A parallel iterative approach for the Stokes–Darcy coupling

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Stokes–Darcy coupling

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Let us start describing the coupled problem. We denote by Ω_S the fluid region, and by Ω_D the ground region. Moreover, $\Gamma := \overline{\Omega_S} \cap \overline{\Omega_D}$ will be the interface between Ω_S and Ω_D.

Stokes–Darcy coupling (cont.)

The Stokes system is given by

$$(1) \qquad \begin{cases} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega_S \\ \text{div}\,\mathbf{u} &= 0 \quad \text{in } \Omega_S\,, \end{cases}$$

where $\mathbf{u}$ is the velocity field, p is the pressure, $\nu > 0$ is the kinematic viscosity and $\mathbf{f}$ is a given force field.

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where $\mathbf{u}$ is the velocity field, p is the pressure, $\nu > 0$ is the kinematic viscosity and $\mathbf{f}$ is a given force field.
The Darcy equation is given by

$$(2) \qquad -\,\text{div}\left(\frac{K}{N}\nabla\varphi\right) = 0 \text{ in } \Omega_D\,,$$

where φ is the piezometric head, K is the hydraulic conductivity tensor and $N > 0$ is the volumetric porosity.

Stokes–Darcy coupling (cont.)

For simplicity, as boundary conditions let us assume that:

$$
\begin{aligned}
\mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega_S \setminus \Gamma \\
\varphi &= 0 \quad \text{on } \partial\Omega_D \setminus \Gamma\,.
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The interface conditions [matching of the normal velocity and of the normal stress] are:

$$
\begin{aligned}
\mathbf{u}\cdot\mathbf{n} &= -\tfrac{K}{N}\nabla\varphi\cdot\mathbf{n} \quad \text{on } \Gamma \\
\mathbf{T}(\mathbf{u},p)\cdot\mathbf{n} &= -g\varphi\,\mathbf{n} \quad \text{on } \Gamma
\end{aligned} \tag{4}
$$

where g is the gravity acceleration, and the fluid stress tensor is given by $T_{ij}(\mathbf{u},p) := \nu(D_i u_j + D_j u_i) - p\delta_{ij}$. Here $\mathbf{n}$ denotes the unit normal vector on Γ, pointing from Ω_S into Ω_D.

Well-posedness and iterative solution algorithms

It can be proved that the coupled problem (1)–(4) has a unique solution [Discacciati and Quarteroni, ENUMATH 2001; Layton, Schieweck and Yotov, SINUM 2003].

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Let us present some of the iterative algorithms that have been proposed for the solution of (1)–(4).

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- pose on Γ

$$\varphi^{m+1} := \theta \varphi^{m+1/2} + (1-\theta)\varphi^m ,$$

where $\theta > 0$ is an acceleration parameter.

Dirichlet/Neumann algorithm (cont.)

[The name Dirichlet/Neumann is somehow arbitrary: indeed, we are solving two Neumann problems. However, for the velocity field $(\mathbf{u}_{|\Omega_S}, -\nabla\varphi_{|\Omega_D})$ the step in Ω_S is a Neumann step, whereas the step in Ω_D is a Dirichlet step (for the normal component...).]

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However:

- for a finite element approximation, the convergence is independent of the mesh parameter h, but depends heavily on the viscosity ν and the conductivity K

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$$\begin{aligned} \mathbf{n} \cdot \mathbf{T}(\mathbf{u}^{m+1}, p^{m+1}) \cdot \mathbf{n} + \gamma_S \mathbf{u}^{m+1} \cdot \mathbf{n} \\ = -g\varphi^{m+1} - \gamma_S \tfrac{K}{N} \nabla \varphi^{m+1} \cdot \mathbf{n} \quad \text{on } \Gamma \\ \boldsymbol{\tau} \cdot \mathbf{T}(\mathbf{u}^{m+1}, p^{m+1}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \end{aligned}$$

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$$\eta^{m+1} := -\mathbf{n} \cdot \mathbf{T}(\mathbf{u}^{m+1}, p^{m+1}) \cdot \mathbf{n} + \gamma_D \mathbf{u}^{m+1} \cdot \mathbf{n}\,.$$

Robin/Robin algorithm 1 (cont.)

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Moreover:

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- for a finite element approximation, the convergence is independent of the mesh parameter h, and, for suitable choices of γ_S and γ_D, in the numerical computations it looks also independent of the viscosity ν and the conductivity K

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The leading idea comes from the Neumann/Neumann algorithm (but it cannot be applied straightforwardly: lack of regularity!).

Recalling that the Neumann/Neumann algorithm for the Laplace operator is indeed given by a couple of Dirichlet solvers, followed by a couple of (homogeneous) Neumann solvers, which play the role of correctors, we propose a double parallel Robin/Robin algorithm.

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- solve in parallel the Stokes problem and the Darcy problem with

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\begin{aligned}
& \mathbf{n} \cdot \mathbf{T}(\mathbf{u}^{m+1}, p^{m+1}) \cdot \mathbf{n} \; - \; \gamma_1 \mathbf{u}^{m+1} \cdot \mathbf{n} = \eta^m \\
& \qquad = -g\varphi^{m+1} + \gamma_1 \tfrac{K}{N} \nabla \varphi^{m+1} \cdot \mathbf{n} \;\; \text{on } \Gamma \\
& \boldsymbol{\tau} \cdot \mathbf{T}(\mathbf{u}^{m+1}, p^{m+1}) \cdot \mathbf{n} = 0 \;\; \text{on} \;\; \Gamma
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- pose

$$
\sigma^{m+1} := \mathbf{u}^{m+1} \cdot \mathbf{n} + \frac{K}{N} \nabla \varphi^{m+1} \cdot \mathbf{n} \tag{6}
$$

Robin/Robin algorithm 2 (cont.)

- solve in parallel the homogeneous ($\mathbf{f} = 0$) Stokes problem and the Darcy problem with

$$
\begin{aligned}
\mathbf{n} \cdot \mathbf{T}(\boldsymbol{\omega}^{m+1}, \pi^{m+1}) \cdot \mathbf{n} \; + \; \gamma_2 \boldsymbol{\omega}^{m+1} \cdot \mathbf{n} &= \gamma_2 \sigma^{m+1} \\
= \quad g\chi^{m+1} \; - \; \gamma_2 \tfrac{K}{N} \nabla \chi^{m+1} \cdot \mathbf{n} \;\; &\text{on } \Gamma \qquad (7) \\
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&\qquad = (+)\, g\chi^{m+1} \,(-)\, \gamma_2 \tfrac{K}{N} \nabla \chi^{m+1} \cdot \mathbf{n} \ \text{ on } \Gamma \\
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$$
\eta^{m+1} := \eta^m - \theta \big(\mathbf{n} \cdot \mathbf{T}(\boldsymbol{\omega}^{m+1}, \pi^{m+1}) \cdot \mathbf{n} + g\chi^{m+1} \big) \,, \qquad (8)
$$

where $\gamma_1 > 0$, $\gamma_2 > 0$ and $\theta > 0$ are acceleration parameters.

Consistency of the Robin/Robin algorithm 2

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At convergence, from (8) it holds

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At convergence, from (8) it holds

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therefore normal stresses of the corrections are matching.
On the other hand, from (7) normal velocities of the corrections are jumping

$$\boldsymbol{\omega}^{\infty} \cdot \mathbf{n} + \frac{K}{N}\nabla\chi^{\infty} \cdot \mathbf{n} = \frac{2g}{\gamma_2}\chi^{\infty}\,.$$

Consistency of the Robin/Robin algorithm 2 (cont.)

But this jump gives an additional positive term in the energy of the problem, namely, one obtains

$$\int_{\Omega_S} 2\nu \sum_{ij} |D_i \omega_j^\infty|^2 + \int_{\Omega_D} \frac{g}{N} K \nabla \chi^\infty \cdot \nabla \chi^\infty$$

$$+ \int_\Gamma \frac{2g^2}{\gamma_2} |\chi^\infty|^2 = 0\,.$$

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Hence, $\boldsymbol{\omega}^\infty = \mathbf{0}$ and $\chi^\infty = 0$, and consequently $\sigma^\infty = 0$.

Consistency of the Robin/Robin algorithm 2 (cont.)

Finally:

- from (6), σ^∞ is the jump of the normal velocities, therefore they are matching:

$$\mathbf{u}^\infty \cdot \mathbf{n} = -\frac{K}{N} \nabla \varphi^\infty \cdot \mathbf{n}$$

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$$\mathbf{T}(\mathbf{u}^\infty, p^\infty) \cdot \mathbf{n} = -g\varphi^\infty \, \mathbf{n}$$

We have found the right solution!

Convergence of the Robin/Robin algorithm 2

The parallel Robin/Robin algorithm can be rewritten as a preconditioned Richardson scheme:

$$\eta^{m+1} = \eta^m + \theta(K_S + K_D)[\psi - (H_S + H_D)\eta^m]\,,$$

for a suitable right hand side ψ.

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The operators K_S, K_D, H_S and H_D are the main building blocks of the algorithm, and are defined as follows.

Convergence of the Robin/Robin algorithm 2 (cont.)

- $K_S\sigma := \mathbf{n} \cdot \mathbf{T}(\boldsymbol{\omega}, \pi) \cdot \mathbf{n}$, where $\boldsymbol{\omega}$ is a solution with the Robin datum

$$\mathbf{n} \cdot \mathbf{T}(\boldsymbol{\omega}, \pi) \cdot \mathbf{n} + \gamma_2 \boldsymbol{\omega} \cdot \mathbf{n} = \gamma_2 \sigma \quad \text{on } \Gamma$$
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- $K_D\sigma := g\chi$, where χ is a solution with the Robin datum

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Convergence of the Robin/Robin algorithm 2 (cont.)

- $H_S\eta := \mathbf{u}\cdot\mathbf{n}$, where $\mathbf{u}$ is a solution (for $\mathbf{f}=\mathbf{0}$) with the Robin datum

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- $H_D\eta := \frac{K}{N}\nabla\varphi\cdot\mathbf{n}$, where φ is a solution with the Robin datum

$$-g\varphi+\gamma_1\tfrac{K}{N}\nabla\varphi\cdot\mathbf{n}=\eta \ \text{ on } \Gamma\,.$$

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Therefore, both the operator $(H_S + H_D)$ and the preconditioner $(K_S + K_D)^{-1}$ are symmetric, continuous and positive definite: convergence is achieved! (for a suitable choice of the parameter θ)

Comments

- It is well-known that the Dirichlet-to-Neumann operator is symmetric, continuous and positive definite from the energy trace space $H^{1/2}(\Gamma)$ into ts dual; here we have seen that:

  the Robin-to-Neumann operator is symmetric, continuous and positive definite in $L^2(\Gamma)$,

  whereas:

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- the above results also hold for the finite element numerical approximation, uniformly with respect to the mesh parameter h