omega)_r} \mathbf{w}_h \cdot \mathbf{n} = 0$ have to be satisfied for all $r = 1, \dots, p_{\partial\Omega}$.
However, it is possible to construct a discrete function $\mathbf{w}_h$ such that

$$\|\mathbf{A} - \mathbf{w}_h\|_W \leq C\|\mathbf{A} - \mathbf{I}_h \mathbf{A}\|_W \ ,$$

where $W = H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$. Therefore, convergence is assured provided that $\mathbf{A}$ is smooth enough.

Vector potential formulation: numerical approximation(cont'd)

- The regularity of $\mathbf{A}$ is a delicate point! In fact, it has to be noted that the regularity of $\mathbf{A}$ is not assured if Ω has reentrant corners or edges, namely, if it is a non-convex polyhedron (see Costabel and Dauge (2000), Costabel, Nicaise and Dauge (2003)). More important, in that case the space $H^1_n(\Omega) := (H^1(\Omega))^3 \cap H_0(\text{curl};\Omega)$ turns out to be a proper closed subspace of $H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ ($H^1_n(\Omega)$ and $H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ coincide if and only if Ω is convex).
  Hence the nodal finite element approximate solution $\mathbf{A}_h \in H^1_n(\Omega)$ cannot approach an exact solution $\mathbf{A} \in H_0(\text{curl};\Omega) \cap H(\text{div};\Omega)$ with $\mathbf{A} \notin H^1_n(\Omega)$, and convergence in $W = H(\text{curl};\Omega) \cap H(\text{div};\Omega)$ is lost: this is a general problem for the nodal finite element approximation of Maxwell equations.

Vector potential formulation: numerical approximation(cont'd)

- Summing up: the nodal finite element approximation is convergent either if the solution is regular (and this information could be available even for a non-convex polyhedron Ω) or else if the domain Ω is a convex polyhedron, as in this case the space of smooth normal vector fields is dense in $H^1_n(\Omega) = H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$, and one can apply Céa lemma in the standard way.
- Let us also note that the assumption that Ω is convex is not a severe restriction, as in most real-life applications $\partial\Omega$ arises from a somehow arbitrary truncation of the whole space. Hence, reentrant corners and edges of Ω can be easily avoided.

Vector potential formulation: numerical approximation(cont'd)

- It is worth noting that a cure for the lack of convergence of nodal finite element approximations in the presence of re-entrant corners and edges has been proposed by Costabel and Dauge (2002). They introduce a special weight in the grad div penalization term, thus permitting to use standard nodal finite elements in a numerically efficient way.
- In numerical implementation, imposing the boundary condition $\mathbf{A}_h \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ is clearly straightforward if the boundary of the computational domain Ω is formed by planar surfaces, parallel to the reference planes.

Vector potential formulation: numerical approximation(cont'd)

- If that is not the case, for each node $\mathbf{p}$ on $\partial\Omega$ introduce a local system of coordinates with one axis aligned with $\mathbf{n}_a$, a suitable average of the normals to the surface elements containing $\mathbf{p}$, and express, through a rotation, the vector $\mathbf{A}_h$ with respect to that system: the condition $\mathbf{A}_h \times \mathbf{n}_a = \mathbf{0}$ is then trivially imposed (see Rodger and Eastham (1985)).
- Another possible approach, which avoids the arbitrariness inherent in the averaging process of the normals at corner points, is described by Bossavit (1999). It is based on imposing $\mathbf{A}_h \times \mathbf{n} = \mathbf{0}$ at the center of the element faces on $\partial\Omega$: the drawback is that it results in a constrained problem, requiring the introduction of as many Lagrange multipliers as the (double of the) number of surface elements on $\partial\Omega$.

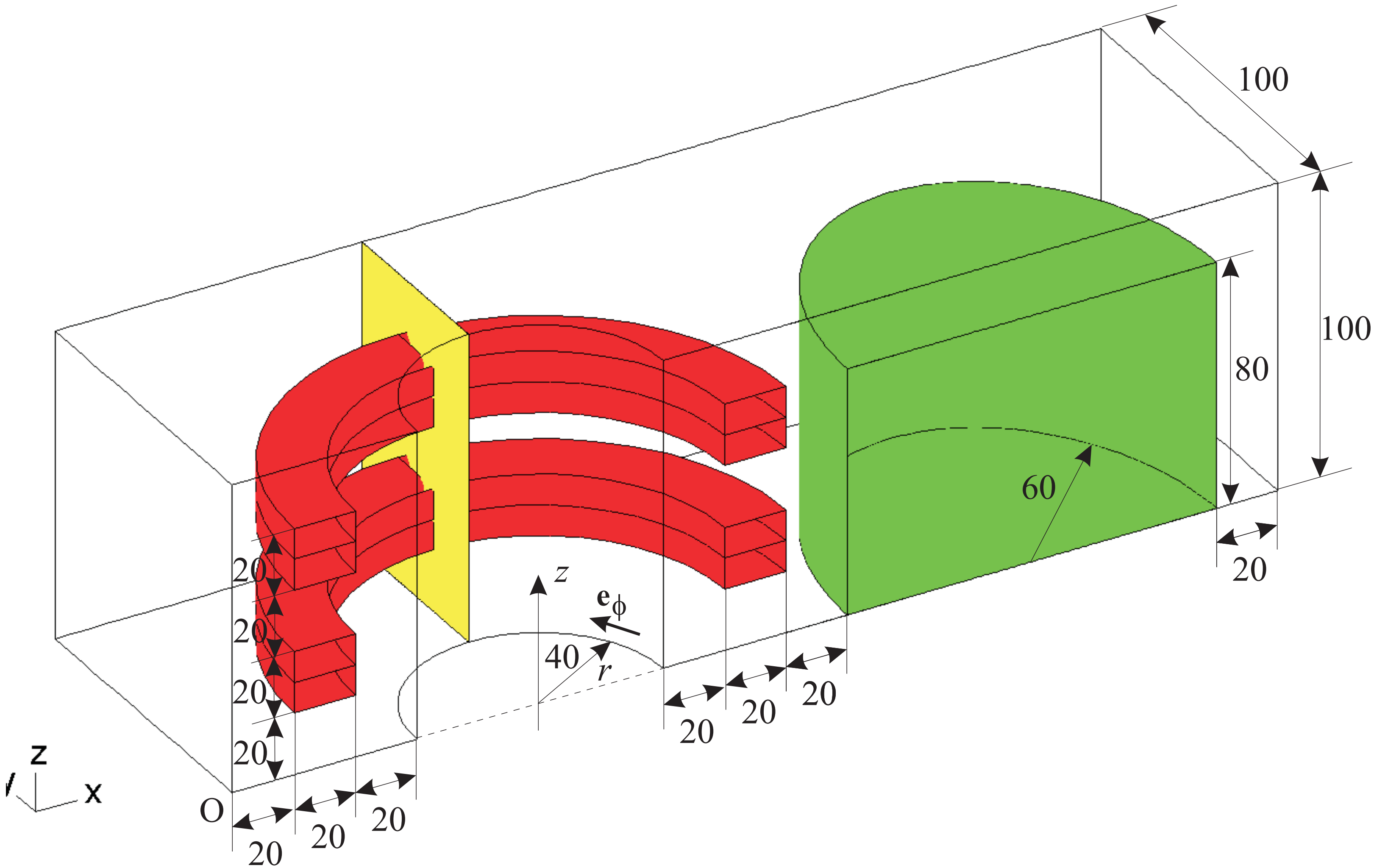
Vector potential formulation: numerical approximation(cont'd)

- Ungauged formulation have been also proposed (see Ren (1996), Kameari and Koganezawa (1997), Bíró (1999)): edge elements are employed for the approximation of the potential $\mathbf{A}$, without requiring that the gauge condition $\operatorname{div} \mathbf{A} = 0$ in Ω is satisfied. Clearly, in this way the resulting linear system is singular: however, in many cases the right-hand sides turn out to be compatible, so that suitable iterative algebraic solvers can still be convergent.
[Warning: lack of a complete theory...]

Numerical results

The numerical results we present here have been obtained in Bíró and V. (2007), for the magnetic boundary conditions (Ω is a torus and Ω_C is a ball-like set).
The employed finite elements are second order hexahedral “serendipity" elements, with 20 nodes (8 at the vertices and 12 at the midpoints of each edge), for all the components of $\mathbf{A}_h$ and for V_h.
The values of the physical coefficients have been assumed as follows: $\mu = \mu_* = 4\pi \times 10^{-7}$ H/m, $\sigma = 5.7 \times 10^7$ S/m, $\omega = 2\pi \times f = 100\pi$ rad/s, i.e., $f = 50$ Hz.
The half of the domain is described here below. The coils (the support of $\mathbf{J}_{e,I}$) are red, while the conductor Ω_C is green; the yellow “cutting" surface Σ_1 is also drawn.

Numerical results (cont'd)



The computational domain [one half].

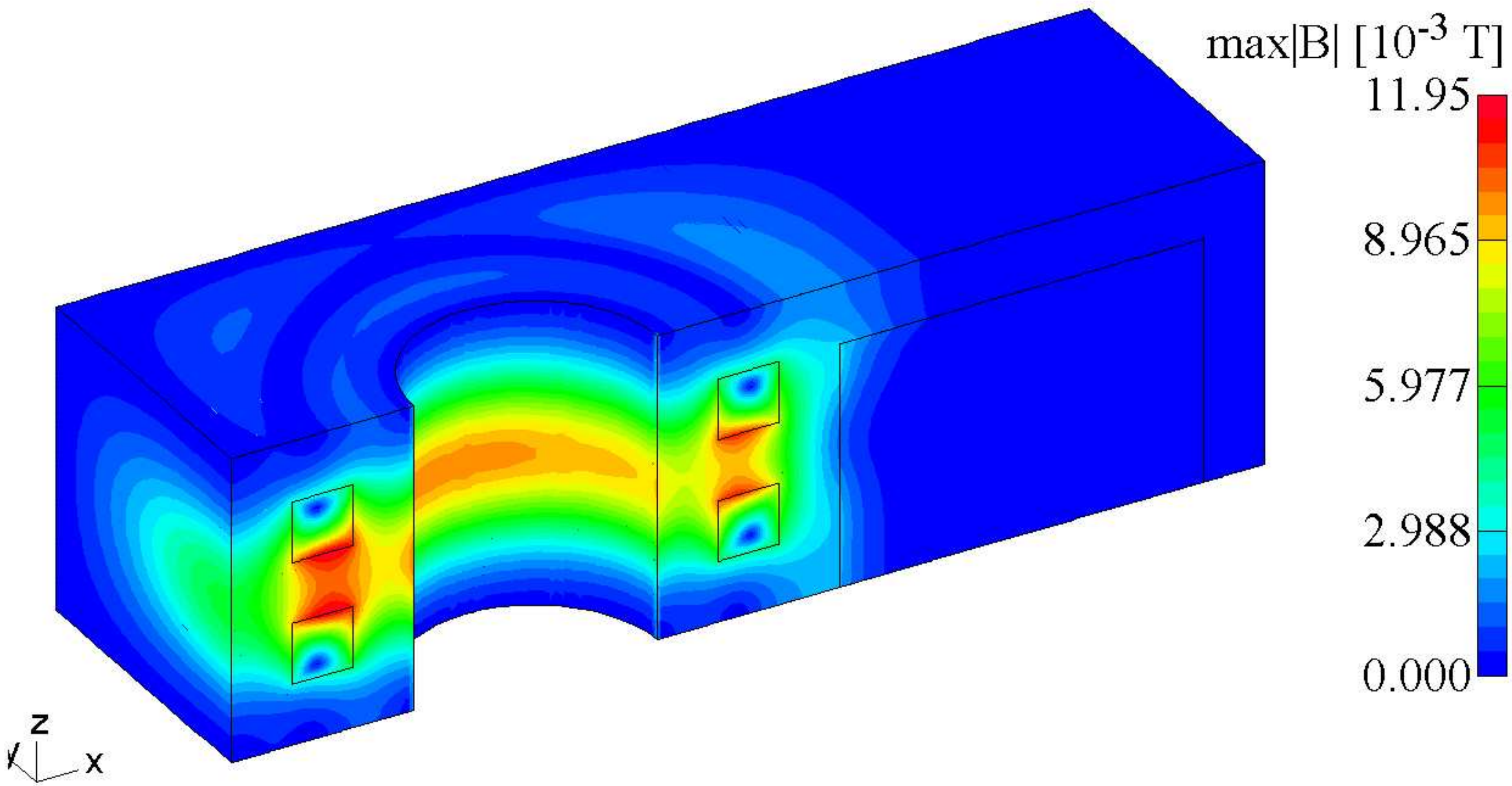
Numerical results (cont'd)

The current density is given by $\mathbf{J}_{e,C} = \mathbf{0}$ and $\mathbf{J}_{e,I} = J_{e,I}\mathbf{e}_\phi$, where $\mathbf{e}_\phi$ is the azymuthal unit vector in the cylindrical system centered at the point (100,0,0), oriented counterclockwise, and

$$J_{e,I} = \begin{cases} 10^6 \text{ A/m}^2 & \text{if } 60 < r < 80\ ,\ 60 < z < 80 \\ -10^6 \text{ A/m}^2 & \text{if } 60 < r < 80\ ,\ 20 < z < 40 \\ 0 & \text{otherwise}\ . \end{cases}$$

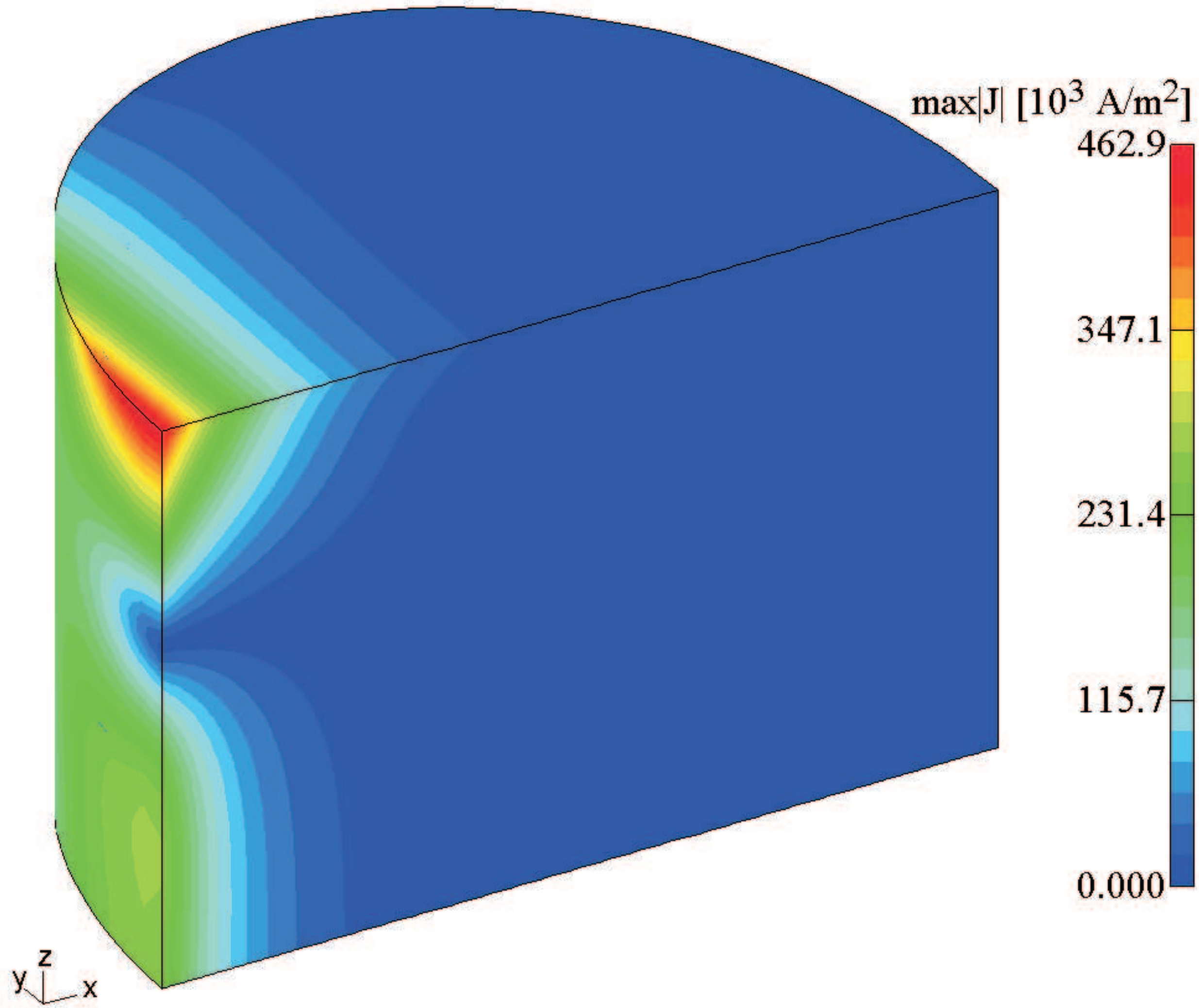
In the two figures below some details of the computed solution are presented: the magnitude of the computed flux density $\mathbf{B}$ in the first figure, the magnitude of the computed current density $\mathbf{J}_C := -i\omega\sigma\mathbf{A}_C - \sigma \operatorname{grad} V_C$ in the second figure.

Numerical results (cont’d)



The magnitude of the flux density B.

Numerical results (cont'd)



The magnitude of the current density
$\mathbf{J}_C := -i\omega\sigma\mathbf{A}_C - \sigma\,\mathrm{grad}\,V_C$.

Pros and cons

- *Pros*
  - standard nodal finite elements for all the unknowns;
  - no difficulty with the topology of the conducting domain;
  - "positive definite" algebraic problem.
- *Cons*
  - many degrees of freedom;
  - lack of convergence for re-entrant corners of the computational domain.

A FEM–BEM approach

Another interesting approach is based on a coupled formulation: variational in Ω_C, by means of potential theory in Ω_I.

In this framework, it is reasonable to consider $\Omega_I := \mathbf{R}^3 \setminus \overline{\Omega_C}$. Moreover, for the sake of simplicity let us require that Ω_C is a simply-connected open set with a connected boundary.

Finally, it is assumed that the applied current density $\mathbf{J}_e$ is vanishing in Ω_I, and that the magnetic permeability $\boldsymbol{\mu}_I$ and the electric permittivity $\boldsymbol{\varepsilon}_I$ are positive constants in Ω_I, say $\mu_0 > 0$ and $\varepsilon_0 > 0$.

A FEM–BEM approach (cont'd)

In terms of the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}_C$ the eddy current problem thus reads

$$\begin{cases} \text{curl}\,\mathbf{E}_C + i\omega\boldsymbol{\mu}_C\mathbf{H}_C = \mathbf{0} & \text{in } \Omega_C \\ \text{curl}\,\mathbf{H}_C - \boldsymbol{\sigma}\mathbf{E}_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\ \text{curl}\,\mathbf{H}_I = \mathbf{0} & \text{in } \Omega_I \\ \text{div}(\mu_0\mathbf{H}_I) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}_C\mathbf{H}_C\cdot\mathbf{n}_C + \mu_0\mathbf{H}_I\cdot\mathbf{n}_I = 0 & \text{on } \Gamma \\ \mathbf{H}_C\times\mathbf{n}_C + \mathbf{H}_I\times\mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\ \mathbf{H}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}|\rightarrow\infty\ . \end{cases} \tag{36}$$

A FEM–BEM approach (cont'd)

[If needed, the electric field $\mathbf{E}_I$ can be computed after having determined $\mathbf{H}_I$ and $\mathbf{E}_C$ in (36), by solving

$$
\begin{cases}
\text{curl}\, \mathbf{E}_I = -i\omega\mu_0 \mathbf{H}_I & \text{in } \Omega_I \\
\text{div}(\varepsilon_0 \mathbf{E}_I) = 0 & \text{in } \Omega_I \\
\mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C & \text{on } \Gamma \\
\int_\Gamma \varepsilon_0 \mathbf{E}_I \cdot \mathbf{n}_I = 0 & \\
\mathbf{E}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \text{ .]}
\end{cases}
$$

A FEM–BEM approach (cont'd)

For obtaining a formulation which is stable with respect to the frequency ω, it is better to look for a vector magnetic potential $\mathbf{A}_C$, a scalar electric potential V_C and a scalar magnetic potential ψ_I such that

$$\boldsymbol{\mu}_C \mathbf{H}_C = \text{curl}\, \mathbf{A}_C \quad , \quad \mathbf{E}_C = -i\omega \mathbf{A}_C - \text{grad}\, V_C \quad , \quad \mathbf{H}_I = \text{grad}\, \psi_I \ .$$

[See Pillsbury (1983), Rodger and Eastham (1983), Emson and Simkin (1983).]

Gauging is necessary only in Ω_C: we require the Coulomb gauge $\text{div}\, \mathbf{A}_C = 0$ in Ω_C, with $\mathbf{A}_C \cdot \mathbf{n}_C = 0$ on Γ. Moreover, we also impose that

$$|\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty \ .$$

A FEM–BEM approach (cont'd)

We have thus obtained the problem

$$
\begin{cases}
\text{curl}(\boldsymbol{\mu}_C^{-1}\,\text{curl}\,\mathbf{A}_C) & \\
\qquad\qquad +i\omega\boldsymbol{\sigma}\mathbf{A}_C + \boldsymbol{\sigma}\,\text{grad}\,V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\
\Delta\psi_I = 0 & \text{in } \Omega_I \\
\text{div}\,\mathbf{A}_C = 0 & \text{in } \Omega_C \\
\mathbf{A}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \\
\text{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C + \mu_0\,\text{grad}\,\psi_I \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\
(\boldsymbol{\mu}_C^{-1}\,\text{curl}\,\mathbf{A}_C) \times \mathbf{n}_C + \text{grad}\,\psi_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\
|\psi_I(\mathbf{x})| + |\,\text{grad}\,\psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty\ ,
\end{cases}
$$

where V_C is determined up to an additive constant.

A FEM–BEM approach (cont'd)

Inserting the Coulomb gauge condition in the Ampère equation as a penalization term, one has

$$
\left\{
\begin{array}{ll}
\operatorname{curl}(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C) - \mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{A}_C & \\
\qquad + i\omega \boldsymbol{\sigma} \mathbf{A}_C + \boldsymbol{\sigma} \operatorname{grad} V_C = \mathbf{J}_{e,C} & \text{in } \Omega_C \\
\Delta \psi_I = 0 & \text{in } \Omega_I \\
\operatorname{div}(i\omega \boldsymbol{\sigma} \mathbf{A}_C + \boldsymbol{\sigma} \operatorname{grad} V_C) = \operatorname{div} \mathbf{J}_{e,C} & \text{in } \Omega_C \\
(i\omega \boldsymbol{\sigma} \mathbf{A}_C + \boldsymbol{\sigma} \operatorname{grad} V_C) \cdot \mathbf{n}_C & \\
\qquad = \mathbf{J}_{e,C} \cdot \mathbf{n}_C & \text{on } \Gamma \\
\mathbf{A}_C \cdot \mathbf{n}_C = 0 & \text{on } \Gamma \\
\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n}_I = 0 & \text{on } \Gamma \\
(\boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C) \times \mathbf{n}_C & \\
\qquad + \operatorname{grad} \psi_I \times \mathbf{n}_I = \mathbf{0} & \text{on } \Gamma \\
|\psi_I(\mathbf{x})| + |\operatorname{grad} \psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \, .
\end{array}
\right.
\tag{37}
$$

A FEM–BEM approach (cont'd)

Since in Ω_I we have to solve the Laplace equation, using potential theory it is possible to transform the problem for ψ_I into a problem on the interface Γ, thus reducing in a significative way the number of unknowns in numerical computations.

We introduce on Γ (in suitable functional spaces...) the single layer and double layer potentials

$$\mathcal{S}(\xi)(\mathbf{x}) := \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\xi(\mathbf{y})dS_y$$

$$\mathcal{D}(\eta)(\mathbf{x}) := \int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3}\cdot\eta(\mathbf{y})\mathbf{n}_C(\mathbf{y})dS_y$$

A FEM–BEM approach (cont'd)

and the hypersingular integral operator

$$\mathcal{H}(\eta)(\mathbf{x}) := -\,\text{grad}\left(\int_\Gamma \frac{\mathbf{x}-\mathbf{y}}{4\pi|\mathbf{x}-\mathbf{y}|^3}\cdot\eta(\mathbf{y})\mathbf{n}_C(\mathbf{y})dS_y\right)\cdot\mathbf{n}_C(\mathbf{x})\ .$$

We also recall that the adjoint operator $\mathcal{D}'$ reads

$$\mathcal{D}'(\xi)(\mathbf{x}) = \left(\int_\Gamma \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3}\,\xi(\mathbf{y})dS_y\right)\cdot\mathbf{n}_C(\mathbf{x})\ .$$

A FEM–BEM approach (cont'd)

We have $\Delta\psi_I = 0$ in Ω_I and $\operatorname{grad}\psi_I \cdot \mathbf{n}_I = -\frac{1}{\mu_0} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C$ on Γ, therefore from potential theory the trace $\psi_\Gamma := \psi_{I|\Gamma}$ satisfies the bounday integral equations

$$\frac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) + \frac{1}{\mu_0}\mathcal{S}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C) = 0 \text{ on } \Gamma \qquad (38)$$

$$\frac{1}{2\mu_0} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C + \frac{1}{\mu_0}\mathcal{D}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}_C) + \mathcal{H}(\psi_\Gamma) = 0 \text{ on } \Gamma\,, \quad (39)$$

and ψ_I has been replaced by its trace ψ_Γ.

We can now devise a weak form of this $(\mathbf{A}_C, V_C) - \psi_\Gamma$ formulation. From the matching condition

$$\mathbf{n}_C \times \boldsymbol{\mu}_C^{-1} \operatorname{curl} \mathbf{A}_C + \mathbf{n}_I \times \operatorname{grad}\psi_I = \mathbf{0} \text{ on } \Gamma$$

A FEM–BEM approach (cont'd)

we find

$$\int_\Gamma \mathbf{n}_C \times \boldsymbol{\mu}_C^{-1} \,\text{curl}\, \mathbf{A}_C \cdot \overline{\mathbf{w}_C} = -\int_\Gamma \mathbf{n}_I \times \text{grad}\, \psi_I \cdot \overline{\mathbf{w}_C}$$
$$= -\int_\Gamma \psi_\Gamma \,\text{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \ ,$$

the last equality coming from standard integration by parts on Γ.
Hence, multiplying by suitable test functions $(\mathbf{w}_C, Q_C, \eta)$ with $\mathbf{w}_C \cdot \mathbf{n}_C = 0$ on Γ, integrating in Ω_C and Γ, and integrating by parts we end up with the following weak problem

A FEM–BEM approach (cont'd)

$$
\begin{aligned}
&\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \,\mathrm{curl}\, \mathbf{A}_C \cdot \mathrm{curl}\, \overline{\mathbf{w}_C} + \mu_*^{-1} \,\mathrm{div}\, \mathbf{A}_C \,\mathrm{div}\, \overline{\mathbf{w}_C}) \\
&\qquad + \int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \boldsymbol{\sigma} \,\mathrm{grad}\, V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad + \int_\Gamma [-\tfrac{1}{2}\psi_\Gamma - \mathcal{D}(\psi_\Gamma) \\
&\qquad\qquad\qquad + \frac{1}{\mu_0} \mathcal{S}(\mathrm{curl}\, \mathbf{A}_C \cdot \mathbf{n}_C)] \,\mathrm{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n}_C \\
&\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{w}_C} \\
&\int_{\Omega_C} (i\omega \boldsymbol{\sigma} \mathbf{A}_C \cdot \mathrm{grad}\, \overline{Q_C} + \boldsymbol{\sigma} \,\mathrm{grad}\, V_C \cdot \mathrm{grad}\, \overline{Q_C}) \\
&\quad = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \mathrm{grad}\, \overline{Q_C} \\
&\int_\Gamma [\tfrac{1}{2} \,\mathrm{curl}\, \mathbf{A}_C \cdot \mathbf{n}_C + \mathcal{D}'(\mathrm{curl}\, \mathbf{A}_C \cdot \mathbf{n}_C) + \mu_0 \mathcal{H}(\psi_\Gamma)] \overline{\eta} = 0 \ ,
\end{aligned}
\tag{40}
$$

having used (38) for obtaining the first equation.

[See Alonso Rodríguez and V. (2009).]

A FEM–BEM approach (cont'd)

- The sesquilinear form at the left hand side is coercive in $[H(\text{curl}; \Omega_C) \cap H_0(\text{div}; \Omega_C)] \times H^1(\Omega_C)/\mathbb{C} \times H^{1/2}(\Gamma)/\mathbb{C}$, uniformly with respect to ω (the case $\omega = 0$ is admitted!). [The crucial point is that $\mathcal{S}$ and $\mathcal{H}$ are coercive; the rest of the proof is similar to that employed for the $(\mathbf{A}, V_C)$-formulation.]
- Existence and uniqueness follow by the Lax–Milgram lemma.
- Having determined $\mathbf{A}_C$ and ψ_Γ (up to an additive constant), then $\psi_I := \mathcal{D}(\psi_\Gamma) - \frac{1}{\mu_0}\mathcal{S}(\text{curl}\,\mathbf{A}_C \cdot \mathbf{n}_C)$.
- Numerical approximation is performed with nodal finite elements in Ω_C and on Γ.

A FEM–BEM approach (cont'd)

- Convergence is assured provided that Ω_C is a convex polyhedron. If this is not true, one can modify the approach, using the vector potential $\mathbf{A}$ on a convex set Ω_A larger than Ω_C, keeping V_C in Ω_C and looking for ψ_{Γ_A} on $\Gamma_A := \partial\Omega_A$.

Other FEM–BEM couplings

- Bossavit and Vérité(1982, 1983) (for the magnetic field, and using the Steklov–Poincaré operator) [numerical code TRIFOU].
- Mayergoyz, Chari and Konrad (1983) (for the electric field, and using special basis functions near Γ).
- Hiptmair (2002) (unknowns: $\mathbf{E}_C$ in Ω_C, $\mathbf{H} \times \mathbf{n}$ on Γ).
- Meddahi and Selgas (2003) (unknowns: $\mathbf{H}_C$ in Ω_C, $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n}$ on Γ).
- Bermúdez, Gómez, Muñiz and Salgado (2007) (for axisymmetric problems associated to the modeling of induction furnaces).

References

A. Alonso and A. Valli: A domain decomposition approach for heterogeneous time-harmonic Maxwell equations. Comput. Methods Appl. Mech. Engrg., 143 (1997), 97-112.

A. Alonso Rodríguez, P. Fernandes and A. Valli: The time-harmonic eddy-current problem in general domains: solvability via scalar potentials. In Computational Electromagnetics (2003), C. Carstensen, S. Funken, W. Hackbusch, R.H.W. Hoppe and P. Monk, eds., Springer, Berlin, 143-163.

A. Alonso Rodríguez, R. Hiptmair and A. Valli: Mixed finite element approximation of eddy current problems. IMA J. Numer. Anal., 24 (2004), 255–271.

A. Alonso Rodríguez, R. Hiptmair and A. Valli: Hybrid formulations of eddy current problems. Report UTM 663 (2004), Department of Mathematics, University of Trento.

References (cont'd)

A. Alonso Rodríguez, R. Hiptmair and A. Valli: A hybrid formulation of eddy current problems. Numer. Methods Partial Differential Equations, 21 (2005), 742–763.

A. Alonso Rodríguez and A. Valli: Mixed finite element approximation of eddy current problems based on the electric field. In ECCOMAS 2004: European Congress on Computational Methods in Applied Sciences and Engineering, P. Neittaanmäki, T. Rossi, K. Majava and O. Pironneau, eds., University of Jyväskylä, Department of Mathematical Information Technology.

A. Alonso Rodríguez and A. Valli: A FEM–BEM approach for electro–magnetostatics and time-harmonic eddy-current problems. Appl. Numer. Math., 59 (2009), 2036–2049.

References (cont'd)

A. Alonso Rodríguez, A. Valli and R. Vázquez Hernández: A formulation of the eddy-current problem in the presence of electric ports. Numer. Math., 113 (2009), 643–672.

A. Bermúdez, D. Gómez, M.C. Muñiz and P. Salgado: A FEM/BEM for axisymmetric electromagnetic and thermal modelling of induction furnaces. Internat. J. Numer. Methods Engrg., 71 (2007), 856–878, 879–882.

A. Bermúdez, R. Rodríguez and P. Salgado: A finite element method with Lagrange multipliers for low-frequency harmonic Maxwell equations. SIAM J. Numer. Anal., 40 (2002), 1823-1849.

A. Bermúdez, R. Rodríguez and P. Salgado: Numerical solution of eddy-current problems in bounded domains using realistic boundary conditions. Comput. Methods Appl. Mech. Engrg., 194 (2005), 411–426.

References (cont'd)

O. Bíró: Edge element formulations of eddy current problems. Comput. Methods Appl. Mech. Engrg., 169 (1999), 391–405.

O. Bíró, K. Preis, G. Buchgraber and I. Tičar: Voltage-driven coils in finite-element formulations using a current vector and a magnetic scalar potential. IEEE Trans. Magn., 40 (2004), 1286–1289.

O. Bíró and A. Valli: The Coulomb gauged vector potential formulation for the eddy-current problem in general geometry: well-posedness and numerical approximation. Comput. Methods Appl. Mech. Engrg., 196 (2007), 1890–1904.

A. Bossavit: On the Lorenz gauge. COMPEL, 18 (1999), 323–336.

References (cont'd)

A. Bossavit: Most general ‘non-local’ boundary conditions for the Maxwell equations in a bounded region. COMPEL, 19 (2000), 239—245.

A. Bossavit and J.C. Vérité: A mixed FEM/BIEM Method to solve eddy-current problems. IEEE Trans. Magn., MAG-18 (1982), 431–435,

A. Bossavit and J.C. Vérité: The TRIFOU code: solving the 3-D eddy-currents problem by using H as state variable. IEEE Trans. Magn., MAG-19 (1983), 2465–2470.

M. Costabel and M. Dauge: Singularities of electromagnetic fields in polyhedral domains. Arch. Rational Mech. Anal., 151 (2000), 221-276.

References (cont'd)

M. Costabel and M. Dauge: Weighted regularization of Maxwell equations in polyhedral domains. A rehabilitation of nodal finite elements. Numer. Math., 93 (2002), 239–277.

M. Costabel, M. Dauge and S. Nicaise: Singularities of eddy current problems. M2AN Math. Model. Numer. Anal., 37 (2003), 807–831.

R.I. Emson and J. Simkin: An optimal method for 3-D eddy currents. IEEE Trans. Magn., MAG-19 (1983), 2450–2452.

P.W. Gross and P.R. Kotiuga: Electromagnetic Theory and Computation. A Topological Approach. Cambridge University Press (2004), New York.

References (cont'd)

R. Hiptmair: Symmetric coupling for eddy current problems. SIAM J. Numer. Anal., 40 (2002), 41–65.

A. Kameari and K. Koganezawa: Convergence of IICG method in FEM using edge elements without gauge condition. IEEE Trans. Magn., 33 (1997), 1223–1226.

P.R. Kotiuga: On making cuts for magnetic scalar potentials in multiply connected regions. J. Appl. Phys., 61 (1987), 3916–3918.

P.R. Kotiuga: Toward an algorithm to make cuts for magnetic scalar potentials in finite element meshes. J. Appl. Phys., 63 (1988), 3357–3359. Erratum: J. Appl. Phys., 64 (1988), 4257.

References (cont'd)

P.R. Kotiuga: An algorithm to make cuts for scalar potentials in tetrahedral meshes based on the finite element method. IEEE Trans. Magn., 25 (1989), 4129–4131.

P.J. Leonard and D. Rodger: A new method for cutting the magnetic scalar potential in multiply connected eddy current problems. IEEE Trans. Magn., 25 (1989), 4132–4134.

I.D. Mayergoyz, M.V.K. Chari and A. Konrad: Boundary Galerkin's method for three-dimensional finite element electromagnetic field computation. IEEE Trans. Magn., MAG-19 (1983), 2333–2336.

S. Meddahi and V. Selgas: A mixed-FEM and BEM coupling for a three-dimensional eddy current problem. M2AN Math. Model. Numer. Anal., 37 (2003), 291–318.

References (cont'd)

R.D. Pillsbury: A three dimensional eddy current formulation using two potentials: the magnetic vector potential and total magnetic scalar potential. IEEE Trans. Magn., MAG-19 (1983), 2284–2287.

Z. Ren: Influence of the R.H.S. on the convergence behaviour of the curl-curl equation. IEEE Trans. Magn., 32 (1996), 655–658.

D. Rodger and J.F. Eastham: A formulation for low frequency eddy current solutions. IEEE Trans. Magn., MAG-19 (1983), 2443–2446.

D. Rodger and J.F. Eastham: The use of transformations in applying boundary conditions to three-dimensional vector field problems. IEE Proc. A, 132 (1985), 165–170.