Topics in Computational Electromagnetism

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Part 1:

- Nodal elements and edge elements
- Models in electromagnetism
- The cavity problem for Maxwell equations: theory and approximation
- Eddy current problems: theory and approximation

Part 2:

- Coupled models and complexity reduction via a scalar magnetic potential
- Source fields, loop fields, harmonic fields
- First de Rham cohomology group and its basis functions
- Efficient numerical approximation of eddy current problems

Part 3:

- A specific application: two inverse problems (EEG and MEG)
- Models: potential equation and Biot–Savart formula, full Maxwell equations, eddy current equations
- Existence theory for the direct problems (potential equation and eddy current equations)
- Methods for the numerical approximation of the direct problem
- Theory and approximation of the MEG inverse problem

A finite element method is an approximation method for variational problems of the form

$$\text{find } u \in V \, : \, a(u,v) = \mathcal{F}(v) \quad \forall \, v \in V \, , \tag{1}$$

where the real/complex vector space V, the bilinear/sesquilinear form $a(\cdot,\cdot)$ and the linear/antilinear functional $\mathcal{F}(\cdot)$ are data of the problem.

Examples:

- Homogeneous Dirichlet problem for the Laplace operator

$$u \in H^1_0(\Omega) \ : \ \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v = \int_\Omega f v \quad \forall\, v \in H^1_0(\Omega)\,.$$

- Homogeneous Dirichlet problem for the $\mathbf{curl}\,\mathbf{curl} + \alpha I$ operator

$$\begin{aligned} \mathbf{u} \in H_0(\mathbf{curl}\,;\Omega) \ : \ & \textstyle\int_\Omega (\mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} + \alpha\, \mathbf{u} \cdot \mathbf{v}) \\ & = \textstyle\int_\Omega \mathbf{F} \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in H_0(\mathbf{curl}\,;\Omega)\,. \end{aligned}$$

- Homogeneous Dirichlet problem for the Stokes operator

$$\mathbf{u} \in (W)^3 \ : \quad \nu \textstyle\int_\Omega \mathbf{Jac}\,\mathbf{u} : \mathbf{Jac}\,\mathbf{v} = \int_\Omega \mathbf{F} \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in (W)^3$$

where $W = \{\mathbf{v} \in (H^1_0(\Omega))^3 \,|\mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega\}$.

The basic ingredients of a finite element method are:

- a triangulation of the computational domain Ω (mesh)
- a (finite dimensional) vector space V_h constituted by piecewise-polynomial functions.

The finite element method thus reads

$$\text{find } u_h \in V_h \ : \ a_h(u_h, v_h) = \mathcal{F}_h(v_h) \quad \forall \, v_h \in V_h \, . \tag{2}$$

Here:

- $a_h(\cdot, \cdot)$ and $\mathcal{F}_h(\cdot)$ are suitable approximations of $a(\cdot, \cdot)$ and $\mathcal{F}(\cdot)$ (often, they coincide with them).

Remark. A first natural requirement is that V_h must be a "good" approximation of V in the sense that

$$\mathrm{dist}(v, V_h) \to 0 \quad \forall\, v \in V\,. \tag{3}$$

- It is not necessary that $V_h \subset V$, but very often this is the case.

- In order to operate with V_h, it is necessary to find a basis of it (easy to construct and suitable for computations...).

Denoting by M_h the dimension of V_h, it is enough to find M_h linear functionals $\mathcal{G}_i$ such that

$$v_h \in V_h \ , \ \mathcal{G}_i(v_h) = 0 \ \ \forall \ i = 1, \dots, M_h \implies v_h = 0 \, . \tag{4}$$

[The $\mathcal{G}_i$ are called degrees of freedom.]

The basis is then given by the functions $\varphi_j \in V_h$ such that

$$\mathcal{G}_i(\varphi_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \, . \end{cases} \tag{5}$$

[Hint: check directly that φ_j are linearly independent...]

A natural choice (not the only possible one... we will see another example later on) of the degrees of freedom is the following: having selected M_h nodes $\mathbf{x}_i$ in the computational domain Ω, define

$$\mathcal{G}_i(\phi) = \phi(\mathbf{x}_i)\,. \tag{6}$$

[This definition requires that the point values of ϕ are well-defined scalar quantities; this is surely true if ϕ is a continuous scalar function, not necessarily if $\phi \in V$...]

Clearly, the choice of the nodes must be co-ordinated with the choice of V_h, in order to satisfy (4).

Let us make precise the context in a specific case.

- Assume that $\Omega \subset \mathbb{R}^3$ and that the elements K of the triangulation are tetrahedra.

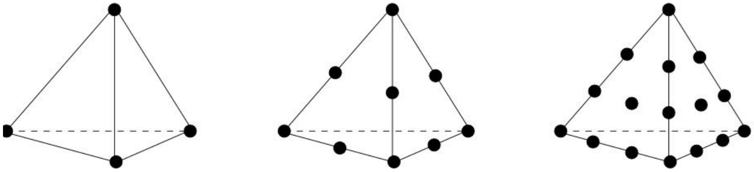
The simplest space of finite elements is the following:

$$V_h = L_h^r := \{ v_h \in C^0(\Omega) \,|\, v_{h|K} \in \mathbb{P}_r \ \forall\ K \} \,, \tag{7}$$

having denoted by $\mathbb{P}_r$ the set of polynomials of degree less than or equal to r, $r \geq 1$.

It is not difficult to determine how to choose the nodes in this situation: for instance,

- $r = 1$: the vertices of all the tetrahedra
- $r = 2$: the vertices of all the tetrahedra and the middle points of all the edges
- $r = 3$: the vertices of all the tetrahedra, all the points dividing an edge in three equal parts and the barycenters of all the faces.



The degrees of freedom for tetrahedra ($r = 1$, $r = 2$, $r = 3$).
Only the visible nodes are indicated.

Exercise. Condition (4) is satisfied. [Hint: show that an element of $\mathbb{P}_r$ vanishing at the nodes of a face must vanish on that face...]

Remark. In the proof of the exercise one verifies that it is possible to construct element-by-element a polynomial $q \in \mathbb{P}_r$ by assigning the value of its nodal degrees of freedom, and that on the interelements it is uniquely determined (if it vanishes on the nodes of a face, then it vanishes on the whole face...).
Hence putting the pieces together one finds a continuous function, namely, an element of the finite element space V_h defined in (7).
This element is uniquely determined by the values of the assigned degrees of freedom: in other words, the total number of the nodal degrees of freedom is equal to the dimension of V_h.

Remark. Indeed, for the finite elements introduced in (7), with nodal degrees of freedom, a more restrictive condition than (4) is satisfied. In fact, denoting by M_K the number of nodes belonging to the element K, one has

$$q \in \mathbb{P}_r \ , \ \mathcal{G}_i(q) = 0 \ \ \forall \ i = 1, \dots, M_K \implies q = 0 \, ,$$

and consequently

$$v_h \in V_h \ , \ \mathcal{G}_i(v_{h|K}) = 0 \ \ \forall \ i = 1, \dots, M_K \implies v_{h|K} = 0 \, . \qquad (8)$$

Therefore, it is easily seen that the basis functions have a "small" support: φ_i is non-vanishing only in the elements K of the triangulation that contain the node $\mathbf{x}_i$.

Question. Having done the choice

$$V_h = L_h^r := \{v_h \in C^0(\Omega) \,|\, v_{h|K} \in \mathbb{P}_r \ \forall \ K\}$$

with nodal degrees of freedom, is condition (3) satisfied?

To find an answer, let us begin with this remark. Denote by $\mathcal{V}$ the space of "smooth" functions and suppose that each function in V can be approximated by an element of $\mathcal{V}$ [this is very often the case for partial differential equations expressed in variational form: but there are exceptions...].

Then, given $v \in V$, a proof of (3) can start observing that

$$\mathrm{dist}(v, V_h) \leq \mathrm{dist}(v, w) + \mathrm{dist}(w, V_h)\,,$$

where $w \in \mathcal{V}$, and $\mathrm{dist}(v, w)$ can be taken arbitrarily small.

On the other hand,

$$\text{dist}(w, V_h) \leq \text{dist}(w, w_h) \quad \forall\; w_h \in V_h \,,$$

therefore the problem is to select a "good" approximation w_h of a smooth function w.

To this end, it is useful to consider the finite element interpolant of a function. It is defined as follows: given a function ϕ (say, continuous), the interpolant $\pi_h \phi$ of ϕ is the unique function belonging to V_h such that

$$(\pi_h \phi)(\mathbf{x}_i) = \phi(\mathbf{x}_i) \quad \forall\; i = 1, \ldots, M_h \,. \tag{9}$$

[Existence and uniqueness of $\pi_h \phi$ are a consequence of (4)...]

The interpolation operator $\pi_h : C^0(\overline{\Omega}) \to V_h$ is then trivially defined as the operator which associates to a function its interpolant:

$$\pi_h : \phi \to \pi_h \phi \,. \tag{10}$$

It is readily seen that

$$\pi_h \phi = \sum_{j=1}^{M_h} \phi(\mathbf{x}_j) \varphi_j \,. \tag{11}$$

[Hint: just check that $\sum_{j=1}^{M_h} \phi(\mathbf{x}_j)\varphi_j(\mathbf{x}_i) = \phi(\mathbf{x}_i)$...]

Let us focus now on the estimate of the interpolation error for a "smooth" function.

An estimate of the interpolation error depends on the characteristics of the space V, namely, depends on the distance defined in V. [Clearly, there are many distances defined in a vector space V: the right one is that making V a Hilbert space...]

Typically, for second order partial differential equations we have that V is a closed subspace of $H^1(\Omega)$, the Sobolev space of first order. (This is not always the case... we will see a different situation later on.)

Therefore one can think that

$$\mathrm{dist}(w, \pi_h w) = \|w - \pi_h w\|_{1,\Omega} \, .$$

It can be proved that for a "regular" family of triangulations and for the choice (7) with nodal degrees of freedom one has

$$\|w - \pi_h w\|_{1,\Omega} \leq C(w)h^r \tag{12}$$

for each "smooth" function w, hence condition (3) is satisfied.

[A family of triangulations $\mathcal{T}_h$, $h > 0$, is said "regular" if

$$\frac{\mathrm{diam}\, K}{\mathrm{diam}\, B_K} \leq \mathrm{const} \quad \forall\ K \in \mathcal{T}_h\ \forall\ h > 0\,,$$

where B_K denotes the largest ball contained in K: namely, the elements are not becoming more and more distorted as the mesh is refined.]

It can be useful to look deeper at the interpolation error estimate (12), in order to make explicit the regularity of w that is sufficient for obtaining the result.

In this respect, it can be proved that (12) holds provided that w belongs to $L^2(\Omega)$ together with all its derivatives up to order $r+1$: in other words, the interpolation error is of order r (with respect to the natural $H^1(\Omega)$-norm) if the (Sobolev) regularity of the solution is equal to $r+1$.

This result will be useful for checking that the order of convergence of the finite element method is related to the (Sobolev) regularity of the exact solution.

What is missing now is an estimate of the discretization error, namely, the distance between the exact solution $u \in V$ of problem (1) and the approximate solution $u_h \in V_h$ of problem (2).

[Clearly, we expect that the approximation condition (3), $\mathrm{dist}\,(v, V_h) \to 0$ for each $v \in V$, is a crucial one; but the discretization error cannot avoid reading also the type of differential problem we have at hand...]

The procedure we present is quite general (for linear problems). However, let us assume for the sake of simplicity that

$$a_h(\cdot,\cdot) = a(\cdot,\cdot) \quad , \quad \mathcal{F}_h(\cdot) = \mathcal{F}(\cdot) \quad , \quad V_h \subset V \, . \qquad (13)$$

[Note that the condition $V_h \subset V$ is clearly satisfied for the choice (7)...]

The argument of the so-called Céa lemma is the following.

By subtracting (2) from (1) (for $v = v_h \in V$) we have

$$a(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h\,. \tag{14}$$

[This property is often called consistency of the finite element scheme.]

Hence

$$\begin{aligned} a(u - u_h, u - u_h) &= a(u - u_h, u) \\ &= a(u - u_h, u - v_h) \quad \forall\, v_h \in V_h\,. \end{aligned} \tag{15}$$

Suppose now that

- V is a Hilbert space
- the (bilinear/sesquilinear) form $a(\cdot, \cdot)$ is
  - continuous, namely

$$|a(w,v)| \leq \gamma \, \|w\|_V \|v\|_V \quad \forall \; w, v \in V \tag{16}$$

  - coercive, namely

$$|a(v,v)| \geq \alpha \, \|v\|_V^2 \quad \forall \; v \in V \, . \tag{17}$$

[In particular, by Lax–Milgram lemma these conditions guarantee that there exists a unique solution u to (1) and a unique solution u_h to (2), for any linear/antilinear and continuous functional $\mathcal{F}$.]

From (15) one has

$$\begin{aligned}\alpha\|u-u_h\|_V^2 &\leq a(u-u_h,u-u_h)\\ &= a(u-u_h,u-v_h)\\ &\leq \gamma\,\|u-u_h\|_V\|u-v_h\|_V \quad \forall v_h\in V_h\,,\end{aligned}$$

hence

$$\|u-u_h\|_V \leq \frac{\gamma}{\alpha}\,\mathrm{dist}\,(u,V_h)\,, \tag{18}$$

and convergence is proved, provided that (3) holds.

Suppose now that V is a closed subspace of $H^1(\Omega)$ and that (16) and (17) are satisfied.

If one is working with the finite elements (7) with nodal degrees of freedom, it is possible to estimate the order of convergence of the finite element method.

In fact, we start from (18) and we find

$$\begin{aligned} \|u - u_h\|_{1,\Omega} &\leq \tfrac{\gamma}{\alpha} \,\mathrm{dist}\,(u, V_h) \\ &\leq \tfrac{\gamma}{\alpha} \,\|u - \pi_h u\|_{1,\Omega} \leq C(u)\, h^r \,, \end{aligned} \tag{19}$$

provided that $\mathcal{T}_h$ is a "regular" family of triangulations and the (Sobolev) regularity of u is equal to $r + 1$.

We will see that the Maxwell equations mainly involve the operator **curl** . This has as a consequence that electromagnetic problems can be approximated by means of a different type of vector finite elements, for which the continuity of all the components is not required.

In fact, what is really needed is that the curl operator is well-defined: not necessarily the gradient operator or the divergence operator.

Therefore, in order that a discrete function $\mathbf{w}_h$ is also an element of the variational space [still to be defined... but only involving the curl operator!], what is needed is the continuity of $\mathbf{w}_h \times \mathbf{n}$ on all the interelements.

- These elements are called edge elements, and have been proposed by Nédélec (1980).

Let us assume that the triangulation is composed by tetrahedra. For $r \geq 1$ denote by $\widetilde{\mathbb{P}}_r$ the space of homogeneous polynomials of degree r and define

$$S_r := \{\mathbf{q} \in (\widetilde{\mathbb{P}}_r)^3 \,|\, \mathbf{q}(\mathbf{x}) \cdot \mathbf{x} = 0\}$$

$$R_r := (\mathbb{P}_{r-1})^3 \oplus S_r \,.$$

The first family of Nédélec finite elements is

$$N_h^r := \{\mathbf{w}_h \in H(\mathbf{curl}\,;\Omega) \,|\, \mathbf{w}_{h|K} \in R_r \ \forall\ K \in \mathcal{T}_h\} \,. \qquad (20)$$

The degrees of freedom are not nodal values, but:

- edge degrees of freedom $m_e(\mathbf{w})$

$$\left\{ \int_e \mathbf{w} \cdot \boldsymbol{\tau}_e \, q \, ds \ \forall \, q \in \mathbb{P}_{r-1}(e) \right\} \tag{21}$$

- face degrees of freedom $m_f(\mathbf{w})$ (for $r \geq 2$)

$$\left\{ \int_f \mathbf{w} \times \mathbf{n}_f \cdot \mathbf{q} \, dS \ \forall \, \mathbf{q} \in (\mathbb{P}_{r-2}(f))^2 \right\} \tag{22}$$

- volume degrees of freedom $m_K(\mathbf{w})$ (for $r \geq 3$)

$$\left\{ \int_K \mathbf{w} \cdot \mathbf{q} \, dV \ \forall \, \mathbf{q} \in (\mathbb{P}_{r-3})^3 \right\} . \tag{23}$$

Here $\boldsymbol{\tau}_e$ denotes a unit vector with the direction of e, while $\mathbf{n}_f$ is the unit normal vector on f.

The total number of degrees of freedom on a tetrahedron K is equal to the dimension of R_r, and it can be shown that, if all the degrees of freedom vanish, then a polynomial $\mathbf{w} \in R_r$ is identically vanishing in K, hence conditions (8) and (4) are satisfied.

It can also be proved that, if a vector function $\mathbf{w} \in R_r$ has all its degrees of freedom vanishing on a face f of K and on the three edges contained in f, then the tangential component of $\mathbf{w}$ vanishes on f. This means that, using these degrees of freedom for identifying a piecewise-polynomial function that locally belongs to R_r, we obtain an element of $H(\mathbf{curl}\,;\Omega)$, hence an element of N_h^r.

- Let us specify the form of Nédélec edge elements and their degrees of freedom for $r = 1$.

The condition $\mathbf{q} \cdot \mathbf{x} = 0$ for $\mathbf{q} \in (\widetilde{\mathbb{P}}_1)^3$ says that $\mathbf{q} = \mathbf{a} \times \mathbf{x}$ with $\mathbf{a} \in \mathbb{R}^3$. Hence the space R_1 is given by the polynomials of the form

$$\mathbf{q}(\mathbf{x}) = \mathbf{b} + \mathbf{a} \times \mathbf{x} \quad , \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \, . \tag{24}$$

For $r = 1$ only edge degrees of freedom are active, and are given by

$$\int_e (\mathbf{b} + \mathbf{a} \times \mathbf{x}) \cdot \boldsymbol{\tau}_e \, ds \tag{25}$$

for the six edges e of the tetrahedron K.

- Let us show that if all the degrees of freedom of $\mathbf{q} = \mathbf{b} + \mathbf{a} \times \mathbf{x}$ on K are equal to 0, then $\mathbf{q} = \mathbf{0}$: in other words, (8) and (4) are satisfied.

A direct computation shows that $\mathbf{curl}\,\mathbf{q} = 2\,\mathbf{a}$. Moreover, from Stokes theorem for each face f we have

$$\begin{aligned} 0 &= \sum_e \int_e \mathbf{q} \cdot \boldsymbol{\tau}_e \, ds = \int_{\partial f} \mathbf{q} \cdot \boldsymbol{\tau} \, ds \\ &= \int_f \mathbf{curl}\,\mathbf{q} \cdot \mathbf{n}_f \, dS = 2\,\mathbf{a} \cdot \mathbf{n}_f \,\mathrm{meas}(f)\,, \end{aligned}$$

hence $\mathbf{a} \cdot \mathbf{n}_f = 0$ on f. Since three of the vectors $\mathbf{n}_f$ are linearly independent, it follows $\mathbf{a} = \mathbf{0}$.

Then for each edge e

$$\begin{aligned} 0 &= \int_e \mathbf{q} \cdot \boldsymbol{\tau}_e \, ds = \int_e \mathbf{b} \cdot \boldsymbol{\tau}_e \, ds \\ &= \mathbf{b} \cdot \boldsymbol{\tau}_e \, \mathrm{length}(e) \,, \end{aligned}$$

and three of the vectors $\boldsymbol{\tau}_e$ are linearly independent, so that $\mathbf{b} = \mathbf{0}$ and in conclusion $\mathbf{q} = \mathbf{0}$.

- Another point is to prove that if the three edge degrees of freedom of $\mathbf{q} = \mathbf{b} + \mathbf{a} \times \mathbf{x}$ on a face f are equal to 0 then $\mathbf{q} \times \mathbf{n}_f = \mathbf{0}$ on f.

We have already seen that $\mathbf{a} \cdot \mathbf{n}_f = 0$ on f. On the other hand,

$$\begin{aligned} \mathbf{q} \times \mathbf{n}_f &= \mathbf{b} \times \mathbf{n}_f + (\mathbf{a} \times \mathbf{x}) \times \mathbf{n}_f \\ &= \mathbf{b} \times \mathbf{n}_f + (\mathbf{a} \cdot \mathbf{n}_f)\, \mathbf{x} - (\mathbf{x} \cdot \mathbf{n}_f)\, \mathbf{a} \,. \end{aligned}$$

Since on a face one has $\mathbf{x} \cdot \mathbf{n}_f =$ const, it follows that $\mathbf{q} \times \mathbf{n}_f$ is equal on f to a constant vector $\mathbf{c}_f$, with $\mathbf{c}_f \cdot \mathbf{n}_f = 0$.

Finally,

$$\begin{aligned} 0 &= \int_e \mathbf{q} \cdot \boldsymbol{\tau}_e \, ds = \int_e (\mathbf{n}_f \times \mathbf{q} \times \mathbf{n}_f) \cdot \boldsymbol{\tau}_e \, ds \\ &= (\mathbf{n}_f \times \mathbf{c}_f) \cdot \boldsymbol{\tau}_e \, \text{length}(e) \, . \end{aligned}$$

Since two of the vectors $\boldsymbol{\tau}_e$ are generating the plane containing f (and the vector $\mathbf{n}_f \times \mathbf{c}_f$), it follows $\mathbf{c}_f = \mathbf{0}$ and consequently $\mathbf{q} \times \mathbf{n}_f = \mathbf{0}$ on f.

- In particular, we have shown that the dimension of N_h^1 is equal to the total number of the edge degrees of freedom (i.e., the total number of edges).

The basis functions are defined as in (5), namely, for each edge e_m we construct the vector function $\mathbf{\Phi}_m$ such that

$$\int_{e_l} \mathbf{\Phi}_m \cdot \boldsymbol{\tau} \, ds = \begin{cases} 1 & \text{if } m = l \\ 0 & \text{if } m \neq l \, . \end{cases} \tag{26}$$

Since (8) is satisfied, the basis functions have a "small" support: $\mathbf{\Phi}_m$ is non-vanishing only in the elements K of the triangulation that contain the edge e_m.

- The explicit construction of a basis for the edge element space N_h^1 is easily done.

In fact, it can be proved that the basis function $\mathbf{\Phi}_{i,j}$ associated to the edge $e_{i,j}$ joining the nodes $\mathbf{x}_i$ and $\mathbf{x}_j$ and satisfying $\int_{e_{i,j}} \mathbf{\Phi}_{i,j} \cdot \boldsymbol{\tau} \, ds = 1$ is given by

$$\mathbf{\Phi}_{i,j} = \varphi_i \, \mathbf{grad} \, \varphi_j - \varphi_j \, \mathbf{grad} \, \varphi_i \, , \tag{27}$$

where φ_i is the piecewise-linear nodal basis function associated to the node $\mathbf{x}_i$.

As usual, the interpolant $\mathbf{r}_h\mathbf{w}$ of a (smooth enough) vector function $\mathbf{w}$ is the unique vector function belonging to N_h^r such that

$$\begin{aligned} m_e(\mathbf{r}_h\mathbf{w}) &= m_e(\mathbf{w}) \\ m_f(\mathbf{r}_h\mathbf{w}) &= m_f(\mathbf{w}) \\ m_K(\mathbf{r}_h\mathbf{w}) &= m_K(\mathbf{w}) \end{aligned} \qquad (28)$$

for each edge e, face f and element K.

The interpolation operator $\mathbf{r}_h : S \to N_h^r$ is defined as

$$\mathbf{r}_h : \mathbf{w} \to \mathbf{r}_h\mathbf{w} \qquad (29)$$

(having denoted by S the space of "smooth enough" vector functions: we will come back to this here below...).

The interpolant $\mathbf{r}_h\mathbf{w}$ can be written as

$$\mathbf{r}_h\mathbf{w} = \sum_e m_e(\mathbf{w})\boldsymbol{\Phi}_e + \sum_f m_f(\mathbf{w})\boldsymbol{\Phi}_f + \sum_K m_K(\mathbf{w})\boldsymbol{\Phi}_K \qquad (30)$$

(having denoted by $\boldsymbol{\Phi}_e$ the set of basis functions associated to the edge e and similarly for the other cases).

- Question: what about the space S, where the interpolation operator is defined?

It is necessary to give a meaning to line integrals and surface integrals, which is not possible for functions belonging to the space $H(\mathbf{curl}\,;\Omega)$.

Up today, the best result is due to Amrouche, Bernardi, Dauge and Girault (1998): if we know that for some $p > 2$ the function $\mathbf{w}$ satisfies $\mathbf{w} \in (L^p(\Omega))^3$ with $\mathbf{curl}\,\mathbf{w} \in (L^p(\Omega))^3$ and $\mathbf{w}_{|K} \times \boldsymbol{\nu} \in ((L^p(\partial K))^3$ for each $K \in \mathcal{T}_h$, then the interpolant $\mathbf{r}_h\mathbf{w}$ is well-defined.

For instance, this is true if $\mathbf{w}$ has a sufficiently large Sobolev regularity, namely, if $\mathbf{w} \in H^s(\mathbf{curl}\,;\Omega)$ for $s > 1/2$, where

$$H^s(\mathbf{curl}\,;\Omega) := \{\mathbf{w} \in (H^s(\Omega))^3 \,|\, \mathbf{curl}\,\mathbf{w} \in (H^s(\Omega))^3\} \,. \tag{31}$$

[Since the exponent s can be non-integer, this space looks a little bit "exotic"... However, it is necessary to take it into consideration, as in general the solutions of Maxwell and eddy current equations are not very regular in the scale of Sobolev spaces: it happens that $s < 1$.]

If the family of triangulations $\mathcal{T}_h$ is regular and $\mathbf{w} \in H^s(\mathbf{curl}\,;\Omega)$, $1/2 < s \leq r$, it is possible to prove the following interpolation error estimate

$$\begin{gathered} \|\mathbf{w} - \mathbf{r}_h\mathbf{w}\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{w} - \mathbf{curl}\,(\mathbf{r}_h\mathbf{w})\|_{0,\Omega} \\ \leq Ch^s(\|\mathbf{w}\|_{s,\Omega} + \|\mathbf{curl}\,\mathbf{w}\|_{s,\Omega}) \end{gathered} \tag{32}$$

(see Alonso and V. (1999)).

Since each vector function belonging to $H(\mathbf{curl}\,;\Omega)$ can be approximated by smooth vector functions, we can conclude that approximation property (3), namely,

$$\mathrm{d}ist(v, V_h) \to 0 \quad \forall\ v \in V$$

is satisfied for $V = H(\mathbf{curl}\,;\Omega)$ and $V_h = N_h^r$.

The complete Maxwell system of electromagnetism reads

$$\begin{cases} \dfrac{\partial \mathcal{D}}{\partial t} + \mathcal{J} = \textbf{curl}\, \mathcal{H} & \text{Maxwell–Ampère equation} \\ \dfrac{\partial \mathcal{B}}{\partial t} + \textbf{curl}\, \mathcal{E} = 0 & \text{Faraday equation} \\ \text{div}\mathcal{D} = \rho & \text{Gauss electrical equation} \\ \text{div}\mathcal{B} = 0 & \text{Gauss magnetic equation .} \end{cases}$$

- $\mathcal{H}$ and $\mathcal{E}$ are the magnetic field and electric field, respectively
- $\mathcal{B}$ and $\mathcal{D}$ are the magnetic induction and electric induction, respectively
- $\mathcal{J}$ and ρ are the (surface) electric current density and (volume) electric charge density, respectively.

These fields are related through some constitutive equations: it is usually assumed a linear dependence like

$$\mathcal{D} = \varepsilon\mathcal{E} \quad , \quad \mathcal{B} = \boldsymbol{\mu}\mathcal{H} \quad , \quad \mathcal{J} = \boldsymbol{\sigma}\mathcal{E} + \mathcal{J}_e \ ,$$

where ε and $\boldsymbol{\mu}$ are the electric permittivity and magnetic permeability, respectively, and $\boldsymbol{\sigma}$ is the electric conductivity.

[In general, ε, $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are not constant, but are symmetric and uniformly positive definite matrices (with entries that are bounded functions of the space variable $\mathbf{x}$). Clearly, the conductivity $\boldsymbol{\sigma}$ is only present in conductors, and is identically vanishing in any insulator.]

- $\mathcal{J}_e$ is the applied electric current density.

As observed in experiments and stated by the Faraday law, a time-variation of the magnetic field generates an electric field. Therefore, in each conductor a current density $\mathbf{J}_{\rm eddy} = \boldsymbol{\sigma}\mathbf{E}$ arises; this term expresses the presence in conducting media of the so-called eddy currents.

This phenomenon, and the related heating of the conductor, was observed and studied in the mid of the nineteenth century by the French physicist L. Foucault, and in fact the generated eddy currents are also known as Foucault currents.

In many real-life applications, the time of propagation of the electromagnetic waves is very small with respect to some characteristic time scale, or, equivalently, their wave length is much larger than the diameter of the physical domain.

Therefore one can think that the speed of propagation is infinite, and take into account only the diffusion of the electromagnetic fields, neglecting electromagnetic waves.

Rephrasing this concept, one can also say that, when considering time-dependent problems in electromagnetism, one can distinguish between "fast" varying fields and "slowly" varying fields. In the latter case, one is led to simplify the set of equations, neglecting time derivatives, or, depending on the specific situation at hand, one time derivative, either $\frac{\partial \mathcal{D}}{\partial t}$ or $\frac{\partial \mathcal{B}}{\partial t}$.

Typically, problems of this type are peculiar of electrical engineering, where low frequencies are involved, but not of electronic engineering, where the frequency ranges in much larger bands.

Let us focus on the case in which the displacement current term $\frac{\partial \mathcal{D}}{\partial t}$ can be disregarded, while the time-variation of the magnetic induction is still important, as well as the related presence of eddy currents in the conductors.

- The resulting equations are called eddy current equations.

A thumb rule for deciding wheter $\frac{\partial \mathcal{D}}{\partial t}$ can be dropped is the following: if L is a typical length in Ω (say, its diameter) and we choose the inverse of the angular frequency ω^{-1} as a typical time, it is possibile to disregard the displacement current term provided that

$$|\mathcal{D}||\omega| \ll |\mathcal{H}|L^{-1} \quad , \quad |\mathcal{D}||\omega| \ll |\boldsymbol{\sigma}\mathcal{E}| \,.$$

Using the Faraday equation, we can write $\mathcal{E}$ is terms of $\mathcal{H}$, finding

$$|\mathcal{E}|L^{-1} \approx |\omega||\boldsymbol{\mu}\mathcal{H}| \,.$$

Hence, recalling that $\mathcal{D} = \varepsilon\mathcal{E}$ and putting everything together, one should have

$$\mu_{\max}\,\varepsilon_{\max}\omega^2 L^2 \ll 1 \quad , \quad \sigma_{\min}^{-1}\varepsilon_{\max}|\omega| \ll 1 \, ,$$

where $\mu_{\max}$ and $\varepsilon_{\max}$ are uniform upper bounds in Ω for the maximum eigenvalues of $\boldsymbol{\mu}(\mathbf{x})$ and $\boldsymbol{\varepsilon}(\mathbf{x})$, respectively, and $\sigma_{\min}$ denotes a uniform lower bound in Ω_C for the minimum eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$.
Since the magnitude of the velocity of the electromagnetic wave can be estimated by $(\mu_{\max}\,\varepsilon_{\max})^{-1/2}$, the first relation is requiring that the wave length λ is large compared to L.

Let us also note that for electrical industry applications some typical values of the parameters involved are $\mu_0 = 4\pi \times 10^{-7}$ H/m, $\varepsilon_0 = 8.9 \times 10^{-12}$ F/m, $\sigma_{\rm copper} = 5.7 \times 10^7$ S/m, $\omega = 2\pi \times 50$ rad/s (power frequency of 50 Hz), hence in that case

$$\lambda = \frac{1}{\sqrt{\mu_0\varepsilon_0}|\omega|} \approx 10^6 \, \mathrm{m} \quad , \quad \sigma_{\rm copper}^{-1}\varepsilon_0|\omega| \approx 4.9 \times 10^{-17} \, ,$$

and dropping the displacement current term looks appropriate.

Though less apparent, the same is true for a typical physiological problem, for which $\varepsilon_{\rm tissue} \approx 10^{-6}$ F/m and $\sigma_{\rm tissue} \approx 10^{-1}$ S/m, giving

$$\lambda \approx 3 \times 10^3 \, \mathrm{m} \quad , \quad \sigma_{\rm tissue}^{-1}\varepsilon_{\rm tissue}|\omega| \approx 3 \times 10^{-3} \, .$$

When interested in time-periodic phenomena, it is assumed that

$$\begin{aligned} \mathcal{J}_e(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{J}_e(\mathbf{x})\exp(i\omega t)] \\ \mathcal{E}(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{E}(\mathbf{x})\exp(i\omega t)] \\ \mathcal{H}(t,\mathbf{x}) &= \mathrm{Re}[\mathbf{H}(\mathbf{x})\exp(i\omega t)]\,. \end{aligned} \tag{33}$$

- $\omega \neq 0$ is the (angular) frequency.

Inserting these relations in the Maxwell equations one obtains the so-called time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - i\omega\varepsilon\mathbf{E} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega\,. \end{cases} \tag{34}$$

[Note that similar equations arise from the backward-Euler time-discretization of Maxwell equations: just substitute $i\omega$ by $\frac{1}{\Delta t}\cdots$]

As a consequence one has $\mathrm{div}(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω, and the electric charge in conductors is defined by $\rho = \mathrm{div}(\varepsilon\mathbf{E})$.

It can be proved that the time-harmonic Maxwell equations have a unique solution (provided that suitable boundary conditions are added, and that the conductor is not empty; we will come back later on to the case in which the conductor is empty).

On the other hand, dropping the displacement current term, the time-harmonic eddy current equations are

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - \boldsymbol{\sigma}\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega\,. \end{cases} \tag{35}$$

Let us spend some more words about eddy current equations.
Since in an insulator one has $\boldsymbol{\sigma} = \mathbf{0}$, it follows that $\mathbf{E}$ is not uniquely determined in that region ($\mathbf{E} + \mathbf{grad}\,\psi$ is still a solution).
Some additional conditions ("gauge" conditions) are thus necessary: the most natural idea is to impose the conditions satisfied by the solution $\mathbf{E}$ of the Maxwell equations.
As in the insulator Ω_I we have no charges, the first additional condition is

$$\mathrm{div}(\varepsilon_I \mathbf{E}_I) = 0 \qquad \text{in } \Omega_I \tag{36}$$

($\mathbf{E}_I$ means $\mathbf{E}_{|\Omega_I}$, and similarly for other quantities).

Other gauge conditions are related to the topology of the insulator Ω_I. Denoting by Ω_C the conductor (strictly contained in the physical domain Ω, and surrounded by the insulator Ω_I) and by $\Gamma := \overline{\Omega_C} \cap \overline{\Omega_I}$, let us define

$$\mathcal{H}_I := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\,\mathbf{G}_I = \mathbf{0}, \mathrm{div}(\varepsilon_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, BC_E(\mathbf{G}_I) = 0 \text{ on } \partial\Omega\} \,,$$

where BC_E denotes the boundary condition imposed on $\mathbf{E}_I$ (see later on for a precise description).
The topological gauge conditions can be written as

$$\varepsilon_I \mathbf{E}_I \perp \mathcal{H}_I \,. \tag{37}$$

Thus these conditions are ensuring that, if in addition one has $\mathbf{curl}\, \mathbf{E}_I = \mathbf{0}$ in Ω_I, $\mathrm{div}(\varepsilon_I \mathbf{E}_I) = 0$ in Ω_I, $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on Γ and $BC_E(\mathbf{E}_I) = 0$ on $\partial\Omega$, then it follows $\mathbf{E}_I = \mathbf{0}$ in Ω_I.

- It can be shown that the orthogonality condition $\varepsilon_I \mathbf{E}_I \perp \mathcal{H}_I$ is equivalent to impose that the flux of $\varepsilon_I \mathbf{E}_I$ is vanishing on a suitable set of surfaces.
  [These surfaces depend on the choice of the boundary condition for $\mathbf{E}_I$; for instance, for $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ they are the connected components of $\partial\Omega \cup \Gamma$.]

We will distinguish between two types of boundary conditions.

- Electric. One imposes $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. [As a consequence, one also has $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$.]
- Magnetic (Maxwell). One imposes $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. [As a consequence, one also has $\boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = -(i\omega)^{-1}\mathbf{J}_e \cdot \mathbf{n}$ on $\partial\Omega$.]
- Magnetic (eddy currents). One imposes $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ and $\boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = 0$ on $\partial\Omega$. [Note that $\mathbf{H} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ implies $\mathbf{J}_e \cdot \mathbf{n} = 0$ on $\partial\Omega$.]

For eddy current equations, the notation $BC_E(\mathbf{E}_I)$ on $\partial\Omega$ therefore refers to $\mathbf{E}_I \times \mathbf{n}$ for the electric boundary condition, and to $\boldsymbol{\varepsilon}_I\mathbf{E}_I \cdot \mathbf{n}$ for the magnetic boundary conditions.

Let us consider a couple of questions.

- If a vector field satisfies $\mathbf{curl}\,\mathbf{v} = \mathbf{0}$ and $\operatorname{div}\mathbf{v} = 0$ in a domain, together with the boundary conditions $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on a part of the boundary and $\mathbf{v} \cdot \mathbf{n} = 0$ on the other part, is it non-trivial, namely, not vanishing everywhere in the domain? [A field like that is called harmonic field.]
- If that is the case, do harmonic fields appear in electromagnetism?

Both questions have an affermative answer.

Let us start from the first question.

If the domain $\mathcal{O}$ is homeomorphic to a three-dimensional ball, a curl-free vector field $\mathbf{v}$ must be a gradient of a scalar function ψ, that must be harmonic due to the constraint on the divergence.

If the boundary condition is $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{O}$, which in this case is a connected surface, then it follows $\psi =$ const. on $\partial\mathcal{O}$, and therefore $\psi =$ const. in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

If the boundary condition is $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\mathcal{O}$, then ψ satisfies a homogeneous Neumann boundary condition and thus $\psi =$ const. in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

The same result follows if the boundary conditions are $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ_D and $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_N, and Γ_D is a connected surface: in fact, we still have $\psi =$ const. on Γ_D and $\mathbf{grad}\, \psi \cdot \mathbf{n} = 0$ on Γ_N, hence ψ satisfies a mixed boundary value problem and we obtain $\psi =$ const. in $\mathcal{O}$ and $\mathbf{v} = \mathbf{0}$ in $\mathcal{O}$.

However, the problem is different in a more general geometry.

In fact, take the magnetic field generated in the vacuum by a current of constant intensity I^0 passing along the x_3-axis: as it is well-known, for $x_1^2 + x_2^2 > 0$ it is given by

$$\mathbf{H}(x_1, x_2, x_3) = \frac{I^0}{2\pi} \left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0 \right) .$$

It is easily checked that, as Maxwell equations require, $\mathbf{curl}\, \mathbf{H} = \mathbf{0}$ and $\mathrm{div} \mathbf{H} = 0$.

Let us consider now the torus $\mathcal{T}$ obtained by rotating around the x_3-axis the disk of centre $(a, 0, 0)$ and radius b, with $0 < b < a$. One sees at once that $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}$; hence we have found a non-trivial harmonic field $\mathbf{H}$ in $\mathcal{T}$ satisfying $\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\mathcal{T}$.

On the other hand, consider now the electric field generated in the vacuum by a pointwise charge ρ_0 placed at the origin. For $\mathbf{x} \neq \mathbf{0}$ it is given by

$$\mathbf{E}(x_1, x_2, x_3) = \frac{\rho_0}{4\pi\varepsilon_0} \frac{\mathbf{x}}{|\mathbf{x}|^3} ,$$

where ε_0 is the electric permittivity of the vacuum.

It satisfies $\mathrm{div}\mathbf{E} = 0$ and $\mathbf{curl}\,\mathbf{E} = \mathbf{0}$, and moreover $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on the boundary of $\mathcal{C} := B_{R_2} \setminus \overline{B_{R_1}}$, where $0 < R_1 < R_2$ and $B_R := \{\mathbf{x} \in \mathbb{R}^3 \,|\, |\mathbf{x}| < R\}$ is the ball of centre $\mathbf{0}$ and radius R. We have thus found a non-trivial harmonic field $\mathbf{E}$ in $\mathcal{C}$ satisfying $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\mathcal{C}$.

These two examples show that the geometry of the domain and the type of boundary conditions play an essential role when considering harmonic fields.

What are the relevant differences between the set $\mathcal{O}$, homeomorphic to a ball, and the sets $\mathcal{T}$ and $\mathcal{C}$?

For the former, the point is that in $\mathcal{T}$ we have a non-bounding cycle, namely, a cycle that is not the boundary of a surface contained in $\mathcal{T}$ (take for instance the circle of centre $\mathbf{0}$ and radius a in the (x_1, x_2)-plane).

In the latter case, the boundary of $\mathcal{C}$ is not connected.

Four types of spaces of harmonic fields are coming into play.

- For the electric field

$$\mathcal{H}_I^{(A)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\, \mathbf{G}_I = \mathbf{0}, \mathrm{div}(\varepsilon_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,$$

$$\mathcal{H}_I^{(B)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\, \mathbf{G}_I = \mathbf{0}, \mathrm{div}(\varepsilon_I \mathbf{G}_I) = 0 \\ \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma, \varepsilon_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,,$$

- For the magnetic field

$$\mathcal{H}_I^{(C)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\,\mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0 \\ \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma, \mathbf{G}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,,$$

$$\mathcal{H}_I^{(D)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\,\mathbf{G}_I = \mathbf{0}, \mathrm{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0 \\ \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \Gamma, \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$$

All are finite dimensional! Their dimension is a topological invariant (precisely,... see below!).

Let us make precise which are the basis functions of $\mathcal{H}_I^{(D)}$ and $\mathcal{H}_I^{(C)}$.

For $\mathcal{H}_I^{(D)}$ one has first to introduce the "cutting" surfaces $\Xi_\alpha^* \subset \Omega_I$, $\alpha = 1, \ldots, g_{\Omega_I}$, with $\partial\Xi_\alpha^* \subset \partial\Omega \cup \Gamma$, such that every curl-free vector field in Ω_I has a global potential in $\Omega_I \setminus \cup_\alpha \Xi_\alpha^*$.

The number g_{Ω_I} is the number of (independent) non-bounding cycles in Ω_I, namely, the first Betti number of Ω_I, or, equivalently, the dimension of the first homology group of $\overline{\Omega_I}$ (this is the quotient space between the cycles in $\overline{\Omega_I}$ and the bounding cycles in $\overline{\Omega_I}$).

These surfaces "cut" the non-bounding cycles in $\overline{\Omega_I}$.

The basis functions $\boldsymbol{\rho}^*_{\alpha,I}$ are the $(L^2(\Omega_I))^3$-extensions of $\mathbf{grad}\, p^*_{\alpha,I}$, where $p^*_{\alpha,I}$ is the solution to

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I \mathbf{grad}\, p^*_{\alpha,I}) = 0 & \text{in } \Omega_I \setminus \Xi^*_\alpha \\ \boldsymbol{\mu}_I \mathbf{grad}\, p^*_{\alpha,I} \cdot \mathbf{n} = 0 & \text{on } (\partial\Omega \cup \Gamma) \setminus \partial\Xi^*_\alpha \\ \left[\boldsymbol{\mu}_I \mathbf{grad}\, p^*_{\alpha,I} \cdot \mathbf{n}_{\Xi^*}\right]_{\Xi^*_\alpha} = 0 & \\ \left[p^*_{\alpha,I}\right]_{\Xi^*_\alpha} = 1 \, , & \end{cases} \tag{38}$$

having denoted by $[\,\cdot\,]_{\Xi^*_\alpha}$ the jump across the surface Ξ^*_α and by $\mathbf{n}_{\Xi^*}$ the unit normal vector on Ξ^*_α.

[Later on we will see another way for constructing the basis functions of $\mathcal{H}_I^{(D)}$.]

The basis functions for $\mathcal{H}_I^{(C)}$ can be defined as follows.

First of all we have $\mathbf{grad}\, z_{I,l}$, the solutions to

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I \mathbf{grad}\, z_{I,l}) = 0 & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \mathbf{grad}\, z_{I,l} \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ z_{I,l} = 0 & \text{on } \partial\Omega \setminus (\partial\Omega)_l \\ z_{I,l} = 1 & \text{on } (\partial\Omega)_l \ , \end{cases} \tag{39}$$

where $l = 1, \ldots, p_{\partial\Omega}$, and $p_{\partial\Omega} + 1$ is the number of connected components of $\partial\Omega$.

To complete the construction of the basis functions we have to proceed further.
For that, as in the preceding case, let us recall that in Ω_I there exists a set of "cutting" surfaces Ξ_q, with $\partial\Xi_q \subset \Gamma$, such that every curl-free vector field in Ω_I with vanishing tangential component on $\partial\Omega$ has a global potential in $\Omega_I \setminus \cup_q \Xi_q$.

These surfaces "cut" the $\partial\Omega$-independent non-bounding cycles in Ω_I (whose number is denoted by n_Γ).

Then introduce the functions $p_{q,I}$, $q = 1, \dots, n_\Gamma$, defined in $\Omega_I \setminus \Xi_q$ and solutions to

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I \,\mathbf{grad}\, p_{q,I}) = 0 & \text{in } \Omega_I \setminus \Xi_q \\ \boldsymbol{\mu}_I \,\mathbf{grad}\, p_{q,I} \cdot \mathbf{n} = 0 & \text{on } \Gamma \setminus \partial\Xi_q \\ p_{q,I} = 0 & \text{on } \partial\Omega \\ [\boldsymbol{\mu}_I \,\mathbf{grad}\, p_{q,I} \cdot \mathbf{n}_\Xi]_{\Xi_q} = 0 & \\ [p_{q,I}]_{\Xi_q} = 1 \,, & \end{cases} \tag{40}$$

having denoted by $[\,\cdot\,]_{\Xi_q}$ the jump across the surface Ξ_q and by $\mathbf{n}_\Xi$ the unit normal vector on Ξ_q.

The other basis functions $\boldsymbol{\rho}_{q,I}$ are the $(L^2(\Omega_I))^3$-extensions of $\mathbf{grad}\, p_{q,I}$ (computed in $\Omega_I \setminus \Xi_q$).

- Edge elements are a suitable tool for numerical approximation of Maxwell and eddy current equations.

In order to give an example, let us consider the cavity problem for the time-harmonic Maxwell equations (34), with electric boundary condition. This means that the computational domain Ω is an empty cavity surrounded by a perfectly conducting medium.

In this situation, it is also reasonable to assume that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are scalar constants, say, $\boldsymbol{\varepsilon} = \varepsilon_0$ and $\boldsymbol{\mu} = \mu_0$, the electric permittivity and the magnetic permeability of the vacuum.

Therefore the problem reads

$$\left\{ \begin{array}{ll} \mathbf{curl}\,\mathbf{H} - i\omega\varepsilon_0\mathbf{E} = \mathbf{J}_e & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\mu_0\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{array} \right. \tag{41}$$

Using the Faraday equation to write $\mathbf{H}$ in terms of $\mathbf{E}$ and substituting the result $\mathbf{H} = -(i\omega\mu_0)^{-1}\mathbf{curl}\,\mathbf{E}$ in the Ampère equation, one is left with

$$\left\{ \begin{array}{ll} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \omega^2\mu_0\varepsilon_0\mathbf{E} = -i\omega\mu_0\mathbf{J}_e & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{array} \right.$$

Introducing the wave number

$$k := |\omega|\sqrt{\mu_0\varepsilon_0}\,, \tag{42}$$

we can finally write

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\mathbf{E} = -i\omega\mu_0\mathbf{J}_e & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases}$$

Splitting $\mathbf{J}_e$ into its real and imaginary parts, we can solve two problems of the same form for the real and imaginary parts of $\mathbf{E}$.

Hence, we can focus on the problem

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\mathbf{E} = \mathbf{F} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,, \end{cases} \tag{43}$$

where all the functions are real valued.

- Problem (43) is associated to a bilinear form that is not coercive in $H(\mathbf{curl}\,;\Omega)$ [$-k^2$ has the "wrong" sign...]. What we can say about existence and uniqueness of a solution?

Consider the Maxwell eigenvalue problem

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \lambda \mathbf{E} & \text{in } \Omega \\ \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{cases} \tag{44}$$

The classical Hilbert–Schmidt theory can be applied to obtain

- Besides $\lambda_0 = 0$, there exists a sequence of positive, increasing and diverging to ∞ eigenvalues λ_m of problem (44) [see, e.g., Leis (1986)].

Fredholm alternative theory can be used to prove

- When $k \neq \sqrt{\lambda_m}$, $m = 0, 1, 2, \ldots$, there exists a unique solution of problem (43).

Numerical approximation of (43) is important in order to simulate the real physical situation and obtain informations for shape optimization (for instance, an electromagnetic cavity is a model for microwave ovens).

[Clearly, to this aim another issue is the numerical simulation of (44); however, here we do not consider this problem, referring to Boffi, Fernandes, Gastaldi and Perugia (1999), Caorsi, Fernandes and Raffetto (2000) and Monk (2003a).]

The variational formulation of (43) is

$$\begin{array}{l} \text{find } \mathbf{E} \in H_0(\mathbf{curl}\,;\Omega) \ : \\ \quad \int_\Omega \mathbf{curl}\,\mathbf{E} \cdot \mathbf{curl}\,\mathbf{w} - k^2 \int_\Omega \mathbf{E} \cdot \mathbf{w} = \int_\Omega \mathbf{F} \cdot \mathbf{w} \\ \forall\ \mathbf{w} \in H_0(\mathbf{curl}\,;\Omega)\,. \end{array} \tag{45}$$

The finite element approximation problem with edge elements reads

$$\begin{array}{l} \text{find } \mathbf{E}_h \in W_h \ : \\ \quad \int_\Omega \mathbf{curl}\,\mathbf{E}_h \cdot \mathbf{curl}\,\mathbf{w}_h - k^2 \int_\Omega \mathbf{E}_h \cdot \mathbf{w}_h = \int_\Omega \mathbf{F} \cdot \mathbf{w}_h \\ \forall\ \mathbf{w}_h \in W_h\,, \end{array} \tag{46}$$

where

$$W_h := N_h^r \cap H_0(\mathbf{curl}\,;\Omega)\,.$$

The existence and uniqueness of the solution to the discrete problem (46) has to be proved. We will do that later on, and for the time being we assume that the solution $\mathbf{E}_h$ does exist.

Let us focus on the convergence of the numerical scheme and on the error estimate, following Monk (2003b) [for different approaches, see Monk and Demkowicz (2001), Boffi and Gastaldi (2002)]. Setting $\mathbf{e}_h := \mathbf{E} - \mathbf{E}_h$, by subtracting (46) from (45) we find

$$\int_\Omega \mathbf{curl}\,\mathbf{e}_h \cdot \mathbf{curl}\,\mathbf{w}_h - k^2 \int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h = 0 \qquad \forall\, \mathbf{w}_h \in W_h\,. \quad (47)$$

A first trivial remark is that $\mathbf{grad}\, L^r_h \subset N^r_h$ (L^r_h defined in (7)), therefore using in (47) $\mathbf{w}_h = \mathbf{grad}\, v_h$ with $v_h \in L^r_h \cap H^1_0(\Omega)$ we have

$$\int_\Omega \mathbf{e}_h \cdot \mathbf{grad}\, v_h = 0\,. \qquad (48)$$

In other words, $\mathbf{e}_h$ is discrete divergence free.

Denote by P_h the orthogonal projection from $H(\mathbf{curl}\,;\Omega)$ onto W_h, by $m(\cdot,\cdot)$ the bilinear form at the left hand side of (47), and by $\|\cdot\|_{\mathrm{curl},\Omega}$ (respectively, $(\cdot,\cdot)_{\mathrm{curl},\Omega}$) the norm (respectively, the scalar product) in $H(\mathbf{curl}\,;\Omega)$. One obtains

$$\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \leq \|\mathbf{E}-P_h\mathbf{E}\|_{\mathrm{curl},\Omega} + (1+k^2) \sup_{\mathbf{w}_h \in W_h} \frac{\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega}}\,. \qquad (49)$$

Let us prove (49). We have

$$
\begin{aligned}
\|\mathbf{e}_h\|^2_{\mathrm{curl},\Omega} &= (\mathbf{e}_h, \mathbf{E} - P_h\mathbf{E})_{\mathrm{curl},\Omega} + (\mathbf{e}_h, P_h\mathbf{E} - \mathbf{E}_h)_{\mathrm{curl},\Omega} \\
&= (\mathbf{e}_h, \mathbf{E} - P_h\mathbf{E})_{\mathrm{curl},\Omega} + m(\mathbf{e}_h, P_h\mathbf{E} - \mathbf{E}_h) \\
&\qquad + (1+k^2)\int_\Omega \mathbf{e}_h \cdot (P_h\mathbf{E} - \mathbf{E}_h) \\
&= (\mathbf{e}_h, \mathbf{E} - P_h\mathbf{E})_{\mathrm{curl},\Omega} + (1+k^2)\int_\Omega \mathbf{e}_h \cdot (P_h\mathbf{E} - \mathbf{E}_h)\,,
\end{aligned}
$$

having used (47).
On the other hand,

$$
\int_\Omega \mathbf{e}_h \cdot (P_h\mathbf{E} - \mathbf{E}_h) \leq \sup_{\mathbf{w}_h \in W_h} \frac{\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega}} \|P_h\mathbf{E} - \mathbf{E}_h\|_{\mathrm{curl},\Omega}\,.
$$

Since $\mathbf{E}_h = P_h\mathbf{E}_h$ and $\|P_h\mathbf{e}_h\|_{\mathrm{curl},\Omega} \leq \|\mathbf{e}_h\|_{\mathrm{curl},\Omega}$, (49) follows at once.

Let us estimate

$$\sup_{\mathbf{w}_h \in W_h} \frac{\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega}} .$$

A Helmholtz orthogonal decomposition result ensures that we can write $\mathbf{e}_h = \mathbf{curl}\, \mathbf{q}_0 + \mathbf{k}_0 + \mathbf{grad}\, p_0$, where $\mathbf{grad}\, p_0$ is the $(L^2(\Omega))^3$-orthogonal projection of $\mathbf{e}_h$ on $\mathbf{grad}\, H_0^1(\Omega)$ (in particular, $p_0 \in H_0^1(\Omega)$), and $\mathbf{k}_0$ is a harmonic field belonging to $\mathcal{H}(e;\Omega)$ (namely, $\mathbf{curl}\, \mathbf{k}_0 = \mathbf{0}$, $\mathrm{div}\mathbf{k}_0 = 0$ and $\mathbf{k}_0 \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$). We set $\mathbf{e}_0 := \mathbf{curl}\, \mathbf{q}_0 + \mathbf{k}_0$, and thus $\mathrm{div}\mathbf{e}_0 = 0$, $\mathbf{curl}\, \mathbf{e}_0 = \mathbf{curl}\, \mathbf{e}_h$, $\mathbf{e}_0 \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.

Since $\mathbf{e}_h$ is discrete divergence free, it follows that $\mathbf{grad}\, p_0$ is discrete divergence free, too.

Due to the properties of orthogonal projections, we also have $\|\mathbf{grad}\, p_0\|_{0,\Omega} \leq \|\mathbf{e}_h\|_{0,\Omega}$.

Similarly, the discrete orthogonal decomposition $\mathbf{w}_h = \mathbf{w}_{0,h} + \mathbf{grad}\, \xi_h$ holds, with $\xi_h \in L_h^r \cap H_0^1(\Omega)$ and $\mathbf{w}_{0,h} \in W_h$. The function $\mathbf{w}_{0,h}$ is discrete divergence free and clearly satisfies $\mathbf{curl}\, \mathbf{w}_{0,h} = \mathbf{curl}\, \mathbf{w}_h$ and $\|\mathbf{w}_{0,h}\|_{0,\Omega} \leq \|\mathbf{w}_h\|_{0,\Omega}$.

Having obtained these preliminaries results, we find

$$\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h = \int_\Omega (\mathbf{e}_0 + \mathbf{grad}\, p_0) \cdot \mathbf{w}_h = \int_\Omega \mathbf{e}_0 \cdot \mathbf{w}_h + \int_\Omega \mathbf{grad}\, p_0 \cdot \mathbf{w}_{0,h}\,.$$

We will see later on how to estimate $\int_\Omega \mathbf{grad}\, p_0 \cdot \mathbf{w}_{0,h}$.

Concerning the term $\int_\Omega \mathbf{e}_0 \cdot \mathbf{w}_h$ we find

$$\int_\Omega \mathbf{e}_0 \cdot \mathbf{w}_h \leq \|\mathbf{e}_0\|_{0,\Omega} \, \|\mathbf{w}_h\|_{0,\Omega} \,, \tag{50}$$

and we need to estimate $\|\mathbf{e}_0\|_{0,\Omega}$.

The required estimate can be obtained by means of a duality argument (see Nitsche (1970), Schatz (1974)). Let $\mathbf{z} \in H(\mathbf{curl}\,;\Omega)$ be the solution to

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{z} - k^2\mathbf{z} = \mathbf{e}_0 & \text{in } \Omega \\ \mathbf{z} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \,, \end{cases} \tag{51}$$

which satisfies the estimate $\|\mathbf{z}\|_{\mathrm{curl},\Omega} \leq C\|\mathbf{e}_0\|_{0,\Omega}$. Since $\mathrm{div}\mathbf{e}_0 = 0$, we also have $\mathrm{div}\mathbf{z} = 0$.

Moreover, **curl z** satisfies

$$\begin{cases} \mathbf{curl}\,(\mathbf{curl}\,\mathbf{z}) = k^2\mathbf{z} + \mathbf{e}_0 & \text{in } \Omega \\ \mathrm{div}(\mathbf{curl}\,\mathbf{z}) = 0 & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{z}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

A couple of regularity results due to Amrouche, Bernardi, Dauge and Girault (1998) say that $\mathbf{z} \in H^s(\Omega)$ with $\mathbf{curl}\,\mathbf{z} \in H^s(\Omega)$ for $s > 1/2$, and the following estimates hold

$$\|\mathbf{z}\|_{s,\Omega} \leq C\|\mathbf{z}\|_{\mathrm{curl},\Omega} \leq C\|\mathbf{e}_0\|_{0,\Omega}$$

$$\begin{aligned} \|\mathbf{curl}\,\mathbf{z}\|_{s,\Omega} &\leq C(\|\mathbf{curl}\,\mathbf{curl}\,\mathbf{z}\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{z}\|_{0,\Omega}) \\ &\leq C(\|\mathbf{z}\|_{\mathrm{curl},\Omega} + \|\mathbf{e}_0\|_{0,\Omega}) \leq C\|\mathbf{e}_0\|_{0,\Omega}\,. \end{aligned}$$

Hence the interpolant $\mathbf{r}_h\mathbf{z}$ is defined and we have

$$\|\mathbf{z} - \mathbf{r}_h\mathbf{z}\|_{\mathrm{curl},\Omega} \leq Ch^s(\|\mathbf{z}\|_{s,\Omega} + \|\mathbf{curl}\,\mathbf{z}\|_{s,\Omega}) \leq Ch^s\|\mathbf{e}_0\|_{0,\Omega} \,.$$

Using (51) we find

$$\|\mathbf{e}_0\|^2_{0,\Omega} = m(\mathbf{z}, \mathbf{e}_0) = m(\mathbf{z}, \mathbf{e}_h - \mathbf{grad}\, p_0) = m(\mathbf{z}, \mathbf{e}_h) \,,$$

since $\mathbf{z}$ is divergence free and $p_{0|\partial\Omega} = 0$.
Moreover, taking into account (47)

$$\begin{aligned} m(\mathbf{z}, \mathbf{e}_h) &= m(\mathbf{z} - \mathbf{r}_h\mathbf{z}, \mathbf{e}_h) \leq C\|\mathbf{z} - \mathbf{r}_h\mathbf{z}\|_{\mathrm{curl},\Omega}\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \\ &\leq Ch^s\|\mathbf{e}_0\|_{0,\Omega}\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \,. \end{aligned}$$

In conclusion,

$$\|\mathbf{e}_0\|_{0,\Omega} \leq Ch^s \|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \, . \qquad (52)$$

Let us come to the estimate of $\int_\Omega \mathbf{grad}\, p_0 \cdot \mathbf{w}_{0,h}$.

Since $\mathbf{w}_{0,h}$ is discrete divergence free, it is possible to find a divergence free vector function $\mathbf{U}_0$ such that

$$\begin{aligned} \|\mathbf{w}_{0,h} - \mathbf{U}_0\|_{0,\Omega} &\leq Ch^s(\|\mathbf{w}_{0,h}\|_{0,\Omega} + \|\mathbf{curl}\, \mathbf{w}_{0,h}\|_{0,\Omega}) \\ &\leq Ch^s(\|\mathbf{w}_h\|_{0,\Omega} + \|\mathbf{curl}\, \mathbf{w}_h\|_{0,\Omega}) \, . \end{aligned}$$

[This can be done by taking the solution $\mathbf{U}_0$ of the problem

$$\begin{cases} \mathbf{curl}\,\mathbf{U}_0 = \mathbf{curl}\,\mathbf{w}_{0,h} & \text{in } \Omega \\ \mathrm{div}\mathbf{U}_0 = 0 & \text{in } \Omega \\ \mathbf{U}_0 \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \int_\Omega \mathbf{U}_0 \cdot \mathbf{grad}\,\psi_l = \int_\Omega \mathbf{w}_{0,h} \cdot \mathbf{grad}\,\psi_l & \forall\ l = 1, \ldots, p_{\partial\Omega}\,, \end{cases}$$

where ψ_l is the discrete function, defined on a fixed coarse mesh, taking value 1 on $(\partial\Omega)_l$ and value 0 on all the other nodes in $\overline{\Omega}$. It can be shown that

$$\begin{aligned} \|\mathbf{U}_0\|_{\mathrm{curl},\Omega} &\leq C(\|\mathbf{curl}\,\mathbf{w}_{0,h}\|_{0,\Omega} + \textstyle\sum_l |\int_\Omega \mathbf{w}_{0,h} \cdot \mathbf{grad}\,\psi_l|) \\ &\leq C\|\mathbf{w}_{0,h}\|_{\mathrm{curl},\Omega}\,, \end{aligned}$$

and that $\mathbf{w}_{0,h} = \mathbf{r}_h\mathbf{U}_0 + \mathbf{grad}\,\phi_h$, with $\phi_h \in L_h^r$ and constant on each $(\partial\Omega)_l$; hence $\mathbf{w}_{0,h} = \mathbf{r}_h\mathbf{U}_0 + \mathbf{grad}\,v_h + \sum_l c_l\mathbf{grad}\,\psi_l$ with $v_h \in L_h^r \cap H_0^1(\Omega)$.

Therefore

$$\begin{aligned}
\|\mathbf{w}_{0,h} - \mathbf{U}_0\|^2_{0,\Omega} &= \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{w}_{0,h} - \mathbf{U}_0) \\
&= \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{w}_{0,h} - \mathbf{r}_h \mathbf{U}_0) \\
&\quad + \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0) \\
&= \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{grad}\, v_h + \textstyle\sum_l c_l \mathbf{grad}\, \psi_l) \\
&\quad + \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0) \\
&= \int_\Omega (\mathbf{w}_{0,h} - \mathbf{U}_0) \cdot (\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0) \\
&\leq \|\mathbf{w}_{0,h} - \mathbf{U}_0\|_{0,\Omega} \|\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0\|_{0,\Omega} \, .
\end{aligned}$$

On the other hand, if $\mathbf{curl}\, \mathbf{U}_0 \in \mathbf{curl}\, W_h$ it can be proved that

$$\begin{aligned}
\|\mathbf{r}_h \mathbf{U}_0 - \mathbf{U}_0\|_{0,\Omega} &\leq C h^s (\|\mathbf{U}_0\|_{s,\Omega} + \|\mathbf{curl}\, \mathbf{U}_0\|_{0,\Omega}) \\
&\leq C h^s \|\mathbf{w}_{0,h}\|_{\mathrm{curl},\Omega} \, . \;]
\end{aligned}$$

Then

$$\begin{aligned}\int_\Omega \mathbf{grad}\, p_0 \cdot \mathbf{w}_{0,h} &= \int_\Omega \mathbf{grad}\, p_0 \cdot (\mathbf{w}_{0,h} - \mathbf{U}_0) \\ &\leq \|\mathbf{grad}\, p_0\|_{0,\Omega} \|\mathbf{w}_{0,h} - \mathbf{U}_0\|_{0,\Omega} \\ &\leq Ch^s \|\mathbf{w}_h\|_{\mathrm{curl},\Omega} \|\mathbf{grad}\, p_0\|_{0,\Omega} \\ &\leq Ch^s \|\mathbf{w}_h\|_{\mathrm{curl},\Omega} \|\mathbf{e}_h\|_{0,\Omega}\,.\end{aligned} \tag{53}$$

In conclusion

$$\sup_{\mathbf{w}_h \in W_h} \frac{\int_\Omega \mathbf{e}_h \cdot \mathbf{w}_h}{\|\mathbf{w}_h\|_{\mathrm{curl},\Omega}} \leq Ch^s \|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \ , \tag{54}$$

and from (49) for h small enough we have

$$\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \leq C \|\mathbf{E} - P_h \mathbf{E}\|_{\mathrm{curl},\Omega}\,. \tag{55}$$

This estimate ensures that for h small enough problem (46) is well-posed. Since it is enough to prove uniqueness, suppose that $\mathbf{E}_h$ is a solution corresponding to $\mathbf{F} = \mathbf{0}$. We know that for this right hand side the exact solution $\mathbf{E}$ of (45) is vanishing, therefore $\mathbf{e}_h = -\mathbf{E}_h$. Using (55) it follows $\mathbf{e}_h = \mathbf{0}$, hence the uniqueness of the solution to (46).

Moreover, since

$$\|\mathbf{E} - P_h\mathbf{E}\|_{\mathrm{curl},\Omega} = \inf_{\mathbf{w}_h \in W_h} \|\mathbf{E} - \mathbf{w}_h\|_{\mathrm{curl},\Omega} ,$$

we have also obtained the quasi-optimal error estimate

$$\|\mathbf{e}_h\|_{\mathrm{curl},\Omega} \leq C \inf_{\mathbf{w}_h \in W_h} \|\mathbf{E} - \mathbf{w}_h\|_{\mathrm{curl},\Omega} , \tag{56}$$

valid for h small enough.

Another example in which edge elements are used is related to the eddy current equations. We have at least two possible approaches:

- **E** formulation

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}) + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{J}_e & \text{in } \Omega \\ \mathrm{div}(\varepsilon_I\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ BC_E(\mathbf{E}_I) = 0 & \text{on } \partial\Omega \\ \varepsilon_I\mathbf{E}_I \perp \mathcal{H}_I \end{cases} \qquad (57)$$

[where the condition $\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ has to be dropped if considering the electric boundary condition].

Once the electric field **E** is available, one sets

$$\mathbf{H} = i\omega^{-1}\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E} \qquad \text{in } \Omega\,. \qquad (58)$$

- **H** formulation

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\sigma}^{-1}\mathbf{curl}\,\mathbf{H}_C) + i\omega\boldsymbol{\mu}_C\mathbf{H}_C \\ \qquad\qquad = \mathbf{curl}\,(\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}) & \text{in } \Omega_C \\ \mathbf{curl}\,\mathbf{H}_I = \mathbf{J}_{e,I} & \text{in } \Omega_I \\ \mathrm{div}(\boldsymbol{\mu}\mathbf{H}) = 0 & \text{in } \Omega \\ BC_H(\mathbf{H}_I) = 0 & \text{on } \partial\Omega \\ \mathbf{H}_I \times \mathbf{n} - \mathbf{H}_C \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \\ TOP(\mathbf{H}) = 0\,, \end{cases} \tag{59}$$

where $BC_H(\mathbf{H}_I)$ means $\boldsymbol{\mu}_I\mathbf{H}_I \cdot \mathbf{n}$ for the electric boundary condition, and $\mathbf{H}_I \times \mathbf{n}$ for the magnetic boundary conditions, and $TOP(\mathbf{H}) = 0$ is a set of topological conditions that have to be satisfied by the magnetic field **H**.

Having determined **H**, the electric field is obtained by setting

$$\mathbf{E}_C = \boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C}) \qquad \text{in } \Omega_C\,, \tag{60}$$

and solving the problem

$$\begin{cases} \mathbf{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}_I\mathbf{H}_I & \text{in } \Omega_I \\ \operatorname{div}(\varepsilon_I\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ BC_E(\mathbf{E}_I) = 0 & \text{on } \partial\Omega \\ \mathbf{E}_I \times \mathbf{n} = \mathbf{E}_C \times \mathbf{n} & \text{on } \Gamma \\ \varepsilon_I\mathbf{E}_I \perp \mathcal{H}_I\,. \end{cases} \tag{61}$$

This last problem is not always solvable, but needs that some compatibility conditions on the data are satisfied.

Besides the conditions $\operatorname{div}(\boldsymbol{\mu}\mathbf{H}) = 0$ in Ω and $\boldsymbol{\mu}_I \mathbf{H}_I \cdot \mathbf{n} = 0$ on $\partial\Omega$ (if $\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$), that are clearly satisfied, it is important to underline that the other needed compatibility conditions are the topological conditions $TOP(\mathbf{H}) = 0$.

Let us make clear their structure. For the sake of definiteness, let us focus on the electric boundary condition. We need to consider again the (finite dimensional) space

$$\mathcal{H}_I^{(D)} := \{\mathbf{G}_I \in (L^2(\Omega_I))^3 \,|\, \mathbf{curl}\, \mathbf{G}_I = \mathbf{0}, \operatorname{div}(\boldsymbol{\mu}_I \mathbf{G}_I) = 0 \\ \boldsymbol{\mu}_I \mathbf{G}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \cup \Gamma\} \,,$$

and its basis functions $\boldsymbol{\rho}^*_{\alpha,I}$, $\alpha = 1, \ldots, g_{\Omega_I}$ [let us recall that g_{Ω_I} is the first Betti number of Ω_I, or, equivalently, the number of (independent) non-bounding cycles in Ω_I].

The topological conditions $TOP(\mathbf{H}) = 0$ mean that

$$\begin{aligned}\int_{\Omega_I} i\omega \boldsymbol{\mu}_I \mathbf{H}_I \cdot \boldsymbol{\rho}^*_{\alpha,I} \\ + \int_\Gamma [\boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \cdot \boldsymbol{\rho}^*_{\alpha,I} = 0\end{aligned} \tag{62}$$

for each $\alpha = 1, \dots, g_{\Omega_I}$. [$\mathbf{n}_C$ is the unit normal vector on Γ, external to Ω_C.]

Note that one has $g_{\Omega_I} \geq 1$ if the conductor Ω_C is not simply-connected, and therefore in that case these conditions have to be taken into account.

- It can be proved that the topological conditions $TOP(\mathbf{H}) = 0$ are equivalent to the integral form of the Faraday equation on each surface that "cuts" a non-bounding cycle [Seifert surface].

The existence and uniqueness of a solution can be proved as follows.
Under the necessary conditions

$$\mathrm{div}\mathbf{J}_{e,I} = 0 \text{ in } \Omega_I \ , \ \mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \ , \ \mathbf{J}_{e,I} \perp \mathcal{H}_I \ ,$$

it can be shown that there exists a field $\mathbf{H}_e \in H(\mathbf{curl}\,;\Omega)$ satisfying

$$\begin{cases} \mathbf{curl}\,\mathbf{H}_{e,I} = \mathbf{J}_{e,I} & \text{in } \Omega_I \\ BC_H(\mathbf{H}_{e,I}) = 0 & \text{on } \partial\Omega \end{cases}$$

[the boundary conditions for $\mathbf{J}_{e,I}$ and $\mathbf{H}_{e,I}$ have to be dropped if considering the electric boundary condition].

Setting

$$V := \{\mathbf{v} \in H(\mathbf{curl}\,;\Omega) \,|\, \mathbf{curl}\,\mathbf{v}_I = \mathbf{0} \text{ in } \Omega_I, \mathbf{v}_I \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$$

[the boundary condition has to be dropped if considering the electric boundary condition], multiplying the Faraday equation by $\overline{\mathbf{v}}$, with $\mathbf{v} \in V$, integrating in Ω and integrating by parts one finds

$$\int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega_I} \mathbf{E}_I \cdot \mathbf{curl}\,\overline{\mathbf{v}_I} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{E} \cdot \overline{\mathbf{v}} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} = 0 \ ,$$

thus

$$\int_{\Omega_C} \mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega} i\omega\boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} = 0 \ ,$$

as $\mathbf{curl}\,\mathbf{v}_I = \mathbf{0}$ in Ω_I.

Using the Ampère equation in Ω_C for expressing $\mathbf{E}_C$, we end up with the following problem

$$\begin{aligned}
&\text{Find } (\mathbf{H} - \mathbf{H}_e) \in V \ : \\
&\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{v}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{H} \cdot \overline{\mathbf{v}} \\
&\qquad = \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl}\, \overline{\mathbf{v}_C} \\
&\text{for each } \mathbf{v} \in V \ .
\end{aligned} \tag{63}$$

This formulation is well-posed via the Lax–Milgram lemma, as the sesquilinear form

$$a_m(\mathbf{u}, \mathbf{v}) := \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{u}_C \cdot \mathbf{curl}\, \overline{\mathbf{v}_C} + \int_\Omega i\omega \boldsymbol{\mu} \mathbf{u} \cdot \overline{\mathbf{v}}$$

is clearly continuous and coercive in V.

Weak **E**-formulation

For deriving the weak **E**-formulation one starts from the Ampère equation: multiplying by $\overline{\mathbf{z}}$, integrating in Ω and integrating by parts one easily sees that

$$\int_\Omega \mathbf{H} \cdot \mathbf{curl}\, \overline{\mathbf{z}} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{H} \cdot \overline{\mathbf{z}} - \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} = \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}}$$

for all $\mathbf{z} \in H(\mathbf{curl}\,;\Omega)$.

The boundary term disappears if **H** satisfies the magnetic boundary condition, or if **z** satisfies the electric boundary condition.

Set

$$\begin{aligned} Z \quad := \{\mathbf{z} \in H(\mathbf{curl}\,;\Omega) \,|\, \mathrm{div}(\varepsilon_I \mathbf{z}_I) = 0 \text{ in } \Omega_I, \\ BC_E(\mathbf{z}_I) = 0,\ \varepsilon_I \mathbf{z}_I \perp \mathcal{H}_I\} \,. \end{aligned}$$

Expressing **H** through the Faraday equation, the weak **E**-formulation finally reads

$$\begin{array}{l} \text{Find } \mathbf{E} \in Z \;\; : \\ \quad \int_\Omega \boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E} \cdot \mathbf{curl}\, \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\ \qquad\qquad\qquad\qquad = -i\omega \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}} \\ \text{for each } \mathbf{z} \in Z\,. \end{array} \tag{64}$$

Though less straightforward, it can be proved that the sesquilinear form

$$a_e(\mathbf{w}, \mathbf{z}) \;\; := \int_\Omega \boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{w} \cdot \mathbf{curl}\, \overline{\mathbf{z}} + i\omega \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{z}_C}$$

is continuous and coercive in Z, and well-posedness of the weak **E**-formulation follows from Lax–Milgram lemma.

For both problems (63) and (64), the variational spaces V and Z are (closed) subspaces of $H(\mathbf{curl}\,;\Omega)$, hence edge elements are the right thing to use for numerical approximation.
However, in (63) and (64) there is a differential constraint: in the former problem on the curl, in the latter on the divergence.

- Numerical approximation needs some care!

Possible ways of attack:

- saddle-point formulations [Lagrange multipliers]
- a scalar potential for $\mathbf{H}_I - \mathbf{H}_{e,I}$.
- a vector potential for $\varepsilon_I \mathbf{E}_I$.

The first choice has been considered by Alonso Rodríguez, Hiptmair and V. (2004a) (for the magnetic field) and by Alonso Rodríguez and V. (2004) (for the electric field); hybrid (coupled) formulations in terms of $(\mathbf{H}_C, \mathbf{E}_I)$ or $(\mathbf{E}_C, \mathbf{H}_I)$ have been also proposed and analyzed (Alonso Rodríguez, Hiptmair and V. (2004b, 2005)). Edge elements are used for the magnetic and electric fields, nodal elements are employed for the (scalar) Lagrange multipliers.

The second possibility, also leading to coupled formulations, will be described here below.

Instead, to our knowledge, the third choice has not been completely exploited. [However, in a different though related situation, one can think to the (classical) approach based on a vector potential for the divergence free vector field $\mu\mathbf{H}$.]

For the sake of definiteness let us consider the electric boundary condition.

The starting point is to consider $\mathbf{H}_e \in H(\mathbf{curl}\,;\Omega)$ satisfying

$$\mathbf{curl}\,\mathbf{H}_{e,I} = \mathbf{J}_{e,I} \qquad \text{in } \Omega_I\,.$$

Then the main step is to use the Helmholtz orthogonal decomposition

$$\mathbf{H}_I - \mathbf{H}_{e,I} = \mathbf{grad}\,\psi_I^* + \sum_{\alpha=1}^{g_{\Omega_I}} \eta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^*\,, \qquad (65)$$

where $\psi_I^* \in H^1(\Omega_I)/\mathbb{C}$ and $\eta_{I,\alpha}^* \in \mathbb{C}$ (the two terms of the decomposition are orthogonal, with respect to the scalar product $(\mathbf{u}_I, \mathbf{v}_I)_{\mu_I,\Omega_I} := \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{u}_I \cdot \mathbf{v}_I$).

Let us recall the Helmholtz decomposition result:

$$\mathbf{v}_I = \boldsymbol{\mu}_I^{-1}\mathbf{curl}\,\mathbf{Q}_I^* + \mathbf{grad}\,\chi_I^* + \sum_{\alpha=1}^{g_{\Omega_I}} \theta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \,.$$

The vector function $\mathbf{Q}_I^*$ is the solution to

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}_I^{-1}\mathbf{curl}\,\mathbf{Q}_I^*) = \mathbf{curl}\,\mathbf{v}_I & \text{in } \Omega_I \\ \mathrm{div}\mathbf{Q}_I^* = 0 & \text{in } \Omega_I \\ \mathbf{Q}_I^* \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \cup \partial\Omega \\ \mathbf{Q}_I^* \perp \mathcal{H}_{I,\varepsilon_0}^{(A)} \end{cases}$$

[$\mathcal{H}_{I,\varepsilon_0}^{(A)}$ denotes $\mathcal{H}_I^{(A)}$ for $\boldsymbol{\varepsilon}_I = \varepsilon_0$, a positive constant].
The scalar function χ_I^* is the solution to the elliptic Neumann boundary value problem

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I\mathbf{grad}\,\chi_I^*) = \mathrm{div}(\boldsymbol{\mu}_I\mathbf{v}_I) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I\mathbf{grad}\,\chi_I^* \cdot \mathbf{n} = \boldsymbol{\mu}_I\mathbf{v}_I \cdot \mathbf{n} & \text{on } \Gamma \cup \partial\Omega \,. \end{cases}$$

Finally the vector $\theta^*_{I,\alpha}$ is the solution of the linear system

$$\sum_{\alpha=1}^{g_{\Omega_I}} A^*_{\beta\alpha}\theta^*_{I,\alpha} = \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{v}_I \cdot \boldsymbol{\rho}^*_{\beta,I} \ ,$$

where

$$A^*_{\beta\alpha} := \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}^*_{\alpha,I} \cdot \boldsymbol{\rho}^*_{\beta,I} \ ,$$

and the harmonic vector fields $\boldsymbol{\rho}^*_{\alpha,I}$ are the basis functions of the space $\mathcal{H}_I^{(D)}$.

Coming back to the scalar potential formulation, in (63) each test function $\mathbf{v} \in V$ can be thus written as

$$\mathbf{v}_I = \mathbf{grad}\,\chi_I^* + \sum_{\alpha=1}^{g_{\Omega_I}} \theta_{I,\alpha}^* \boldsymbol{\rho}_{\alpha,I}^* \,. \qquad (66)$$

Inserting (65) and (66) in (63) and using orthogonality one easily finds, for the unknowns $\mathbf{Z}_C := \mathbf{H}_C - \mathbf{H}_{e,C}$, ψ_I^*, $\eta_{I,\alpha}^*$,

$$\begin{aligned}
&\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{curl}\,\mathbf{Z}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}_C} + \int_{\Omega_C} i\omega\boldsymbol{\mu}_C \mathbf{Z}_C \cdot \overline{\mathbf{v}_C} \\
&\quad + \int_{\Omega_I} i\omega\boldsymbol{\mu}_I \mathbf{grad}\,\psi_I^* \cdot \mathbf{grad}\,\overline{\chi_I^*} + i\omega[A^*\boldsymbol{\eta}_I^*, \boldsymbol{\theta}_I^*] \\
&= -\int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{curl}\,\mathbf{H}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{v}_C} - \int_{\Omega_C} i\omega\boldsymbol{\mu}_C \mathbf{H}_{e,C} \cdot \overline{\mathbf{v}_C} \\
&\quad - \int_{\Omega_I} i\omega\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot (\mathbf{grad}\,\overline{\chi_I^*} + \textstyle\sum_{\alpha=1}^{g_{\Omega_I}} \overline{\theta_{I,\alpha}^*}\boldsymbol{\rho}_{\alpha,I}^*) \\
&\quad + \int_{\Omega_C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{v}_C} \,,
\end{aligned} \qquad (67)$$

where we recall that the matrix A^* is defined by

$$A^*_{\beta\alpha} := \int_{\Omega_I} \boldsymbol{\mu}_I \boldsymbol{\rho}^*_{\alpha,I} \cdot \boldsymbol{\rho}^*_{\beta,I} \, ,$$

and is symmetric and positive definite (the fields $\boldsymbol{\rho}^*_{\alpha,I}$ form a basis for the space $\mathcal{H}_I^{(D)}$).
Clearly, the solutions $\mathbf{Z}_C$, ψ^*_I and $\boldsymbol{\eta}^*_I$ have to satisfy on Γ the matching condition

$$\mathbf{Z}_C \times \mathbf{n} - \mathbf{grad}\, \psi^*_I \times \mathbf{n} - \sum_{\alpha=1}^{g_{\Omega_I}} \eta^*_{I,\alpha} \boldsymbol{\rho}^*_{\alpha,I} \times \mathbf{n} = \mathbf{0} \, .$$

The same holds for the test functions $\mathbf{v}_C$, χ^*_I and $\boldsymbol{\theta}^*_I$.

The left hand side in (67) is a continuous and coercive sesquilinear form, therefore the problem is well-posed.

A coupled formulation in terms of $\mathbf{E}_C$, ψ_I^* and $\boldsymbol{\eta}_I^*$ is also possible.

From the Ampère equation in Ω_C, multiplying by $\overline{\mathbf{z}_C}$, integrating in Ω_C and integrating by parts one finds

$$\begin{gathered}\int_{\Omega_C} \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{z}_C} + \int_\Gamma \mathbf{n}_C \times \mathbf{H}_C \cdot \overline{\mathbf{z}_C} - \int_{\Omega_C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C} \\ = \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C}\,.\end{gathered}$$

Using the Faraday equation for expressing $\mathbf{H}_C$ and recalling that $\mathbf{n}_C \times \mathbf{H}_C = \mathbf{n}_C \times \mathbf{H}_I$ on Γ, it holds

$$\begin{aligned}&\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{z}_C} + i\omega \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C}) \\ &\qquad + i\omega \int_\Gamma \mathbf{H}_I \times \mathbf{n}_C \cdot \overline{\mathbf{z}_C} = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C}\,.\end{aligned}$$

On the other hand, multiplying the Faraday equation in Ω_I by a test function $\overline{\mathbf{v}_I}$ such that $\mathbf{curl}\,\mathbf{v}_I = \mathbf{0}$ in Ω_I and recalling that $\mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C$ on Γ, by integration by parts one has

$$i\omega \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \overline{\mathbf{v}_I} = -\int_{\Omega_I} \mathbf{curl}\,\mathbf{E}_I \cdot \overline{\mathbf{v}_I} = -\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \overline{\mathbf{v}_I}\,.$$

Setting

$$V_I(\mathbf{G}) := \{\mathbf{v}_I \in H(\mathbf{curl}\,;\Omega_I) \mid \mathbf{curl}\,\mathbf{v}_I = \mathbf{G} \text{ in } \Omega_I\}\,,$$

we are thus looking for $\mathbf{E}_C \in H(\mathbf{curl}\,;\Omega_C)$ and $\mathbf{H}_I \in V_I(\mathbf{J}_{e,I})$ such that

$$\int_{\Omega_C} (\boldsymbol{\mu}_C^{-1} \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{z}_C} + i\omega \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}_C}) \\ - i\omega \int_\Gamma \overline{\mathbf{z}_C} \times \mathbf{n}_C \cdot \mathbf{H}_I = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} \\ -i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \overline{\mathbf{v}_I} + \omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_I \cdot \overline{\mathbf{v}_I} = 0 \,, \tag{68}$$

where $\mathbf{z}_C \in H(\mathbf{curl}\,; \Omega_C)$ and $\mathbf{v}_I \in V_I(\mathbf{0})$.

Using in (68) the orthogonal decompositions of $\mathbf{H}_I - \mathbf{H}_{e,I}$ and $\mathbf{v}_I$ one finds

$$\mathcal{K}((\mathbf{E}_C, \psi_I^*, \boldsymbol{\eta}_I^*), (\mathbf{z}_C, \chi_I^*, \boldsymbol{\theta}_I^*)) \\ = -i\omega \int_{\Omega_C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}_C} + i\omega \int_\Gamma \mathbf{H}_{e,I} \cdot \overline{\mathbf{z}_C} \times \mathbf{n}_C \\ -\omega^2 \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot (\mathbf{grad}\, \overline{\chi_I^*} + \textstyle\sum_{\alpha=1}^{g_{\Omega_I}} \overline{\theta_{I,\alpha}^*} \boldsymbol{\rho}_{\alpha,I}^*) \,, \tag{69}$$

where the sesquilinear form $\mathcal{K}(\cdot,\cdot)$, that can be proved to be continuous and coercive, is given by

$$
\begin{aligned}
&\mathcal{K}((\mathbf{E}_C,\psi_I^*,\boldsymbol{\eta}_I^*),(\mathbf{z}_C,\chi_I^*,\boldsymbol{\theta}_I^*)) \\
&\quad := \int_{\Omega_C}(\boldsymbol{\mu}_C^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{z}_C} + i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{z}_C}) \\
&\qquad -i\omega\int_\Gamma(\mathbf{grad}\,\psi_I^* + \textstyle\sum_{\alpha=1}^{g_{\Omega_I}}\eta_{I,\alpha}^*\boldsymbol{\rho}_{\alpha,I}^*)\cdot\overline{\mathbf{z}_C}\times\mathbf{n}_C \\
&\qquad -i\omega\int_\Gamma(\mathbf{grad}\,\overline{\chi_I^*} + \textstyle\sum_{\alpha=1}^{g_{\Omega_I}}\overline{\theta_{I,\alpha}^*}\boldsymbol{\rho}_{\alpha,I}^*)\cdot\mathbf{E}_C\times\mathbf{n}_C \\
&\qquad +\omega^2\int_{\Omega_I}\boldsymbol{\mu}_I\mathbf{grad}\,\psi_I^*\cdot\mathbf{grad}\,\overline{\chi_I^*} \\
&\qquad +\omega^2[A^*\boldsymbol{\eta}_I^*,\boldsymbol{\theta}_I^*]\,.
\end{aligned}
\tag{70}
$$

Note that the interaction between $\mathbf{E}_C$ and $\mathbf{H}_I$ is driven in a weak way by boundary integrals, and no strong matching conditon on Γ has to be imposed: non-matching meshes can be employed!

Domain decomposition approaches can be devised. Let us specify one of them.

Given $\mathbf{e}_\Gamma^{\rm old}$ on Γ, find the solutions to

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I \mathbf{grad}\, \psi_I^*) = -\mathrm{div}(\boldsymbol{\mu}_I \mathbf{H}_{e,I}) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \mathbf{grad}\, \psi_I^* \cdot \mathbf{n}_C = i\omega^{-1} \mathrm{div}_\tau \mathbf{e}_\Gamma^{\rm old} & \\ \qquad\qquad\qquad -\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \mathbf{n}_C & \text{on } \Gamma \\ \boldsymbol{\mu}_I \mathbf{grad}\, \psi_I^* \cdot \mathbf{n} = -\boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \mathbf{n} & \text{on } \partial\Omega \end{cases} \tag{71}$$

$$\begin{aligned} (A^* \boldsymbol{\eta}_I^*)_\beta = i\omega^{-1} & \int_\Gamma \mathbf{e}_\Gamma^{\rm old} \cdot \boldsymbol{\rho}_{\beta,I}^* \\ & - \int_{\Omega_I} \boldsymbol{\mu}_I \mathbf{H}_{e,I} \cdot \boldsymbol{\rho}_{\beta,I}^* \quad \forall\, \beta = 1, \ldots, g_{\Omega_I} \end{aligned} \tag{72}$$

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\sigma}^{-1}\mathbf{curl}\,\mathbf{H}_C) + i\omega\boldsymbol{\mu}_C\mathbf{H}_C \\ \quad = \mathbf{curl}\,(\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}) & \text{in } \Omega_C \\ \mathbf{H}_C \times \mathbf{n}_C = \mathbf{grad}\,\psi_I^* \times \mathbf{n}_C \\ \qquad + \sum_{\alpha=1}^{g_{\Omega_I}} \eta_{I,\alpha}^*\,\boldsymbol{\rho}_{\alpha,I}^* \times \mathbf{n}_C + \mathbf{H}_{e,I} \times \mathbf{n}_C & \text{on } \Gamma\,, \end{cases} \tag{73}$$

finally set

$$\mathbf{e}_\Gamma^{\mathrm{new}} = (1-\delta)\mathbf{e}_\Gamma^{\mathrm{old}} + \delta\,[\boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n}_C \quad \text{on } \Gamma \tag{74}$$

and iterate until convergence ($\delta > 0$ is an acceleration parameter). At convergence one has $\mathbf{e}_\Gamma^\infty = \mathbf{E}_C \times \mathbf{n}_C$ on Γ, the right tangential value of the electric field on Γ.

This iteration-by-subdomain procedure has shown good convergence properties (convergence rate independent of the mesh size [Alonso and V. (1997)]).

The numerical approximation follows some well-drawn lines:

- edge finite elements in Ω_C
- nodal finite elements in Ω_I.

In addition, one looks for

- other g_{Ω_I} degrees of freedom (expressing the line integrals of $\mathbf{H}_I - \mathbf{H}_{e,I}$ along the non-bounding cycles contained in $\overline{\Omega_I}$).

Convergence is ensured by Céa lemma.

[Bermúdez, Rodríguez and Salgado (2002), Alonso Rodríguez, Fernandes and V. (2003).]

Some remarks about implementation issues:

- The matching condition on the interface Γ is easily imposed by eliminating the degrees of freedom of $\mathbf{v}_{C,h}$ associated to the edges on Γ in terms of those of $\mathbf{grad}\,\chi^*_{I,h} + \sum_{\alpha=1}^{g_{\Omega_I}} \theta^*_{I,\alpha}\boldsymbol{\rho}^*_{\alpha,I}$.
- The construction of the vector $\mathbf{H}_{e,I}$ can be done through the Biot–Savart formula

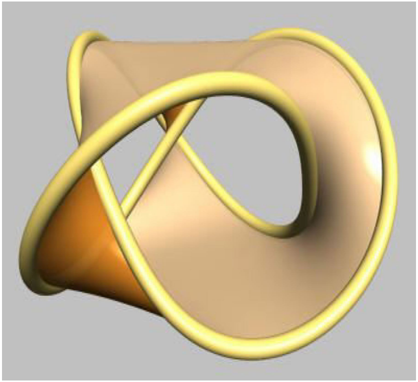
$$\begin{aligned} \mathbf{H}_{e,I}(\mathbf{x}) &:= \mathbf{curl}\left(\int_{\Omega_I} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \mathbf{J}_{e,I}(\mathbf{y})\, d\mathbf{y} \right) \\ &= \int_{\Omega_I} \frac{\mathbf{y}-\mathbf{x}}{4\pi|\mathbf{x}-\mathbf{y}|^3} \times \mathbf{J}_{e,I}(\mathbf{y})\, d\mathbf{y} \end{aligned}$$

[at least for $\mathbf{J}_{e,I}\cdot\mathbf{n} = 0$ on $\partial\Omega \cup \Gamma$; if this is not satisfied, one has to extend $\mathbf{J}_{e,I}$ on a set larger than Ω_I, in such a way that $\mathbf{J}_{e,I}$ is tangential on the boundary of this set].

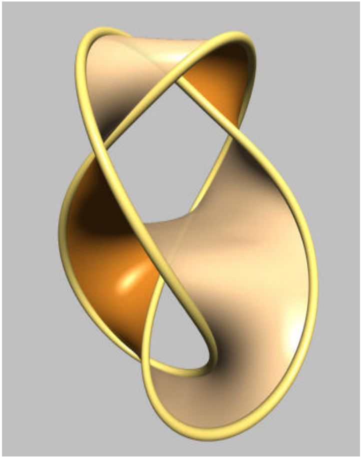
- The construction of the fields $\boldsymbol{\rho}^*_{\alpha,I}$ (or of a suitable approximation of them) is not needed. It is enough to construct g_{Ω_I} interpolants λ^*_α, each one jumping by 1 on a "cutting" surface (and continuous across all the others). One looses orthogonality properties, but everything works well.
- One needs to determine the "cutting" surfaces of the non-bounding cycles (their knowledge is necessary for constructing the basis functions $\boldsymbol{\rho}^*_{\alpha,I}$ or the interpolants λ^*_α). This can be easy in many situations, but for a general topological domain it can be computationally expensive.

Some algorithms have been proposed to the aim of constructing "cutting" surfaces: see Kotiuga (1987, 1988, 1989), Leonard and Rodger (1989) and the book by Gross and Kotiuga (2004).

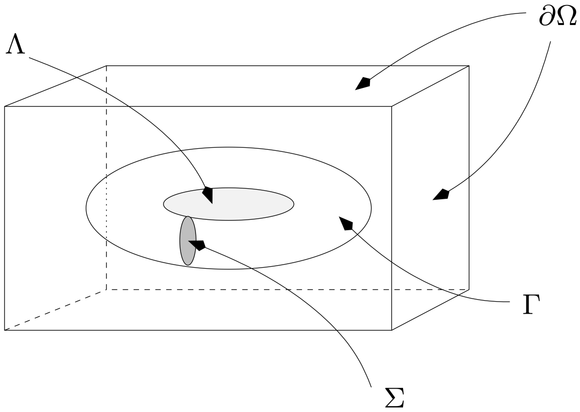
Let us see two pictures of the "cutting" surface when Ω_C is the trefoil knot (or the 4_1-knot) and Ω is a box containing it (thanks to J.J. van Wijk).



"Cutting" surfaces (cont'd)



Instead, if Ω_C is a torus, we have the "cutting" surface Λ:



Pros:

- complexity reduction:
  - few degrees of freedom;
  - "positive definite" algebraic problem.

Cons:

- need of computing in advance a vector potential of the current density;
- some difficulties coming from the topology of the computational domain, in particular of the conductor [construction of the "cutting" surfaces].

Using a scalar magnetic potential has thus led to a complexity reduction, as the total number of degrees of freedom has become much smaller.
However, we have noted that there are two weak points: the need of finding a vector potential of the current density $\mathbf{J}_{e,I}$, and the need of determining the "cutting" surfaces (or, better, the interpolants λ^*_α, jumping by 1 on the "cutting" surfaces).

Rephrasing in the engineering language, we need a source field $\mathbf{H}_{e,I}$ and a suitable set of loop fields (namely, curl-free vector fields $\mathbf{T}^*_0$ that cannot be expressed in Ω as the gradient of any single-valued scalar potential; in other words, there exists a loop in Ω such that the line integral of $\mathbf{T}^*_0$ on it is different from 0).

[Note that the gradients of the interpolants λ^*_α are loop fields: thus finding "cutting" surface is a way for determining suitable loop fields.]

At the discretization level, it is enough to construct an edge element source field, corresponding to a suitable finite element approximation $\mathbf{J}_h$ of the current density $\mathbf{J}_{e,I}$. Moreover, also the loop fields can be edge elements.

Following Alonso Rodríguez, Bertolazzi, Ghiloni and V. (2013), our aim now is to present an efficient computational method for determining edge element source fields and loop fields (and, as a byproduct, an edge element approximation of the space $\mathcal{H}_I^{(D)}$ of harmonic fields).

- A suitable set of loop fields furnishes a basis of the first de Rham cohomology group of Ω_I (the quotient space between curl-free vector fields and gradients defined in Ω_I).

Let us start from the edge element approximation of harmonic fields: if we have a set of g_{Ω_I} linearly independent loop fields $\mathbf{T}^*_{0,\alpha}$, a basis of $\mathcal{H}_I^{(D)}$ is obtained by taking $\mathbf{T}^*_{0,\alpha} + \mathbf{grad}\, \xi_\alpha$, where ξ_α solves the Neumann problem

$$\begin{cases} \mathrm{div}(\boldsymbol{\mu}_I \,\mathbf{grad}\, \xi_\alpha) = -\mathrm{div}(\boldsymbol{\mu}_I \mathbf{T}^*_{0,\alpha}) & \text{in } \Omega_I \\ \boldsymbol{\mu}_I \,\mathbf{grad}\, \xi_\alpha \cdot \mathbf{n} = -\boldsymbol{\mu}_I \mathbf{T}^*_{0,\alpha} \cdot \mathbf{n} & \text{on } \partial\Omega \cup \Gamma\,. \end{cases} \quad (75)$$

Therefore, an approximation of $\mathcal{H}_I^{(D)}$ is easily obtained through a nodal finite element approximation of (75).

For determining a basis of discrete loop fields we propose the following procedure.

Tools:

- homology theory
  - generators of the first homology group of $\partial\Omega \cup \Gamma$, $\overline{\Omega_I}$ and $\mathbb{R}^3 \setminus \Omega_I$
- graph theory applied to the mesh
  - a spanning tree of the graph given by the edges of the mesh in $\overline{\Omega_I}$
- direct elimination procedure
  - a direct algorithm of Webb and Forghani (1989)
  - an explicit formula for the discrete loop fields in terms of *linking numbers*.

Similarly, for discrete source fields:

- proceed as for the loop fields [generators of the homology group on $\partial\Omega \cup \Gamma$, $\overline{\Omega_I}$ and $\mathbb{R}^3 \setminus \Omega_I$, spanning tree of the mesh in $\overline{\Omega_I}$, Webb–Forghani algorithm]
- when the algorithm stops, introduce a dual graph for the remaining edges
- use a direct solver for the final (small and sparse) system.

We denote by $\mathcal{T}_h$ a triangulation of $\overline{\Omega}$ composed by tetrahedra. We assume that it induces a triangulation in both $\overline{\Omega_C}$ and $\overline{\Omega_I}$.
We also need the space $RT_h^1 \subset H(\mathrm{div}; \Omega_I)$ of Raviart–Thomas finite elements of degree 1 [locally: $\mathbf{a} + b\,\mathbf{x}$]. Its dimension is n_f, the number of faces in $\mathcal{T}_h \cap \overline{\Omega_I}$.
Finally, we set

$$H^0(\mathbf{curl}\,; \Omega_I) = \{\mathbf{v}_I \in H(\mathbf{curl}\,; \Omega_I) \,|\, \mathbf{curl}\,\mathbf{v}_I = \mathbf{0} \text{ in } \ \Omega_I\} \, .$$

We assume that we have:

- a basis σ_n of the first homology group of $\overline{\Omega_I}$
- a basis $\widehat{\sigma}_\alpha$ of the first homology group of $\mathbb{R}^3 \setminus \Omega_I$
- a spanning tree $\mathcal{S}_h$ of the graph given by the edges of $\mathcal{T}_h \cap \overline{\Omega_I}$.

We focus now on our *main problem*: given $\mathbf{J}_h \in RT_h^1$ satisfying the necessary conditions, find $\mathbf{Z}_h \in N_h^1$ such that

$$\begin{array}{ll} \mathbf{curl}\, \mathbf{Z}_h = \mathbf{J}_h & \text{in } \Omega_I \\ \oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n & \forall\, n = 1, \dots, g \\ \int_{e'} \mathbf{Z}_h \cdot \boldsymbol{\tau} = 0 & \forall\, e' \in \mathcal{S}_h \,, \end{array} \tag{76}$$

where $\kappa_1, \dots, \kappa_g$ are real numbers. [For simplicity, we have denoted by g the first Betti number g_{Ω_I} of Ω_I.]

[Note that the number of edges e' in $\mathcal{S}_h$ is $n_v - 1$, n_v being the number of vertices in $\mathcal{T}_h \cap \overline{\Omega_I}$; therefore $(76)_3$ can be seen as a "filtre" for gradients.]

Theorem (1)

Problem (76) has a solution and this solution is unique.

Proof. Uniqueness: the difference of two solutions satisfies $\mathbf{Z}_h - \tilde{\mathbf{Z}}_h \in N_h^1$, $\mathbf{curl}\,(\mathbf{Z}_h - \tilde{\mathbf{Z}}_h) = \mathbf{0}$ and $\oint_{\sigma_n}(\mathbf{Z}_h - \tilde{\mathbf{Z}}_h) \cdot d\mathbf{s} = 0$ for all $n = 1, \ldots, g$, hence from de Rham theorem $\mathbf{Z}_h - \tilde{\mathbf{Z}}_h = \mathbf{grad}\,\psi_h$ with $\psi_h \in L_h^1$. For each $e' \in \mathcal{S}_h$ we have $0 = \int_{e'} \mathbf{grad}\,\psi_h \cdot d\mathbf{s} = \psi_h(v_b) - \psi_h(v_a)$, thus ψ_h is constant in Ω_I because $\mathcal{S}_h$ is a spanning tree.

Existence: we can see that the solution $\mathbf{Z}_h$ is given by $\mathbf{Z}_h = \mathbf{R}_h + \Pi^{N_h^1}\mathbf{H}^*$, where $\mathbf{H}^*$ is a source field of $\mathbf{J}_h$ and $\mathbf{R}_h \in N_h^1 \cap H^0(\mathbf{curl}\,;\Omega_I)$ is the solution of

$$
\begin{array}{ll}
\oint_{\sigma_n} \mathbf{R}_h \cdot d\mathbf{s} = \kappa_n - \oint_{\sigma_n} \Pi^{N_h^1}\mathbf{H}^* \cdot d\mathbf{s} & \forall\, n = 1, \ldots, g \\
\int_{e'} \mathbf{R}_h \cdot \boldsymbol{\tau} = -\int_{e'} \Pi^{N_h^1}\mathbf{H}^* \cdot \boldsymbol{\tau} & \forall\, e' \in \mathcal{S}_h \,.
\end{array}
$$

□

Clearly,

- a discrete source field $\mathbf{H}_{e,I}^{(h)}$ can be computed by solving (76), being $\mathbf{J}_h$ a suitable approximation of $\mathbf{J}_{e,I}$ and for any choice of κ_n.

But also (see the following lemma):

- a set of cohomologically independent finite element loop fields $\mathbf{T}_{0,\alpha}^{*}$ in Ω_I (namely, a basis of the first de Rham cohomology group) can be determined by solving (76) with $\mathbf{J}_h = \mathbf{0}$ and $\kappa_n = m_{n,j}$, for any choice of a non-singular matrix $M = (m_{n,j})$
- a basis of $N_h^1 \cap H^0(\mathbf{curl}\,; \Omega_I)$ can be computed starting from $\{\varphi_1, \ldots, \varphi_{n_v}\}$, a basis of L_h^1, and using these loop fields.

Lemma (2)

Let $\mathbf{T}^*_{0,\alpha}$, $\alpha = 1, \dots, g$, be the solutions to problem (76) with $\mathbf{J}_h = \mathbf{0}$ and $\kappa_n = m_{n,\alpha}$, where the matrix $M = (m_{n,\alpha})$ is non-singular, and let φ_i, $i = 1, \dots, n_v$, be a basis of L^1_h. Then the fields $\mathbf{T}^*_{0,\alpha}$ are cohomogically independent loop fields in Ω_I and the set

$$\{\mathbf{grad}\,\varphi_1, \dots, \mathbf{grad}\,\varphi_{n_v-1}\} \cup \{\mathbf{T}^*_{0,1}, \dots, \mathbf{T}^*_{0,g}\}$$

is a basis of $N^1_h \cap H^0(\mathbf{curl}\,; \Omega_I)$.

Proof. The dimension of $N_h^1 \cap H^0(\mathbf{curl}\,; \Omega_I)$ is equal to $g + n_v - 1$, hence is enough to prove linear independence. If we have $\sum_{i=1}^{n_v-1} p_i \mathbf{grad}\, \varphi_i + \sum_{j=1}^{g} q_j \mathbf{T}^*_{0,\alpha} = \mathbf{0}$, it follows

$$0 = \sum_{i=1}^{n_v-1} p_i \oint_{\sigma_n} \mathbf{grad}\, \varphi_i \cdot d\mathbf{s} + \sum_{j=1}^{g} q_j \oint_{\sigma_n} \mathbf{T}^*_{0,\alpha} \cdot d\mathbf{s} = \sum_{j=1}^{g} q_j m_{n,j}$$

for all $n = 1, \ldots, g$, hence $q_j = 0$ for each $j = 1, \ldots, g$. We thus have $\sum_{i=1}^{n_v-1} p_i \mathbf{grad}\, \varphi_i = \mathbf{0}$, hence $\sum_{i=1}^{n_v-1} p_i \varphi_i = \text{const}$; the conclusion follows from the fact that $\varphi_i(v_{n_v}) = 0$ for each $i = 1, \ldots, n_v - 1$.
The proof that the loop fields $\mathbf{T}^*_{0,\alpha}$ are cohomologically independent follows the same argument. $\square$

When solving the problem $\mathbf{curl}\,\mathbf{Z}_h = \mathbf{J}_h$ we have to match two Raviart–Thomas elements, hence their fluxes across each face of $\mathcal{T}_h$ have to be the same.
Since the Stokes theorem assures that

$$\int_{e_1} \mathbf{Z}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{Z}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{Z}_h \cdot \boldsymbol{\tau} = \int_f \mathbf{J}_h \cdot \boldsymbol{\nu}\,, \qquad (77)$$

where $\partial f = e_1 \cup e_2 \cup e_3$ and $\boldsymbol{\nu}$ is the unit normal vector on f (with consistent orientation), we deduce that the corresponding linear system has exactly three non-zero values for each row.

Webb and Forghani (1989) proposed the following solution algorithm:

1. set value 0 to the unknowns corresponding to an edge belonging to the spanning tree
2. take a face f for which at least one edge unknown has not yet been assigned
    1. if exactly one edge unknown is not determined, compute its value from the Stokes relation (77)
    2. if two or three edge unknowns are not determined, pass to another face
3. if the iterations stop before the end, check if one of the "homological" equations $\oint_{\sigma_n} \mathbf{Z}_h \cdot d\mathbf{s} = \kappa_n$ permits to restart.

The Webb–Forghani algorithm is a simple elimination procedure for solving the linear system at hand, and it is quite efficient, as the computational costs is linearly dependent on the number of unknowns.

The weak point is that:

- it strongly depends on the choice of the spanning tree and it can stop without having determined all the edge unknowns (even in simple topological situations!)

(see Dłotko and Specogna (2010)).

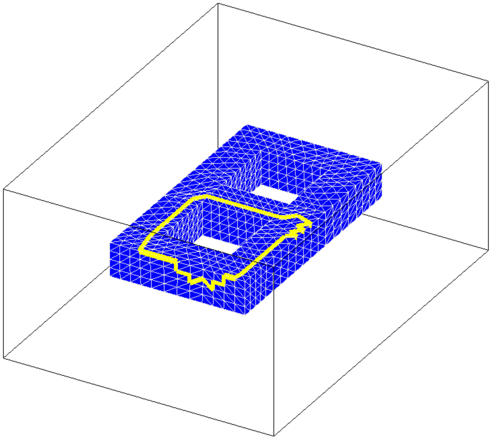


Figure : Case A: 2-torus (one homological cycle σ_n is drawn).

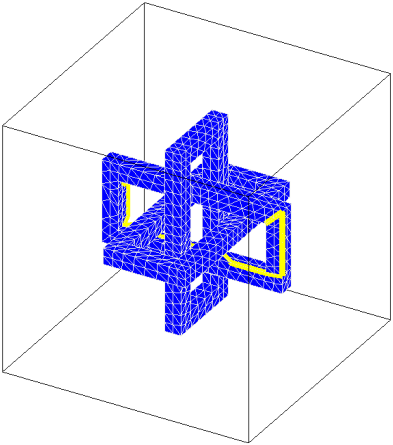


Figure : Case B: Borromean rings (one homological cycle σ_n is drawn).

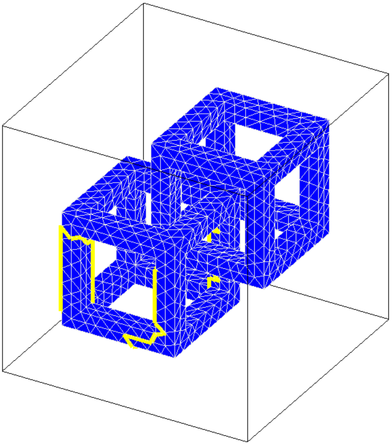


Figure : Case C: two-5-tori link (one homological cycle σ_n is drawn).

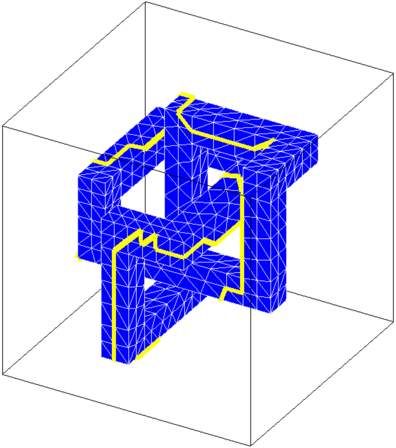


Figure : Case D: trefoil knot (one homological cycle σ_n is drawn).

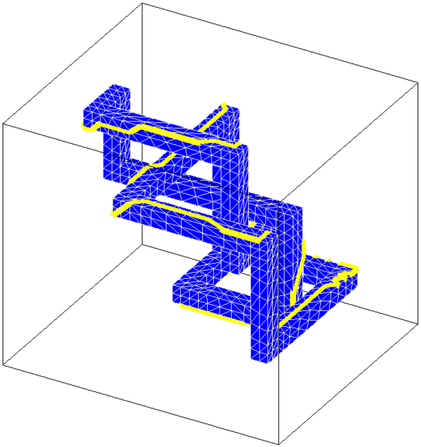


Figure : Case E: knot 4_1 (one homological cycle σ_n is drawn).

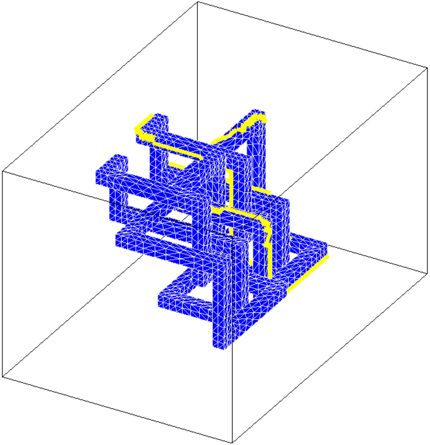


Figure : Case F: two-4_1-knots link (one homological cycle σ_n is drawn).

	n_e	$n_e^{(2)}$ breadth-first	$n_e^{(2)}$ depth-first
Test A	42200	0	27912
Test B	35380	0	23595
Test C	25768	0	15707
Test D	15349	2092	9554
Test E	34372	6002	22776
Test F	80504	12916	53488

Table : Dependence of the reduction of the unknowns on the choice of the spanning tree.

[Remind - A: 2-torus; B: Borromean rings; C: two-5-tori link; D: trefoil knot; E: knot 4_1; F: two-4_1-knots link.]

If you prefer: breadth-first algorithm is like the fox, and depth-first algorithm is like the hedgehog.

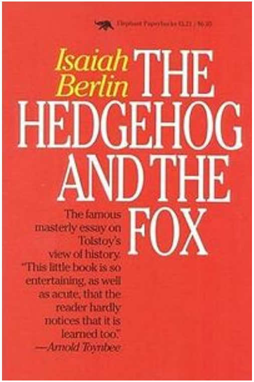


Figure : A new denomination for the spanning tree choice?

If $\mathbf{J}_h = \mathbf{0}$ we devise an explicit formula for the solution to (76).

The idea is the following: the Biot–Savart law gives the magnetic field generated by a unitary density current concentrated along the edge cycle $\widehat{\sigma}_\alpha$ (a generator of the first homology group of $\mathbb{R}^3 \setminus \Omega_I$) by means of the formula:

$$\widehat{\mathbf{H}}(\mathbf{x}) = \frac{1}{4\pi} \oint_{\widehat{\sigma}_\alpha} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y \quad , \quad \mathbf{x} \notin \widehat{\sigma}_\alpha \, .$$

Since the cycle $\widehat{\sigma}_\alpha$ can be chosen external to $\overline{\Omega_I}$, one has $\mathbf{curl}\, \widehat{\mathbf{H}} = \mathbf{0}$ in Ω_I. Moreover, on each cycle $\gamma \subset \overline{\Omega_I}$ that is linking the current passing in $\widehat{\sigma}_\alpha$ one finds $\oint_\gamma \widehat{\mathbf{H}} \cdot d\mathbf{s} \neq 0$, hence $\widehat{\mathbf{H}}$ is a loop field in Ω_I.

[There are cycles γ with the required property: for instance, one of the generators of the first homology group of $\overline{\Omega_I}$.]

Clearly, the Nédélec interpolant $\Pi^{N_h^1}\widehat{\mathbf{H}}$ is a finite element loop field in Ω_I. For each $e \in \mathcal{T}_h$, its degrees of freedom are given by

$$\widehat{q}_e = \frac{1}{4\pi} \int_e \left(\oint_{\widehat{\sigma}_\alpha} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot \boldsymbol{\tau}_x \, .$$

This resembles the formula for computing the linking number between $\widehat{\sigma}_\alpha$ and another disjoint cycle σ:

$$LK(\sigma, \widehat{\sigma}_\alpha) = \frac{1}{4\pi} \oint_\sigma \left(\oint_{\widehat{\sigma}_\alpha} \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x \, .$$

- The linking number is an integer that represents the number of times that each cycle winds around the other.

- Is it possible to reduce the definition of the finite element loop field to the computation of suitable linking numbers?

Consider the spanning tree $\mathcal{S}_h$, its root v_*, and define in the vertices of $\mathcal{T}_h \cap \overline{\Omega_I}$ the scalar function $\phi_h \in L^1_h$ as $\phi_h(v_*) = 0$ and

$$\phi_h(v_b) = \phi_h(v_a) + \widehat{q}_{[v_a,v_b]} \quad \forall e' = [v_a, v_b] \in \mathbf{S}_h \, .$$

The Nédélec finite element $\mathbf{Z}_h = \Pi^{N^1_h}\widehat{\mathbf{H}} - \mathbf{grad}\, \phi_h$ is a loop field in Ω_I, and its degrees of freedom are equal to 0 for all the edges e' of the spanning tree $\mathcal{S}_h$.

For each $e \in \mathcal{T}_h \cap \overline{\Omega_I}$, define now by D_e the edge cycle constituted by: the edges from the root of the spanning tree $\mathcal{S}_h$ to the first vertex v_e^- of e, the edge e, the edges from the second vertex v_e^+ of e to the root of the spanning tree $\mathcal{S}_h$. In particular, $D_{e'}$ is a trivial cycle if $e' \in \mathcal{S}_h$.

When $e \notin \mathcal{S}_h$ the cycle D_e is constituted by edges all belonging to the spanning tree (except e): hence we have

$$\begin{aligned} &\tfrac{1}{4\pi} \oint_{D_e} \left(\oint_{\widehat{\sigma}_\alpha} \tfrac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x \\ &= \widehat{q}_e + \textstyle\sum_{e' \in D_e \cap \mathcal{S}_h} \widehat{q}_{e'} \\ &= \widehat{q}_e + \textstyle\sum_{e' \in D_e \cap \mathcal{S}_h} (\phi_h(v_{e'}^+) - \phi_h(v_{e'}^-)) \\ &= \widehat{q}_e + (\phi_h(v_e^-) - \phi_h(v_e^+)) = \int_e \mathbf{Z}_h \cdot \boldsymbol{\tau} \,, \end{aligned}$$

and thus the degrees of freedom of $\mathbf{Z}_h$ are given by

$$\int_e \mathbf{Z}_h \cdot \boldsymbol{\tau} = LK(D_e, \widehat{\sigma}_\alpha) \,.$$

In particular, the loop field $\mathbf{Z}_h$ thus defined satisfies problem (76) with $\kappa_n = m_{n,\alpha} = LK(\sigma_n, \widehat{\sigma}_\alpha)$, a non-singular matrix.

- Selecting $\alpha = 1, \ldots, g$ we have an explicit formula for a basis of the first de Rham cohomology group.

Since a linking number is a sum of simple double integrals, its computation can be done efficiently (see Bertolazzi and Ghiloni (2013)).
However, for a fine mesh it is too expensive if used for all the edges (not belonging to the spanning tree...).

- Recipe: when the Webb–Forghani algorithm stops, use the formula for computing the value of one single unknown, and restart the algorithm.

Numerical experiments show that the use of the explicit formula is necessary very few times [one for Test D and Test E, four for Test F].

	Mesh 1		Mesh 2		Mesh 3	
	n_e	ms	n_e	ms	n_e	ms
Test A	42200	138	325904	868	2560416	6770
Test B	35380	93	273348	586	2147096	4397
Test C	25768	293	195256	1318	1517328	7434
Test D	15349	79	116170	294	902388	2016
Test E	34372	144	264548	749	2073688	4760
Test F	80504	310	624352	2671	4913792	12723

Table : CPU time for computing all the homological cycles σ_n and $\widehat{\sigma}_n$.

	n_e	$n_e - \#L$	$n_e^{(1)}$	$n_e^{(2)}$	$\#CC$
Test A	2560416	2185729	58987	0	-
Test B	2147096	1832896	110245	0	-
Test C	1517328	1292168	124239	0	-
Test D	902388	768384	54273	34506	30
Test E	2073688	1769408	150694	98603	107
Test F	4913792	4196608	275832	212088	145

Table : Reduction of the number of unknowns.

[n_e: number of edges; $\#L$: number of spanning tree edges;
$n_e^{(1)}$: number of unknowns left after the algorithm has stopped;
$n_e^{(2)}$: number of unknowns left after having used the homological equations; $\#CC$: number of connected components of dual graph.]

	n_e	loop fields		source field
Test A	2560416	(2)	9659	9937
Test B	2147096	(3)	9447	8822
Test C	1517328	(10)	28187	6322
Test D	902388	(1)	3759	3814
Test E	2073688	(1)	8705	8907
Test F	4913792	(2)	37338	22210

Table : CPU time (ms) for computing all the loop fields (their number is indicated in parenthesis) and one source field.

We are now in position for giving a more efficient numerical approximation of the eddy current problem in terms of a scalar potential.
As a preprocessing, by the Webb–Forghani algorithm we have computed:

- an edge element source field $\mathbf{H}_e^{(h)}$, constructed in Ω_I in correspondence with $\mathbf{J}_h$ (a suitable approximation of the current density $\mathbf{J}_e$) and extended by 0 on the edges internal to Ω_C;
- an edge element basis $\mathbf{T}^*_{0,\alpha}$ of the first de Rham cohomology group of Ω_I.

Variational formulation (63) refers to the space

$$V = \{\mathbf{v} \in H(\mathbf{curl}\,;\Omega) \,|\, \mathbf{curl}\,\mathbf{v}_I = \mathbf{0} \text{ in } \Omega_I\} \,.$$

A suitable finite element approximation of it is clearly given by

$$V_h = N_h^1 \cap V \,.$$

As a basis for the space V_h we choose the following functions:

- for all the edges internal to Ω_C (and not on Γ), the Nédélec basis function of the lowest degree, extended by 0 on the edges on Γ and internal to Ω_I;
- for all the nodes internal to Ω_I and on Γ (except one), the gradient of the Lagrange basis function of degree 1, extended by 0 on the edges internal to Ω_C;
- the de Rham cohomology basis functions $\mathbf{T}^*_{0,\alpha}$, $\alpha = 1, \ldots, g$, extended by 0 on the edges internal to Ω_C.

Clearly, all these functions are elements of N_h^1, and are curl-free in Ω_I. They form a basis of V_h, as in Lemma (2) we have seen that the gradients of the Lagrange basis functions and the de Rham cohomology basis functions are a basis for $N_h^1 \cap H^0(\mathbf{curl}\,;\Omega_I)$.

The approximate problem simply reads:

Find $(\mathbf{H}_h - \mathbf{H}_e^{(h)}) \in V_h$:

$$\int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{H}_{C,h} \cdot \mathbf{curl}\, \overline{\mathbf{v}_{C,h}} + \int_{\Omega} i\omega\boldsymbol{\mu} \mathbf{H}_h \cdot \overline{\mathbf{v}_h} = \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl}\, \overline{\mathbf{v}_{C,h}} \qquad (78)$$

for each $\mathbf{v}_h \in V_h$.

The algebraic structure is easily found by expressing $\mathbf{H}_h - \mathbf{H}_e^{(h)}$ in terms of the Nédélec basis functions $\mathbf{\Phi}_m$, the gradients of nodal basis functions φ_i and the de Rham loop fields $\mathbf{T}^*_{0,\alpha}$.

Convergence of the finite element scheme is also readily proved. Let us first recall that we have set

$$a_m(\mathbf{u},\mathbf{v}) := \int_{\Omega_C} \boldsymbol{\sigma}^{-1} \mathbf{curl}\, \mathbf{u}_C \cdot \mathbf{curl}\, \overline{\mathbf{v}_C} + \int_\Omega i\omega\boldsymbol{\mu}\mathbf{u} \cdot \overline{\mathbf{v}} \,.$$

Let us also define

$$V_h^\star = \{\mathbf{v}_h \in N_h^1 \,|\, \mathbf{curl}\, \mathbf{v}_{I,h} = \mathbf{J}_h \text{ in } \Omega_I\} \,.$$

We note that $a_m(\mathbf{H} - \mathbf{H}_h, \mathbf{v}_h) = 0$ for all $\mathbf{v}_h \in V_h$, hence

$$\begin{aligned} &\|\mathbf{H}_C - \mathbf{H}_{C,h}\|^2_{\mathrm{curl},\Omega_C} + \|\mathbf{H}_I - \mathbf{H}_{I,h}\|^2_{0,\Omega_I} \\ &\quad \leq C_1\, a_m(\mathbf{H} - \mathbf{H}_h, \mathbf{H} - \mathbf{H}_h) \\ &\quad = C_1\, a_m(\mathbf{H} - \mathbf{H}_h, \mathbf{H} - \mathbf{z}_h) \,, \end{aligned} \tag{79}$$

for each $\mathbf{z}_h \in V_h^\star$.

Thus we have the optimal error estimate

$$\begin{aligned} &\|\mathbf{H}_C - \mathbf{H}_{C,h}\|^2_{\mathrm{curl},\Omega_C} + \|\mathbf{H}_I - \mathbf{H}_{I,h}\|^2_{0,\Omega_I} \\ &\qquad \leq C \inf_{\mathbf{z}_h \in V_h^\star} (\|\mathbf{H}_C - \mathbf{z}_{C,h}\|^2_{\mathrm{curl},\Omega_C} + \|\mathbf{H}_I - \mathbf{z}_{I,h}\|^2_{0,\Omega_I}) \,. \end{aligned} \tag{80}$$

Choosing as an approximation of the current density the Raviart–Thomas interpolant, namely, $\mathbf{J}_h = \Pi^{RT_h^1} \mathbf{J}_{e,I}$, it clearly follows $\Pi^{N_h^1}\mathbf{H} \in V_h^\star$, as $\mathbf{curl}\,(\Pi^{N_h^1}\mathbf{H}_I) = \Pi^{RT_h^1}(\mathbf{curl}\,\mathbf{H}_I)$. Therefore we end with

$$\begin{aligned} \|\mathbf{H} - \mathbf{H}_h\|^2_{\mathrm{curl},\Omega} \leq C\,(&\|\mathbf{H}_C - \Pi^{N_h^1}\mathbf{H}_C\|^2_{\mathrm{curl},\Omega_C} \\ &+\|\mathbf{H}_I - \Pi^{N_h^1}\mathbf{H}_I\|^2_{0,\Omega_I} \\ &+\|\mathbf{J}_{e,I} - \Pi^{RT_h^1}\mathbf{J}_{e,I}\|^2_{0,\Omega_I}) \,, \end{aligned} \tag{81}$$

as $\mathbf{curl}\,\mathbf{H}_I = \mathbf{J}_{e,I}$ and $\mathbf{curl}\,\mathbf{H}_{I,h} = \mathbf{J}_h = \Pi^{RT_h^1}\mathbf{J}_{e,I}$.

Remarks:

- at the finite dimensional level, the described approach does not require imposing that the matching condition on the interface holds, as it is automatically satisfied by the choice of the discrete space (which is given by functions that have a well-defined curl in the whole Ω).
- once having constructed $\mathbf{H}_e^{(h)}$ and $\mathbf{T}_{0,\alpha}$, $\alpha = 1, \ldots, g$, one could also use a domain decomposition scheme like (71)–(74) (suitably modified as the de Rham loop fields are not orthogonal to gradients). [One could also devise a domain decomposition scheme starting form problem (69): in this case non-matching meshes on Γ could be employed.]

Electroencephalography (EEG) and magnetoencephalography (MEG) are two non-invasive techniques used to localize electric activity in the brain from measurements of external electromagnetic signals.

EEG measures the electric potential (on the scalp), while MEG measures the magnetic flux (closed but external to the head).

The electromagnetic activity of the brain is due to the movements of ions within activated regions of the cortex sheet, the so-called impressed currents (or primary currents). In addition, Ohmic currents are generated in the surrounding medium, the so-called return currents.

The measures of EEG and MEG correspond to both impressed and return currents, but the source of interest are the impressed currents, as they represent the area of neural activity associated to a sensory stimulus.

- First EEG in man: H. Berger (1924)
- First MEG in man: D. Cohen (late 1960s).

It is worth noting that the magnetic signal related to brain activity is extremely weak, about 100 times lower than the earths geomagnetic field. Its measurement only became possible with the SQUID (SuperconductingQUantum Interface Devices) magnetometer (1970).

Source localization is an inverse problem: knowing the value of the magnetic field or of the electric field on the surface of the head (or, possibly, external to the head, but close to its surface), the aim is to determine the position and some physical characteristics of the current density that has given rise to that value.

Since the current distribution inside a conductor cannot be retrieved uniquely from knowledge of the electromagnetic field outside the conductor, the mathematical problem does not have a unique solution unless some additional conditions on the source model are assumed.

- Two different approaches are mainly used to reconstruct the brain neural sources: dipolar and distributed source models.

In the dipolar model the primary current distribution is represented as a point source located at $\mathbf{x}_0$ with moment $\mathbf{p}$, namely,

$$\mathbf{J}_e(\mathbf{x}) = \mathbf{p}\,\delta(\mathbf{x} - \mathbf{x}_0)\,,$$

where δ is the Dirac delta distribution.

The dipole is a convenient representation for a uni-directional impressed current due to the activation of a large number of cells (in real situations may indeed extend over several square centimeters of the cortex). More generally, it is assumed that a primary current source can be decomposed as the sum of (few) current dipoles.

The distributed source model (also called imaging approach) assumes that a lot of dipoles are located perpendicularly to the cortical surface. The geometry of the cortical surface can be extracted from brain magnetic resonance imaging (MRI) data. A tessellation of this surface is constructed and a current dipole is placed on each element with its orientation normal to the surface.

The inverse problem in this case turns out to be linear: only the magnitudes of the dipole moments have to be reconstructed, and not the location nor the orientation. Proceeding in this way the number of unknowns is typically greater than the number of measured data and the inverse problem is solved using regularization schemes, such as a truncated singular value decomposition of the Tikhonov regularization.

In both cases, a preliminary step for the solution of the inverse problem is an efficient resolution of the forward problem.

In fact, the procedure is essentially the following: given a source $\mathbf{J}_e$, solve the forward problem, thus determining the electric and magnetic fields generated by $\mathbf{J}_e$, and then minimize in a suitable way the difference between the computed and the measured data.

The current density $\mathbf{J}_e^*$ which achieves the minimum is the source we are trying to determine.

Let us focus now on the forward problem.

Due to its complicated detailed structure, the human brain is indeed a heterogeneous anisotropic medium, with physical parameters that depend on the spatial variable and that may be tensors.

The frequency spectrum for electrophysiological signals in MEG is tipically below 1000 Hz, and most studies deal with frequency between 0.1 and 100 Hz.

However, let us start with the static approximation of Maxwell equations

$$\begin{aligned} \mathbf{curl}\,\mathbf{H} &= \mathbf{J}_e + \sigma\mathbf{E} \\ \operatorname{div}\mathbf{B} &= 0 \\ \mathbf{curl}\,\mathbf{E} &= \mathbf{0}\,, \end{aligned} \tag{82}$$

neglecting not only the displacement current but also the electromagnetic diffusion.

- Note that in this way the electric field $\mathbf{E}$ can be determined independently from the magnetic field $\mathbf{H}$.

From Ohm law the total current density $\mathbf{J}$ is the sum of the impressed currents plus the return currents

$$\mathbf{J} = \mathbf{J}_e + \sigma\mathbf{E} = \mathbf{J}_e - \sigma\,\mathbf{grad}\,U\,,$$

where U is the electric scalar potential.

From the first equation in (82) it follows that

$$0 = \operatorname{div} \mathbf{J} = \operatorname{div} (\mathbf{J}_e - \boldsymbol{\sigma} \operatorname{\mathbf{grad}} U) \,.$$

Hence U can be obtained by solving the Poisson equation with Neumann boundary condition

$$\begin{cases} \operatorname{div} (\boldsymbol{\sigma} \operatorname{\mathbf{grad}} U) = \operatorname{div} \mathbf{J}_e & \text{in } \Omega_C \\ \boldsymbol{\sigma} \operatorname{\mathbf{grad}} U \cdot \mathbf{n} = \mathbf{J}_e \cdot \mathbf{n} & \text{on } \partial\Omega_C \,, \end{cases} \tag{83}$$

where the boundary condition is a consequence of the fact that outside the head Ω_C the magnetic field is supposed to be curl-free (the source $\mathbf{J}_e$ is located inside the head, and the conductivity is vanishing outside the head, so that $\mathbf{J}_I = \mathbf{0}$ in Ω_I and consequently $\mathbf{J}_C \cdot \mathbf{n} = 0$ on $\partial\Omega_C$).

For EEG this is the point: solving this elliptic problem gives the potential of the electric field, and the inverse problem of source localization can be dealt with.

For MEG, one has to go further. Assuming that the magnetic permeability is equal to μ_0, the free-space permeability, $\mathbf{B}$ is given by the Biot–Savart law

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_{\Omega_C} [\mathbf{J}_e(\mathbf{y}) - \boldsymbol{\sigma}(\mathbf{y})\mathbf{grad}\, U(\mathbf{y})] \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y}\,. \qquad (84)$$

Clearly, this formula furnishes a direct way to compute the magnetic induction $\mathbf{B}$ only if the electric scalar potential U has been already determined through (83).

However, in some cases solving the elliptic problem (83) can be avoided.

In fact, the simplified model assumes that the head can be described by three (scalp, skull and brain) to five (scalp, skull, cerebrospinal fluid, gray matter and white matter) contiguous layers Ω_j, $j = 1, \ldots, n$. The different layers are separated by the surfaces S_j, $j = 1, \ldots, n$, S_1 being the outermost one. Assuming that the $\boldsymbol{\sigma}_{|\Omega_k}$ is a scalar constant, by classical results of potential theory (see Sarvas (1987)) it is possible to derive a surface integral equation for $U_k := U_{|S_k}$, $k = 1, \ldots, n$,

$$\begin{aligned} \tfrac{\sigma_k^- + \sigma_k^+}{2} \, U_k(\mathbf{x}) &= U_\infty(\mathbf{x}) \\ &\quad - \tfrac{1}{4\pi} \textstyle\sum_{j=1}^n (\sigma_j^- - \sigma_j^+) \int_{S_j} U_j(\mathbf{y}) \, \mathbf{n}_j(\mathbf{y}) \cdot \tfrac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \, dS_y \,, \end{aligned} \tag{85}$$

where

$$U_\infty(\mathbf{x}) := \frac{1}{4\pi} \int_\Omega \mathbf{J}_e(\mathbf{y}) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{y} \, ,$$

$\mathbf{n}_j$ is the unit outward normal vector to S_j, σ_j^- is the inside conductivity and σ_j^+ is the outside conductivity, with $\sigma_1^+ = 0$ and, clearly, $\sigma_j^- = \sigma_{j+1}^+$, $j = 1, \dots, n-1$.

[For a current dipole, one has

$$U_\infty(\mathbf{x}) = \frac{1}{4\pi} \, \mathbf{p} \cdot \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|^3} \, .]$$

Integration by parts in (84) yields the Geselowitz formula (see Geselowitz (1970))

$$\mathbf{B}(\mathbf{x}) = \mathbf{B}_\infty(\mathbf{x}) - \frac{\mu_0}{4\pi} \sum_{j=1}^{n} (\sigma_j^- - \sigma_j^+) \int_{S_j} U_j(\mathbf{y})\, \mathbf{n}_j(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, dS_y \,, \tag{86}$$

where

$$\mathbf{B}_\infty(\mathbf{x}) := \frac{\mu_0}{4\pi} \int_\Omega \mathbf{J}_e(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y} \,.$$

[For a current dipole, one has

$$\mathbf{B}_\infty(\mathbf{x}) = \frac{\mu_0}{4\pi} \mathbf{p} \times \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3} \,.]$$

At this stage, for MEG the main point turns out to be the determination of the functions U_j on the surfaces S_j, which furnish the magnetic induction $\mathbf{B}$ via the explicit formula (86).

Hence a boundary element approach can be introduced, with the aim of finding a solution to the discrete approximation of (85), then inserting the obtained results in (86).

- A more accurate source reconstruction needs a more realistic model (for instance, anisotropy of the conductivity in the skull and brain must be taken into account).

From a numerical point of view this means that one has to go back to the numerical solution of (83), and this can be done by using a finite element scheme (see Wolters, Grasedyck and Hackbusch (2004)).

- However, we already noted that a modelization through the elliptic equation (83) is not completely satisfactory, as the physiological frequency ranges between 0.1 and 100 Hz, and in general cannot be assumed to vanish.

Since in terms of the electric field $\mathbf{E}$ the time-harmonic Maxwell equations can be written as

$$\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}) - \omega^2\boldsymbol{\varepsilon}\mathbf{E} + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{J}_e\,,$$

we have already seen that a thumb rule that drives the choice of the model could be formulated as follows: if L is a typical length (say, the diameter of the physical domain), it is possible to disregard the displacement current term provided that

$$\mu\,\varepsilon\,\omega^2 L^2 \ll 1 \quad , \quad \sigma^{-1}\,\varepsilon\,|\omega| \ll 1\,.$$

[Let us also recall that the wavelength can be expressed by

$$\lambda = \frac{1}{|\omega|\sqrt{\mu\,\varepsilon}}\,.]$$

On the other hand, it seems reasonable to utilize the static model when, in addition,

$$\mu\,\sigma\,|\omega| L^2 \ll 1\,.$$

For physiological problems, we have

$$\begin{aligned}
&\omega = 2\pi \times 50 \ rad/s \\
&\mu = 4\pi \times 10^{-7} \ H/m \\
&\sigma = 0.1 \ S/m \\
&L = 0.3 \ m \,,
\end{aligned}$$

while the electric permittivity can vary with the frequency, and a reasonable value can be

$$\varepsilon \approx 10^{-6} \ F/m \,.$$

Therefore we have

$$\mu \, \varepsilon \, \omega^2 L^2 \approx 10^{-8} \quad , \quad \sigma^{-1} \, \varepsilon \, |\omega| \approx 3 \times 10^{-3}$$

$$\lambda \approx 3 \times 10^3 m \quad , \quad \mu \, \sigma \, |\omega| L^2 \approx 3.5 \times 10^{-6} \,.$$

- The first two values and the estimate of the wavelength say that it seems suitable to disregard the displacement current term, adopting the eddy current model.
- From the estimate of $|\omega| \, \mu \, \sigma L^2$ it seems also possible to utilize the static model. However, it is easy to construct source current densities $\mathbf{J}_e$ for which the electric field given by the static model is vanishing, while the electric field solution of the eddy current model is large (it is enough to take $\mathrm{div}\mathbf{J}_e = 0$).
- Hence the static model is not really satisfactory, and it is qualitatively different from the non-static ones. An accurate description of the problem seems requiring the eddy current model (or, possibly, for larger values of the frequency and of the electric permittivity, on the full Maxwell model).

He and Romanov (1998) and Ammari, Bao and Fleming (2002) use the full Maxwell system in $\mathbb{R}^3$ for a current dipole:

$$\begin{cases} \mathbf{curl}\,\mathbf{H} - i\omega\varepsilon\mathbf{E} = \boldsymbol{\sigma}\mathbf{E} + \mathbf{p}\,\delta(\mathbf{x} - \mathbf{x}_0) \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0}\,, \end{cases} \tag{87}$$

with the Silver–Müller radiation condition

$$\lim_{|\mathbf{x}|\to+\infty} |\mathbf{x}| \left(\sqrt{\mu_0}\,\mathbf{H} \times \frac{\mathbf{x}}{|\mathbf{x}|} - \sqrt{\varepsilon_0}\,\mathbf{E} \right) = \mathbf{0}\,,$$

ε_0 being the free-space electric permittivity.

An intermediate situation is the one based on the eddy current equations (say, in a topologically simple bounded region Ω containing a topologically simple conductor Ω_C): always referring to a current dipole, the problem reads

$$\begin{cases} \mathbf{curl}\,\mathbf{H} = \boldsymbol{\sigma}\mathbf{E} + \mathbf{p}\,\delta(\mathbf{x} - \mathbf{x}_0) & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \operatorname{div}(\varepsilon_I \mathbf{E}_I) = 0 & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \varepsilon_I \mathbf{E}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{88}$$

Due to the singularity of the right hand side, the existence theory for the potential equation (83) or the eddy current system (88) when the current density is a dipole is not completely straightforward.

Let us start from the potential equation (83).
The problem reads

$$\begin{cases} \mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,U) = \mathrm{div}(\mathbf{p}\,\delta_{\mathbf{x}_0}) & \text{in } \Omega_C \\ (\boldsymbol{\sigma}\,\mathbf{grad}\,U)\cdot\mathbf{n} = 0 & \text{on } \partial\Omega_C\,, \end{cases} \tag{89}$$

where $\mathbf{x}_0 \in \Omega_C$ and we have set, for simplicity, $\delta_{\mathbf{x}_0}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0)$.
Clearly, the solution U is defined up to an additive constant.

Following V. (2012), we want to give a weak formulation of problem (89).

- We assume the local Lipschitz regularity condition for the conductivity:

$$\text{there exists } r_0 > 0 \text{ such that } \boldsymbol{\sigma} \in W^{1,\infty}(B_{r_0}(\mathbf{x}_0))\,. \tag{90}$$

Introduce the linear space

$$X_q := \{\varphi \in H^1(\Omega_C) \,|\, \varphi \in C^1(B_{r_*}(\mathbf{x}_0)), \mathrm{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, \varphi) \in L^q(\Omega_C), \\ (\boldsymbol{\sigma} \,\mathbf{grad}\, \varphi) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_C\} \,,$$

where $0 < r_* < r_0$ is a fixed number, q is a fixed number satisfying $3 < q < +\infty$, and p is its Hölder dual exponent defined by $\frac{1}{p} + \frac{1}{q} = 1$ (hence $1 < p < \frac{3}{2}$).

Multiplying the first equation in (89) by $\varphi \in X_q$, integrating in Ω_C and integrating by parts we readily find

$$
\begin{aligned}
\int_{\Omega_C} & \operatorname{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, U)\, \varphi \\
& = -\int_{\Omega_C} (\boldsymbol{\sigma} \,\mathbf{grad}\, U) \cdot \mathbf{grad}\, \varphi + \int_{\partial\Omega_C} (\boldsymbol{\sigma} \,\mathbf{grad}\, U) \cdot \mathbf{n}\, \varphi \\
& = \int_{\Omega_C} U \operatorname{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, \varphi) - \int_{\partial\Omega_C} U\, (\boldsymbol{\sigma} \,\mathbf{grad}\, \phi) \cdot \mathbf{n} \\
& \quad + \int_{\partial\Omega_C} (\boldsymbol{\sigma} \,\mathbf{grad}\, U) \cdot \mathbf{n}\, \varphi \\
& = \int_{\Omega_C} U \operatorname{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, \varphi)
\end{aligned}
$$

and

$$
\int_{\Omega_C} \operatorname{div}(\mathbf{p}\, \delta_{\mathbf{x}_0})\, \varphi = -\int_{\Omega_C} \mathbf{p} \cdot \mathbf{grad}\, \varphi\, \delta_{\mathbf{x}_0} = -\mathbf{p} \cdot \mathbf{grad}\, \varphi(\mathbf{x}_0)\,,
$$

having taken into account the boundary conditions satisfied by U and φ.

Note that, for duality, the term $\int_{\Omega_C} U \operatorname{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, \varphi)$ has a meaning also for $U \in L^p(\Omega_C)$.

We are now in a position to describe the weak formulation of (89) that we consider:

$$\begin{cases} \text{find } U \in L^p(\Omega_C) \ : \\ \int_{\Omega_C} U \,\mathrm{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, \varphi) = -\,\mathbf{p} \cdot \mathbf{grad}\, \varphi(\mathbf{x}_0) \quad \forall\, \varphi \in X_q \\ \int_{\Omega_C} U = 0\,. \end{cases} \tag{91}$$

The following theorem gives the proof of the existence of the solution U of problem (91).

Theorem (3)

There exists a solution U to (91).

Proof. We use an approximation argument. Let us denote by δ_k a sequence of functions such that $\delta_k \in C_0^\infty(B_{r_*}(\mathbf{x}_0))$, $\delta_k \geq 0$, $\int_{\Omega_C} \delta_k = 1$ and $\int_{\Omega_C} \delta_k \, \xi \to \xi(\mathbf{x}_0)$ for each $\xi \in C^0(B_{r_*}(\mathbf{x}_0))$. We consider the solution $U_k \in H^1(\Omega_C)$ of the Neumann problem

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \operatorname{\mathbf{grad}} U_k) = \operatorname{div}(\mathbf{p} \, \delta_k) & \text{in } \Omega_C \\ (\boldsymbol{\sigma} \operatorname{\mathbf{grad}} U_k) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_C \\ \int_{\Omega_C} U_k = 0 \, . \end{cases}$$

The existence and uniqueness of U_k is assured as $\int_{\Omega_C} \operatorname{div}(\mathbf{p} \, \delta_k) = \int_{\partial\Omega_C} \mathbf{p} \cdot \mathbf{n} \, \delta_k = 0$, hence the compatibility condition is satisfied.

By integrating by parts we see that U_k satisfies

$$\int_{\Omega_C} U_k \,\mathrm{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, \varphi) = - \int_{\Omega_C} \mathbf{p} \cdot \mathbf{grad}\, \varphi \, \delta_k \quad \forall \, \varphi \in X_q \,.$$

Take now $\psi \in L^q(\Omega_C)$: we want to find an uniform estimate of $\int_{\Omega_C} U_k \, \psi$. Consider the solution $\hat{\varphi}$ of the Neumann problem

$$\begin{cases} \mathrm{div}(\boldsymbol{\sigma} \,\mathbf{grad}\, \hat{\varphi}) = \psi - \frac{1}{\mathrm{meas}(\Omega_C)} \left(\int_{\Omega_C} \psi \right) & \text{in } \Omega_C \\ (\boldsymbol{\sigma} \,\mathbf{grad}\, \hat{\varphi}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_C \\ \int_{\Omega_C} \hat{\varphi} = 0 \,. \end{cases} \tag{92}$$

Since $\int_{\Omega_C} \left[\psi - \frac{1}{\mathrm{meas}(\Omega_C)} \left(\int_{\Omega_C} \psi \right) \right] = 0$, we have a unique solution $\hat{\varphi} \in H^1(\Omega_C)$.

On the other hand, we have $\left[\psi - \frac{1}{\mathrm{meas}(\Omega_C)}\left(\int_{\Omega_C} \psi\right)\right] \in L^q(\Omega_C)$ and the regularity results for elliptic problems yield $\hat{\varphi} \in W^{2,q}(B_{r_*}(\mathbf{x}_0))$. The Sobolev embedding theorem also gives $\hat{\varphi} \in C^1(\overline{B_{r_*}(\mathbf{x}_0)})$, hence $\hat{\varphi} \in X_q$. Moreover, $\|\hat{\varphi}\|_{C^1(\overline{B_{r_*}(\mathbf{x}_0)})} \leq c_0 \|\psi\|_{L^q(\Omega_C)}$, where c_0 depends on $\boldsymbol{\sigma}$, q, r_*, but not on ψ.

We are now in a position to obtain the needed estimate. We have

$$\begin{aligned}
\left|\int_{\Omega_C} U_k\,\psi\right| &= \left|\int_{\Omega_C} U_k\left[\psi - \frac{1}{\mathrm{meas}(\Omega_C)}\left(\int_{\Omega_C}\psi\right)\right]\right| \\
&= \left|\int_{\Omega_C} U_k \mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\hat{\varphi})\right| = \left|-\int_{\Omega_C} \mathbf{p}\cdot\mathbf{grad}\,\hat{\varphi}\,\delta_k\right| \\
&\leq |\mathbf{p}|\,\|\mathbf{grad}\,\hat{\varphi}\|_{C^0(\overline{B_{r_*}(\mathbf{x}_0)})}\int_{\Omega_C}\delta_k \leq c_0\,|\mathbf{p}|\|\psi\|_{L^q(\Omega_C)}\ .
\end{aligned}$$

In other words,

$$\|U_k\|_{L^p(\Omega_C)} := \sup_{\psi\in L^q(\Omega_C)} \frac{\left|\int_{\Omega_C} U_k\,\psi\right|}{\|\psi\|_{L^q(\Omega_C)}} \leq c_0\,|\mathbf{p}|\,.$$

We can thus select a subsequence (still denoted by U_k) that converges weekly in $L^p(\Omega_C)$ to $U \in L^p(\Omega_C)$. In particular, for each $\varphi \in X_q$

$$\begin{aligned} &\textstyle\int_{\Omega_C} U_k \,\mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\varphi) \to \int_{\Omega_C} U \,\mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\varphi)\,, \\ &-\textstyle\int_{\Omega_C} \mathbf{p}\cdot\mathbf{grad}\,\varphi\,\delta_k \to -\,\mathbf{p}\cdot\mathbf{grad}\,\varphi(\mathbf{x}_0)\,. \end{aligned}$$

Finally,

$$0 = \int_{\Omega_C} U_k \to \int_{\Omega_C} U\,,$$

and U is a solution to (91). □

Theorem (4)

The solution U to (91) is unique.

Proof. Let U be any solution to (91). For each $\psi \in L^q(\Omega_C)$, consider the solution $\hat{\varphi}$ of (92). Using it in (91) we find

$$\begin{aligned} |\int_{\Omega_C} U\,\psi| &= |\int_{\Omega_C} U\Big[\psi - \frac{1}{\mathrm{meas}(\Omega_C)}\Big(\int_{\Omega_C}\psi\Big)\Big]| \\ &= |\int_{\Omega_C} U\,\mathrm{div}(\boldsymbol{\sigma}\,\mathbf{grad}\,\hat{\varphi})| = |-\mathbf{p}\cdot\mathbf{grad}\,\hat{\varphi}(\mathbf{x}_0)| \\ &\leq |\mathbf{p}|\,\|\mathbf{grad}\,\hat{\varphi}\|_{C^0(\overline{B_{r_*}(\mathbf{x}_0)})} \leq c_0\,|\mathbf{p}|\|\psi\|_{L^q(\Omega_C)}\ , \end{aligned}$$

hence $\|U\|_{L^p(\Omega_C)} \leq c_0\,|\mathbf{p}|$, and uniqueness follows. □

- Since for $3 < s < q$ one has $X_s \supset X_q$ and $L^r(\Omega_C) \subset L^p(\Omega_C)$ (here $\frac{1}{r} + \frac{1}{s} = 1$), from the uniqueness result it follows readily that the solution U to (91) is the same for all finite values $s, q > 3$. Therefore we have solved the problem

$$
\begin{aligned}
&\text{find } U \in \textstyle\bigcap_{q>3} L^p(\Omega_C) \text{ with } \int_{\Omega_C} U = 0 \, : \\
&\quad \int_{\Omega_C} U \, \mathrm{div}(\boldsymbol{\sigma} \, \mathbf{grad} \, \varphi) = - \, \mathbf{p} \cdot \mathbf{grad} \, \varphi(\mathbf{x}_0) \\
&\forall \, \varphi \in \bigcup_{q>3} X_q \, .
\end{aligned}
\tag{93}
$$

Let us come now to the eddy current system with a dipole source. We write it in terms of the electric field $\mathbf{E}$ only:

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}) + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{p}\delta_{\mathbf{x}_0} & \text{in } \Omega \\ \mathrm{div}(\boldsymbol{\varepsilon}_I\mathbf{E}_I) = 0 & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \boldsymbol{\varepsilon}_I\mathbf{E}_I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{94}$$

The existence proof here below is presented in Alonso Rodríguez, Camaño and V. (2012).

- We assume that the magnetic permeability $\boldsymbol{\mu}$ and the conductivity $\boldsymbol{\sigma}$ satisfy the isotropic homogeneity condition: there exist $r_0 > 0$, $\mu_0 > 0$ and $\sigma_0 > 0$ such that

$$\boldsymbol{\mu}(\mathbf{x}) = \mu_0 I \text{ and } \boldsymbol{\sigma}(\mathbf{x}) = \sigma_0 I \quad \text{for each } \mathbf{x} \in B_{r_0}(\mathbf{x}_0)\,, \tag{95}$$

where I is the identity matrix.

Some preliminary results are needed. Set $\kappa^2 = -i\omega\mu_0\sigma_0$ and $\mathbf{q} = -i\omega\mu_0\mathbf{p}$.

Theorem (5)

The fundamental solution $\mathbf{K}$ *of the operator* **curl curl** $-\kappa^2 I$, *that is, the solution to*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{K} - \kappa^2\mathbf{K} = \mathbf{q}\delta_{\mathbf{0}}\,,$$

is given by

$$\mathbf{K}(\mathbf{x}) = \mathbf{q}\frac{e^{i\kappa|\mathbf{x}|}}{4\pi|\mathbf{x}|} + \frac{1}{\kappa^2}(\mathbf{q}\cdot\mathbf{grad}\,)\mathbf{grad}\,\frac{e^{i\kappa|\mathbf{x}|}}{4\pi|\mathbf{x}|}\,. \quad (96)$$

Proof. We start from the fundamental solution Φ of the Helmholtz operator

$$-\Delta\Phi - \kappa^2\Phi = \delta_{\mathbf{0}}\,,$$

which, as it is well-known, is given by

$$\Phi(\mathbf{x}) = \frac{e^{i\kappa|\mathbf{x}|}}{4\pi|\mathbf{x}|}\,.$$

From this we get at once

$$-\Delta(\mathbf{q}\Phi) - \kappa^2(\mathbf{q}\Phi) = \mathbf{q}\delta_{\mathbf{0}}\,.$$

Then we look for $\mathbf{K}$ in the form

$$\mathbf{K} = \mathbf{q}\Phi + \mathbf{Y}\,.$$

We have

$$
\begin{aligned}
&\mathbf{curl}\,\mathbf{curl}\,\mathbf{K} - \kappa^2\mathbf{K} \\
&\qquad = -\Delta(\mathbf{q}\Phi) + \mathbf{grad}\,\mathrm{div}(\mathbf{q}\Phi) - \kappa^2(\mathbf{q}\Phi) + \mathbf{curl}\,\mathbf{curl}\,\mathbf{Y} - \kappa^2\mathbf{Y} \\
&\qquad = \mathbf{q}\delta_{\mathbf{0}} + \mathbf{grad}\,\mathrm{div}(\mathbf{q}\Phi) + \mathbf{curl}\,\mathbf{curl}\,\mathbf{Y} - \kappa^2\mathbf{Y}\,.
\end{aligned}
$$

Hence $\mathbf{Y}$ has to satisfy

$$
\mathbf{curl}\,\mathbf{curl}\,\mathbf{Y} - \kappa^2\mathbf{Y} = -\mathbf{grad}\,\mathrm{div}(\mathbf{q}\Phi)\,,
$$

and we easily find

$$
\mathbf{Y} = \frac{1}{\kappa^2}\mathbf{grad}\,\mathrm{div}(\mathbf{q}\Phi)\,.
$$

In conclusion, we have obtained

$$\begin{aligned}\mathbf{K}(\mathbf{x}) &= \mathbf{q}\Phi(\mathbf{x}) + \tfrac{1}{\kappa^2}\,\mathbf{grad}\,\mathrm{div}(\mathbf{q}\Phi(\mathbf{x})) \\ &= \mathbf{q}\frac{e^{i\kappa|\mathbf{x}|}}{4\pi|\mathbf{x}|} + \frac{1}{\kappa^2}(\mathbf{q}\cdot\mathbf{grad}\,)\mathbf{grad}\,\frac{e^{i\kappa|\mathbf{x}|}}{4\pi|\mathbf{x}|}\,,\end{aligned}$$

namely, the representation formula (96). □

Note that the fundamental solution $\mathbf{K}$ is much more singular than the fundamental solution of the Laplace or the Helmholtz operator: while the first term belongs to L^2_{loc}, the second one has a singularity like $|\mathbf{x}|^{-3}$.

It can be also remarked that, setting $\hat{\mathbf{K}}(\mathbf{x}) := \mathbf{K}(\mathbf{x}-\mathbf{x}_0)$ we have $\hat{\mathbf{K}} \in H^{-2}(\Omega)$, the dual space of $H^2_0(\Omega)$; however, $\hat{\mathbf{K}}$ is a regular function far from $\mathbf{x} = \mathbf{x}_0$, in particular it is regular in $\overline{\Omega_I}$.

Theorem (6)

Assuming that condition (95) is satisfied, there exists a solution $\mathbf{E} \in H^{-2}(\Omega)$ *to (94), satisfying* $(\mathbf{E} - \hat{\mathbf{K}}) \in H(\mathbf{curl}\,;\Omega)$. *It is unique among all the solutions* $\mathbf{E}^*$ *such that* $(\mathbf{E}^* - \hat{\mathbf{K}}) \in H(\mathbf{curl}\,;\Omega)$.

Proof. We split the solution to (94) as $\mathbf{E}(\mathbf{x}) = \hat{\mathbf{K}}(\mathbf{x}) + \mathbf{Q}(\mathbf{x})$. It is easily seen that we have to look for the solution $\mathbf{Q} \in H(\mathbf{curl}\,;\Omega)$ to

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{Q}) + i\omega\boldsymbol{\sigma}\mathbf{Q} = \mathbf{J} & \text{in } \Omega \\ \mathrm{div}(\boldsymbol{\varepsilon}_I\mathbf{Q}_I) = -\mathrm{div}(\boldsymbol{\varepsilon}_I\hat{\mathbf{K}}_I) & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{Q}) \times \mathbf{n} = -(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\hat{\mathbf{K}}) \times \mathbf{n} & \text{on } \partial\Omega \\ \boldsymbol{\varepsilon}_I\mathbf{Q}_I \cdot \mathbf{n} = -\boldsymbol{\varepsilon}_I\hat{\mathbf{K}}_I \cdot \mathbf{n} & \text{on } \partial\Omega\,, \end{cases} \tag{97}$$

where

$$\mathbf{J}(\mathbf{x}) := \begin{cases} \mathbf{0} & \text{if } \mathbf{x} \in B_{r_0}(\mathbf{x}_0) \\ -\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\hat{\mathbf{K}})(\mathbf{x}) - i\omega\boldsymbol{\sigma}\hat{\mathbf{K}}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega \setminus \overline{B_{r_0}(\mathbf{x}_0)}\,. \end{cases}$$

We introduce now the solution $\eta_I \in H^1(\Omega_I)$ of the mixed problem

$$\begin{cases} \mathrm{div}(\varepsilon_I \mathbf{grad}\, \eta_I) = -\mathrm{div}(\varepsilon_I \hat{\mathbf{K}}_I) & \text{in } \Omega_I \\ \varepsilon_I \mathbf{grad}\, \eta_I \cdot \mathbf{n} = -\varepsilon_I \hat{\mathbf{K}}_I \cdot \mathbf{n} & \text{on } \partial\Omega \\ \eta_I = 0 & \text{on } \Gamma\,, \end{cases}$$

which exists and is unique as $\hat{\mathbf{K}}_I$ is smooth in $\overline{\Omega_I}$. The extension of η_I obtained by setting the value 0 in Ω_C will be called η; clearly, one has $\eta \in H^1(\Omega)$.
The solution $\mathbf{Q}$ to (97) is found in the form $\mathbf{Q} = \mathbf{Q}^* + \mathbf{grad}\, \eta$, where $\mathbf{Q}^* \in H(\mathbf{curl}\,; \Omega)$ is the solution to

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{Q}^*) + i\omega\boldsymbol{\sigma} \mathbf{Q}^* = \mathbf{J} & \text{in } \Omega \\ \mathrm{div}(\varepsilon_I \mathbf{Q}_I^*) = 0 & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{Q}^*) \times \mathbf{n} = -(\boldsymbol{\mu}^{-1} \mathbf{curl}\, \hat{\mathbf{K}}) \times \mathbf{n} & \text{on } \partial\Omega \\ \varepsilon_I \mathbf{Q}_I^* \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

The existence and uniqueness of such a solution follows from the fact that the compatibility conditions

$$\mathrm{div}\,\mathbf{J}_I = -\mathrm{div}[\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\hat{\mathbf{K}}_I)] = 0 \text{ in } \Omega_I$$
$$\mathbf{J}_I \cdot \mathbf{n} = -\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\hat{\mathbf{K}}) \cdot \mathbf{n} = -\mathrm{div}_\tau(\mu^{-1}\mathbf{curl}\,\hat{\mathbf{K}} \times \mathbf{n}) \text{ on } \partial\Omega$$

are satisfied.
We have thus found a solution $\mathbf{E} = \hat{\mathbf{K}} + \mathbf{grad}\,\eta + \mathbf{Q}^*$ of (94).

For showing its uniqueness, suppose that we have another solution $\mathbf{E}^*$ such that $(\mathbf{E}^* - \hat{\mathbf{K}}) \in H(\mathbf{curl}\,;\Omega)$. We can write it as $\mathbf{E}^* = \hat{\mathbf{K}} + (\mathbf{E}^* - \hat{\mathbf{K}})$, and it is readily verified that $\mathbf{E}^* - \hat{\mathbf{K}}$ is a solution to (97), a problem for which uniqueness holds in $H(\mathbf{curl}\,;\Omega)$. Therefore $\mathbf{E}^* - \hat{\mathbf{K}} = \mathbf{Q} = \mathbf{E} - \hat{\mathbf{K}}$, and uniqueness is proved. □

- The existence and uniqueness of a solution to the potential equation (83) could also be proved by an approach similar to the one just presented for the eddy current problem: first construct a fundamental solution $K^\sharp$, namely, a function satisfying

$$\mathrm{div}(\boldsymbol{\sigma}_0 \mathbf{grad}\, K^\sharp) = \mathrm{div}(\mathbf{p}\, \delta_{\mathbf{x}_0}) \,,$$

then look for $U = K^\sharp + \mathcal{U}$, $\mathcal{U}$ being a solution of

$$\begin{cases} \mathrm{div}\,(\boldsymbol{\sigma} \mathbf{grad}\, \mathcal{U}) = g & \text{in } \Omega_C \\ \boldsymbol{\sigma} \mathbf{grad}\, \mathcal{U} \cdot \mathbf{n} = -\boldsymbol{\sigma} \mathbf{grad}\, K^\sharp \cdot \mathbf{n} & \text{on } \partial\Omega_C \,, \end{cases} \tag{98}$$

where

$$g(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \in B_{r_0}(\mathbf{x}_0) \\ -\mathrm{div}(\boldsymbol{\sigma} \mathbf{grad}\, K^\sharp)(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_C \setminus \overline{B_{r_0}(\mathbf{x}_0)} \,. \end{cases}$$

This is called the subtraction approach, proposed and analyzed in Wolters, Köstler, Möller, Härdtlein, Grasedyck and Hackbusch (2007).
It needs the non-isotropic homogeneity condition: there exist $r_0 > 0$ and a constant symmetric and positive definite matrix $\boldsymbol{\sigma}_0$ such that

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}_0 \quad \text{for each } \mathbf{x} \in B_{r_0}(\mathbf{x}_0)\,. \tag{99}$$

Note that this condition is stronger than the local Lipschitz regularity condition (90).

The fundamental solution is given by

$$K^\sharp(\mathbf{x}) = \frac{1}{4\pi\sqrt{\det \boldsymbol{\sigma}_0}} \frac{\mathbf{p} \cdot \boldsymbol{\sigma}_0^{-1}(\mathbf{x} - \mathbf{x}_0)}{[\boldsymbol{\sigma}_0^{-1}(\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)]^{3/2}}$$

(see Sauter and Schwab (2011)).

In principle, the numerical approximation of the potential equation can be based on both formulations (91) or (98). Since the solution of the latter is more regular, it should be easier to find an accurate approximate solution.

However, this formulation requires the non-isotropic homogeneity condition (99), which is stronger than the local Lipschitz regularity condition (90), and this has an influence on the efficiency of numerical computations.

In particular, when considering a head model in which the conductivity is jumping (and this is indeed the realistic case, as the conductivity is quite different in the skull or in the brain: in the skull it is from ten to one hundred times smaller), the subtraction method has shown some instabilities when the position $\mathbf{x}_0$ of the dipole is quite close to the discontinuity surface.

Therefore, a direct finite element approach could be suitable.

Following Alonso Rodríguez, Camaño, Rodríguez and V. (2013), we analyze the simplest finite element approximation of (91), namely:

$$\begin{cases} \text{find } U_h \in L_h^1 \text{ with } \int_{\Omega_C} U_h = 0 \ : \\ \quad \int_{\Omega_C} \boldsymbol{\sigma} \, \mathbf{grad}\, U_h \cdot \mathbf{grad}\, \varphi_h = \mathbf{p} \cdot \mathbf{grad}\, (\varphi_h|_{T_0})(\mathbf{x}_0) \\ \forall \, \varphi_h \in L_h^1 \, , \end{cases} \tag{100}$$

where $\mathbf{x}_0 \in T_0$ (if $\mathbf{x}_0$ belongs to many elements, just choose one of them) and

$$L_h^1 := \{\varphi_h \in C^0(\Omega_C) \, | \, \varphi_{h|K} \in \mathbb{P}_1 \ \forall \ K\} \, .$$

To find an a priori error estimate in $L^p(\Omega_C)$, with $1 < p < 3/2$, a duality argument is used, and the solution $\hat{\varphi} \in H^1(\Omega_C)$ of problem (92) comes into play.

For utilizing the duality argument, we need that $\hat{\varphi} \in W^{2,q}(\Omega_C)$ for a suitable $q > 3$ ($q > 2$ in the two-dimensional case). This is true under some assumptions.

- In the two-dimensional case, we require that $\boldsymbol{\sigma} \in C^1(\overline{\Omega_C})$ and that Ω_C is convex; then the regularization result is true for all q such that $2 < q < q_0$, for a suitable $q_0 > 2$ (for the Laplace operator, $q_0 = \frac{2}{2-\pi/\theta}$, $\theta > \frac{\pi}{2}$ being the largest inner angle of Ω_C).
- In the three-dimensional case, we require that $\boldsymbol{\sigma} = \sigma_0 I$ ($\sigma_0 > 0$ a constant and I the identity matrix) and that Ω_C is a cubic domain (namely, a parallepiped with right angles); then the regularization result is true for all $q > 3$.

Finally we have:

Theorem (7)

Let $\mathcal{T}_h$ be a quasi-uniform family of triangulations of Ω_C, and let $\boldsymbol{\sigma}$ and Ω_C satisfy the assumptions stated above. Let U and U_h be the solutions to problems (91) *and* (100)*. Then there exists $h_0 > 0$ such that*

$$\|U - U_h\|_{0,p,\Omega_C} \leq \begin{cases} Ch^{2/p-1} & \text{when } d = 2 \\ Ch^{3/p-2} & \text{when } d = 3 \end{cases}$$

for all $0 < h < h_0$, with p such that $\frac{1}{p} + \frac{1}{q} = 1$, and q the exponent such that the solution $\hat{\varphi}$ of problem (92) belongs to $W^{2,q}(\Omega_C)$ (hence, $2 < q < q_0$ in the two-dimensional case, $q > 3$ in the three-dimensional case).

The variational structure of problems (91) and (100) permits to perform an a posteriori error analysis, and therefore to devise an adaptive mesh procedure.

From now on let us restrict ourselves to the three-dimensional case. Let $\mathcal{F}_{h,i}$ be the set of all the inner faces and $\mathcal{F}_{h,e}$ that of external faces of the mesh $\mathcal{T}_h$. Let $\mathcal{F}_h := \mathcal{F}_{h,i} \cup \mathcal{F}_{h,e}$. For all $T \in \mathcal{T}_h$ we define

$$\widehat{\varrho}_{T,p} := \Big(\tfrac{1}{2} \textstyle\sum_{F \in \mathcal{F}(T) \cap \mathcal{F}_{h,i}} (\mathrm{meas}(F))^{(p+3)/2} |[\![\, \mathbf{grad}\, U_h \cdot \mathbf{n}_F \,]\!]|^p \\ + \textstyle\sum_{F \in \mathcal{F}(T) \cap \mathcal{F}_{h,e}} (\mathrm{meas}(F))^{(p+3)/2} |\mathbf{grad}\, U_h \cdot \mathbf{n}_F|^p \Big)^{1/p} \, ,$$

where $\mathcal{F}(T)$ is the set of faces of T and $[\![\cdot]\!]$ denotes the jump across the face F.

We define the local a posteriori error indicator $\widehat{\eta}_{T,p}$ for all $T \in \mathcal{T}_h$ by

$$\widehat{\eta}_{T,p} := \begin{cases} \left(h_{T_0}^{3-2p} + \widehat{\varrho}_{T_0,p}^p\right)^{1/p} & \text{if } T = T_0\,, \\ \widehat{\varrho}_{T,p} & \text{otherwise}\,, \end{cases}$$

and the global error estimator from these indicators as follows:

$$\widehat{\eta}_p := \left(\sum_{T \in \mathcal{T}_h} \widehat{\eta}_{T,p}^p \right)^{1/p} .$$

Set now $\omega_T := \{T' \in \mathcal{T}_h \mid T' \cap T \neq \emptyset\}$. We have:

Theorem (8)

Let $\mathcal{T}_h$ be a regular family of triangulations of Ω_C, and assume that $\boldsymbol{\sigma} = \sigma_0 I$ ($\sigma_0 > 0$ a constant and I the identity matrix) and that Ω_C is a cubic domain. Let U and U_h be the solutions of (91) *and* (100)*, respectively. Then the following estimates hold true:*

$$\|U - U_h\|_{0,p,\Omega} \leq C\widehat{\eta}_p$$

and

$$\widehat{\eta}_{T,p} \leq C\|U - U_h\|_{0,p,\omega_T}$$

for all $T \in \mathcal{T}_h$.

Referring again to Alonso Rodríguez, Camaño and V. (2012), we want to analyze now some inverse source problems for the eddy current equations. Let us rewrite them, in terms of the electric $\mathbf{E}$ field only:

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}) + i\omega\boldsymbol{\sigma}\mathbf{E} = -i\omega\mathbf{J}_e & \text{in } \Omega \\ \operatorname{div}(\varepsilon\mathbf{E}) = 0 & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \varepsilon\mathbf{E}\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{101}$$

We already know that, under suitable assumptions, there exists a unique solution $\mathbf{E}$ (and moreover $\mathbf{H} = -(i\omega)^{-1}\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}$ in Ω).

Integration by parts in Ω_C easily yields

$$-i\omega \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{z}} = \int_{\Omega_C} \mathbf{E} \cdot [i\omega\boldsymbol{\sigma}\overline{\mathbf{z}} + \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\overline{\mathbf{z}})] + \int_\Gamma [\mathbf{n}_C \times \mathbf{E} \cdot (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) - i\omega \mathbf{n}_C \times \mathbf{H} \cdot \overline{\mathbf{z}}]\,.$$

Therefore, if $\mathbf{z} \in H(\mathbf{curl}\,; \Omega_C)$ satisfies

$$\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\boldsymbol{\sigma}\mathbf{z} = \mathbf{0} \quad \text{in } \Omega_C\,,$$

the current density $\mathbf{J}_e$ satisfies the representation formula

$$-i\omega \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{z}} = \int_\Gamma \mathbf{n}_C \times \mathbf{E} \cdot (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) - i\omega \int_\Gamma \mathbf{n}_C \times \mathbf{H} \cdot \overline{\mathbf{z}}\,. \qquad (102)$$

Let us define

$$\mathcal{W} = \{\mathbf{z} \in H(\mathbf{curl}\,; \Omega_C) \,|\, \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\boldsymbol{\sigma}\mathbf{z} = \mathbf{0} \quad \text{in } \Omega_C\}$$

and W the closure of $\mathcal{W}$ in $(L^2(\Omega_C))^3$. (Note that $\mathcal{W}$ is not a trivial subspace.)
We have the orthogonal splitting

$$(L^2(\Omega_C))^3 = W \oplus W^\perp \,.$$

Let us give a more explicit description of the elements of $W^\perp$.

Lemma (9)

Consider $\boldsymbol{\eta} \in C_0^\infty(\Omega_C)$ and set $\phi = \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\eta) + i\omega\boldsymbol{\sigma}\boldsymbol{\eta}$. Then $\phi \in W^\perp$ (and $W^\perp$ is not a trivial subspace).

Proof. Take $\mathbf{z} \in \mathcal{W}$. Then

$$\begin{aligned}\int_{\Omega_C} \phi \cdot \overline{\mathbf{z}} &= \int_{\Omega_C} [\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\eta) + i\omega\boldsymbol{\sigma}\eta] \cdot \overline{\mathbf{z}} \\ &= \int_{\Omega_C} \boldsymbol{\eta} \cdot [\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) + i\omega\boldsymbol{\sigma}\overline{\mathbf{z}}] = 0\,.\end{aligned}$$

The result follows by a density argument. □

Let us split the current density $\mathbf{J}_e$ as

$$\mathbf{J}_e = \mathbf{J}_e^\sharp + \mathbf{J}_e^\perp \quad , \quad \mathbf{J}_e^\sharp \in W \ , \ \mathbf{J}_e^\perp \in W^\perp \, .$$

We have

Theorem (10)

(i) Let us assume that $\mathbf{J}_e = \mathbf{J}_e^\sharp \in W$ *and that* $\mathbf{E}^\sharp$ *is the corresponding solution of the eddy current problem. Then the knowledge of* $\mathbf{E}^\sharp \times \mathbf{n}_C$ *on* Γ *uniquely determines* $\mathbf{J}_e^\sharp$*.*
(ii) Let us assume that $\mathbf{J}_e = \mathbf{J}_e^\perp \in W^\perp$ *and that* $\mathbf{E}^\perp$ *is the corresponding solution of the eddy current problem. Then* $\mathbf{E}^\perp \times \mathbf{n}_C = \mathbf{0}$ *and* $\mathbf{H}^\perp \times \mathbf{n}_C = \mathbf{0}$ *on* Γ*, namely,* $\mathbf{J}_e^\perp$ *is a on-radiating source.*

Proof. (i) The electric field in the insulator satisfy

$$\begin{array}{ll} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}^\sharp) = \mathbf{0} & \text{in } \Omega_I \\ \text{div}(\varepsilon\mathbf{E}^\sharp) = 0 & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}^\sharp) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \varepsilon\mathbf{E}^\sharp \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{array}$$

If $\mathbf{E}^\sharp \times \mathbf{n}_\Gamma = \mathbf{0}$ on Γ, multiplying the first equation by $\mathbf{E}^\sharp$ and integrating by parts one easily finds $\mathbf{curl}\,\mathbf{E}^\sharp = \mathbf{0}$, then $\mathbf{E}^\sharp = \mathbf{0}$ in Ω_I. Consequently, $\mathbf{H}^\sharp = -(i\omega\boldsymbol{\mu})^{-1}\mathbf{curl}\,\mathbf{E}^\sharp = \mathbf{0}$ in Ω_I and in particular $\mathbf{H}^\sharp \times \mathbf{n}_\Gamma = \mathbf{0}$ on Γ.
Therefore from (102) we know that $\int_{\Omega_C} \mathbf{J}_e^\sharp \cdot \overline{\mathbf{z}} = 0$ for each $\mathbf{z} \in \mathcal{W}$, hence, by a density argument, for each $\mathbf{z} \in W$. Taking $\mathbf{z} = \mathbf{J}_e^\sharp \in W$, the thesis follows.

(ii) Since $\mathbf{J}_e^\perp \in W^\perp$, taking $\mathbf{z} \in W$ from (102) we have that

$$\int_\Gamma \mathbf{n}_C \times \mathbf{E}^\perp \cdot (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) - i\omega \int_\Gamma \mathbf{n}_C \times \mathbf{H}^\perp \cdot \overline{\mathbf{z}} = 0\,. \qquad (103)$$

For each $\boldsymbol{\eta} \in H^{-1/2}_{\mathrm{div},\tau}(\Gamma)$ we denote by $\mathbf{Z} \in H(\mathbf{curl}\,;\Omega)$ the solution to

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{Z}) - i\omega\boldsymbol{\sigma}\mathbf{Z} = \mathbf{0} & \text{in } \Omega_C \cup \Omega_I \\ \mathrm{div}(\boldsymbol{\varepsilon}\mathbf{Z}) = 0 & \text{in } \Omega_I \\ (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{Z}) \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \boldsymbol{\varepsilon}\mathbf{Z} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{Z})_{|\Omega_C} \times \mathbf{n}_C = (\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{Z})_{|\Omega_I} \times \mathbf{n}_C + \boldsymbol{\eta} & \text{on } \Gamma\,. \end{cases}$$

We can select $\mathbf{Z}_{|\Omega_C} \in W$ as a test function in (103) and obtain

$$
\begin{aligned}
&\int_\Gamma \mathbf{n}_C \times \mathbf{E}^\perp \cdot \boldsymbol{\mu}^{-1} \mathbf{curl}\, \overline{\mathbf{Z}_{|\Omega_C}} \\
&\quad = \int_\Gamma \mathbf{E}^\perp \cdot \overline{\boldsymbol{\eta}} - \int_\Gamma \mathbf{E}^\perp \cdot (\mathbf{n}_C \times \boldsymbol{\mu}^{-1} \mathbf{curl}\, \overline{\mathbf{Z}_{|\Omega_I}}) \\
&\quad = \int_\Gamma \mathbf{E}^\perp \cdot \overline{\boldsymbol{\eta}} - \int_{\Omega_I} \boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E}^\perp \cdot \mathbf{curl}\, \overline{\mathbf{Z}_{|\Omega_I}}
\end{aligned}
$$

$$
\begin{aligned}
&-i\omega \int_\Gamma \mathbf{n}_C \times \mathbf{H}^\perp \cdot \overline{\mathbf{Z}_{|\Omega_C}} \\
&\quad = -\int_\Gamma \boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E}^\perp \cdot \mathbf{n}_C \times \overline{\mathbf{Z}_{|\Omega_I}} \\
&\quad = \int_{\Omega_I} \boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E}^\perp \cdot \mathbf{curl}\, \overline{\mathbf{Z}_{|\Omega_I}} .
\end{aligned}
$$

In conclusion, we have obtained

$$
\int_\Gamma \mathbf{E}^\perp \cdot \overline{\boldsymbol{\eta}} = 0
$$

for each $\boldsymbol{\eta} \in H^{-1/2}_{\mathrm{div},\tau}(\Gamma)$, hence $\mathbf{n}_C \times \mathbf{E}^\perp \times \mathbf{n}_C = \mathbf{0}$ on Γ.
Proceeding as in the proof of (i) we show that $\mathbf{E}^\perp \times \mathbf{n}_C = \mathbf{0}$ on Γ implies $\mathbf{H}^\perp \times \mathbf{n}_C = \mathbf{0}$ on Γ, and the proof is complete. □

We consider a surface current $\mathbf{J}_* \in H^{-1/2}_{\mathrm{div},\tau}(\partial B)$, where B is a (known) open connected set with Lipschitz boundary ∂B and satisfying $\overline{B} \subset \Omega_C$.
The direct problem reads

$$\begin{cases} \mathbf{curl}\,\mathbf{E} + i\omega\boldsymbol{\mu}\mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{curl}\,\mathbf{H} = \boldsymbol{\sigma}\mathbf{E} & \text{in } B \cup (\Omega \setminus \overline{B}) \\ \mathrm{div}(\boldsymbol{\varepsilon}\mathbf{E}) = 0 & \text{in } \Omega_I \\ \mathbf{H} \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\ \boldsymbol{\varepsilon}\mathbf{E} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \mathbf{H}_{|B} \times \mathbf{n}_B - \mathbf{H}_{|\Omega \setminus \overline{B}} \times \mathbf{n}_B = \mathbf{J}_* & \text{on } \partial B\,, \end{cases} \tag{104}$$

where $\mathbf{n}_B$ is the unit normal vector on ∂B, pointing outward B.
It is easy to see that, for each given $\mathbf{J}_* \in H^{-1/2}_{\mathrm{div},\tau}(\partial B)$, this problem has unique solution.

Theorem (11)

Assume that the coefficients $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are piecewise C^1-functions, and that the discontinuity surfaces are Lipschitz surfaces. Let $\mathbf{E}_*$ be the solution of the eddy current problem driven by the surface current $\mathbf{J}_* \in H^{-1/2}_{\mathrm{div},\tau}(\partial B)$. The knowledge of $\mathbf{E}_* \times \mathbf{n}_C$ on Γ uniquely determines $\mathbf{J}_*$.

Proof. As in the preceding case, by solving the problem in Ω_I we easily show that $\mathbf{E}_* \times \mathbf{n}_C = \mathbf{0}$ on Γ also gives $\mathbf{E}_* = \mathbf{0}$ in Ω_I, $\mathbf{H}_* = \mathbf{0}$ in Ω_I and in particular $\mathbf{H}_* \times \mathbf{n}_C = \mathbf{0}$ on Γ.
As a consequence of the unique continuation principle we have $\mathbf{E}_* = \mathbf{0}$ and $\mathbf{H}_* = \mathbf{0}$ in $\Omega \setminus \overline{B}$ (the assumptions on the coefficients μ and σ play a role here).

For each $\mathbf{z} \in H(\mathbf{curl}\,; B)$ with $\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{z}) \in (L^2(B))^3$ we have

$$\begin{aligned}\int_B \sigma \mathbf{E}_* \cdot \overline{\mathbf{z}} &= \int_B \mathbf{curl}\,\mathbf{H}_* \cdot \overline{\mathbf{z}} \\ &= \int_{\partial B} \mathbf{n}_B \times \mathbf{H}_{*|B} \cdot \overline{\mathbf{z}} \\ &\qquad -(i\omega)^{-1} \int_{\partial B} \mathbf{n}_B \times \mathbf{E}_* \cdot (\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) \\ &\qquad -(i\omega)^{-1} \int_B \mathbf{E}_* \cdot \mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}})\,.\end{aligned}$$

Taking into account that $\mathbf{H}_{*|B} \times \mathbf{n}_B - \mathbf{H}_{*|\Omega\setminus\overline{B}} \times \mathbf{n}_B = \mathbf{J}_*$ on ∂B, we obtain the representation formula

$$\begin{aligned}-i\omega \int_{\partial B} \mathbf{J}_* \cdot \overline{\mathbf{z}} &= \int_{\partial B} \mathbf{n}_B \times \mathbf{E}_* \cdot (\mu^{-1}\mathbf{curl}\,\overline{\mathbf{z}}) \\ &\qquad -i\omega \int_{\partial B} \mathbf{n}_B \times \mathbf{H}_{*|\Omega\setminus\overline{B}} \cdot \overline{\mathbf{z}}\end{aligned} \qquad (105)$$

for each $\mathbf{z} \in H(\mathbf{curl}\,; B)$ such that $\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\sigma\mathbf{z} = \mathbf{0}$ in B. Since we know that $\mathbf{E}_* = \mathbf{0}$ and $\mathbf{H}_* = \mathbf{0}$ in $\Omega \setminus \overline{B}$, it follows from (105) that $\int_{\partial B} \mathbf{J}_* \cdot \overline{\mathbf{z}} = 0$.

For each $\boldsymbol{\rho} \in H^{-1/2}_{\mathrm{curl},\tau}(\Gamma)$ we can choose $\mathbf{z} \in H(\mathbf{curl}\,; B)$, the solution to

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\boldsymbol{\sigma}\mathbf{z} = \mathbf{0} & \text{in } B \\ \mathbf{z} \times \mathbf{n}_B = \boldsymbol{\rho} \times \mathbf{n}_B & \text{on } \partial B\,. \end{cases}$$

Hence $\int_{\partial B} \mathbf{J}_* \cdot \overline{\boldsymbol{\rho}} = 0$ for each $\boldsymbol{\rho} \in H^{-1/2}_{\mathrm{curl},\tau}(\Gamma)$, and this space is the dual space of $H^{-1/2}_{\mathrm{div},\tau}(\Gamma)$. This ends the proof. □

Suppose that the source is a finite sum of dipoles, in different positions and with non-vanishing polarizations, namely,

$$\mathbf{J}_{\dagger}(\mathbf{x}) = \sum_{k=1}^{M} \mathbf{p}_k \delta(\mathbf{x} - \mathbf{x}_k)\,, \tag{106}$$

where $\mathbf{x}_k \in \Omega_C$, $\mathbf{x}_k \neq \mathbf{x}_j$ for $k \neq j$, $\mathbf{p}_k \neq \mathbf{0}$, and δ is the Dirac delta distribution.

We have already seen that, under suitable assumptions, there exists a unique solution of the eddy current equations with this type of current density.

Theorem (12)

Assume that $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are smooth enough. Let $\mathbf{E}_\dagger$ be the solution of the eddy current problem (101) driven by the surface current $\mathbf{J}_\dagger$ introduced in (106). The knowledge of $\mathbf{E}_\dagger \times \mathbf{n}_C$ on Γ uniquely determines $\mathbf{J}_\dagger$, namely, the number, the position and the polarization of the dipoles.

Proof. We start proving that the number and the position of the dipoles are uniquely determined.

By contradiction, let us denote by Q_1 and Q_2 two different sets of points where the dipoles are located, and by $\mathbf{E}_{\dagger,1}$, $\mathbf{H}_{\dagger,1}$ and $\mathbf{E}_{\dagger,2}$, $\mathbf{H}_{\dagger,2}$ the corresponding solutions, with the same value $\mathbf{E}_\dagger \times \mathbf{n}_C$ on Γ. As in the preceding cases, by solving the problem in Ω_I with datum $\mathbf{E}_\dagger \times \mathbf{n}_C$ on Γ we obtain that $\mathbf{E}_{\dagger,1} = \mathbf{E}_{\dagger,2}$ and $\mathbf{H}_{\dagger,1} = \mathbf{H}_{\dagger,2}$ in Ω_I.

By the unique continuation principle it follows $\mathbf{E}_{\dagger,1} = \mathbf{E}_{\dagger,2}$ in $\Omega \setminus (Q_1 \cup Q_2)$ [the smoothness of the coefficients $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ plays a role here]. Let $\mathbf{x}_*$ a point belonging, say, to Q_1 but not to Q_2. We have that $\mathbf{E}_{\dagger,2}$ is bounded in a neighborhood of $\mathbf{x}_*$, while $\mathbf{E}_{\dagger,1}$ is unbounded there, a contradiction since $\mathbf{E}_{\dagger,1}$ and $\mathbf{E}_{\dagger,2}$ coincide around $\mathbf{x}_*$. Therefore $Q_1 = Q_2$.

Let us prove now that the polarizations are uniquely determined. Since the problem is linear, we can assume that $\mathbf{E}_\dagger = \mathbf{0}$ in $\Omega \setminus Q_1$. Therefore, in the sense of distributions in Ω we have $\mathbf{E}_\dagger = \mathbf{0}$ and $\mathbf{curl}\, \mathbf{H}_\dagger = \mathbf{0}$, and in particular the equation

$$\sum_{k=1}^{M} \mathbf{p}_k \delta(\mathbf{x} - \mathbf{x}_k) = \mathbf{0} \, .$$

By choosing test functions in $C^\infty(\Omega)$ supported around each point $\mathbf{x}_j$ we obtain $\mathbf{p}_j = \mathbf{0}$ for each $j = 1, \ldots, M$. $\square$

For the sake of simplicity, consider a source given by only one dipole.
Assume that μ and σ are constants. Proceeding as in the proof of (102), one obtains the representation formula

$$\begin{aligned} &-i\omega \mathbf{p}_1 \cdot \overline{\mathbf{z}}(\mathbf{x}_1) \\ &\qquad = \int_\Gamma \mathbf{n}_C \times \mathbf{E}_\dagger \cdot (\mu^{-1}\mathbf{curl}\, \overline{\mathbf{z}}) - i\omega \int_\Gamma \mathbf{n}_C \times \mathbf{H}_\dagger \cdot \overline{\mathbf{z}} \,, \end{aligned} \tag{107}$$

for each $\mathbf{z} \in H(\mathbf{curl}\,; \Omega_C)$ satisfying

$$\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\mathbf{z}) - i\omega\sigma\mathbf{z} = \mathbf{0} \quad \text{in } \Omega_C \,. \tag{108}$$

To determine the source, we have to find the polarization $\mathbf{p}_1$ and the position $\mathbf{x}_1$: therefore, six parameters. The natural idea is to choose in a suitable way some (at least six...) functions $\mathbf{z}$ in (107), and solve the corresponding nonlinear system.
The usual choice is to take $\mathbf{z}(\mathbf{x}) = \mathbf{b}e^{i\kappa\mathbf{d}\cdot\mathbf{x}}$, with $\kappa \in \mathbb{C}$, $\mathbf{b} \in \mathbb{R}^3$, $\mathbf{d} \in \mathbb{R}^3$. It is not restrictive to assume $|\mathbf{d}| = |\mathbf{b}| = 1$; in order that $\mathbf{z}$ is a solution to (108) we need

$$\kappa^2 = i\omega\mu\sigma \quad , \quad \mathbf{b} \cdot \mathbf{d} = 0 \,.$$

It can be shown that $\mathbf{p}_1$ and $\mathbf{x}_1$ are uniquely determined by solving the nonlinear system (107) obtained by suitable selections of $\mathbf{b}$ and $\mathbf{d}$.

Before finishing, let us make a few comments on some related results.

- Bleistein and Cohen (1977) have shown the existence of non-radiating sources for the Maxwell equations with constant coefficients.
- He and Romanov (1998) has solved the inverse problem for the (vector) Helmholtz equation with a dipole source.
- Ammari, Bao and Fleming (2002) has solved the inverse problem for the Maxwell equations with a dipole source.
- Albanese and Monk (2006) has solved the inverse problem for the Maxwell equations with a distributed source, a surface current and a superposition of dipole sources.

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Last but not least, for eddy current problems:

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