Mixed Finite Element Approximation of Eddy Current Problems

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Abstract

Finite element approximations of eddy current problems that are entirely based on the magnetic field $\mathbf{H}$ are haunted by the need to enforce the algebraic constraint $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ in non-conducting regions. As an alternative to techniques employing combinatorial Seifert (cutting) surfaces in order to introduce a scalar magnetic potential, we propose mixed multi-field formulations, which enforce the constraint in the variational formulation. In light of the fact that the computation of cutting surfaces is expensive, the mixed finite element approximation is a viable option despite the increased number of unknowns.

1 Introduction

The governing equations of electromagnetic fields and currents $\mathcal{E}, \mathcal{H}, \mathcal{B}, \mathcal{D}, \mathcal{J}$ are Maxwell's equations completed by constitutive laws in order to model the field-matter interaction. In what follows we shall restrict ourselves to the "Maxwell model of memoryless linear materials with Ohm's law" (see [9]):

$$\begin{array}{rclcrcl} -\partial_t \mathcal{D} + \mathbf{curl}\,\mathcal{H} & = & \mathcal{J} = \mathcal{J}_e + \boldsymbol{\sigma}\mathcal{E}\ , & \quad & \mathcal{D} & = & \boldsymbol{\epsilon}\mathcal{E}, \\ \partial_t \mathcal{B} + \mathbf{curl}\,\mathcal{E} & = & 0\ , & & \mathcal{B} & = & \boldsymbol{\mu}\mathcal{H}\ . \end{array} \qquad (1)$$

Here $\boldsymbol{\mu}$ is the magnetic permeability, $\boldsymbol{\epsilon}$ the dielectric tensor, and $\boldsymbol{\sigma}$ stands for conductivity. $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ are assumed to be uniformly positive definite symmetric 3×3-matrices, whereas $\boldsymbol{\sigma}$ is supposed to be symmetric and uniformly positive definite inside the conducting region Ω^C, but vanishes in the "air region" Ω^I. All the material parameters are functions of the spatial variable $\mathbf{x}$ only. Under these circumstances, if the source current $\mathcal{J}_e$ is of the form $\mathcal{J}_e(t, \mathbf{x}) = \mathrm{Re}[\mathbf{J}_e(\mathbf{x}) \exp(i\omega t)]$, where $\mathbf{J}_e$ is a complex-valued vector field and $\omega \neq 0$ is a

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fixed angular frequency, also the fields $\mathcal{E}, \mathcal{H}, \mathcal{B}, \mathcal{D}$ have this harmonic dependence on time. Maxwell model with Ohm's law then assumes the following strong form.

$$\begin{aligned} -i\omega\mathbf{D} + \mathbf{curl}\,\mathbf{H} &= \mathbf{J}_e + \boldsymbol{\sigma}\mathbf{E}\,, & \mathbf{D} &= \boldsymbol{\epsilon}\mathbf{E}\,, \\ i\omega\mathbf{B} + \mathbf{curl}\,\mathbf{E} &= \mathbf{0}\,, & \mathbf{B} &= \boldsymbol{\mu}\mathbf{H}\,. \end{aligned} \tag{2}$$

The unknows now are the complex amplitudes $\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}$ independent of time.

In many situations it is possible to consider simpler quasi-static models that still offer a sufficiently accurate description of electromagnetic phenomena. The most popular among these simplified models is the so-called "eddy current model", which consists in neglecting the term $-i\omega\mathbf{D}$ in (2) [5, 12].

Then compliance with Ampere's law entails

$$\operatorname{div}\mathbf{J}_{e,I} = 0 \quad \text{in } \Omega^I \quad , \quad \int_{\Gamma^j} \mathbf{J}_{e,I}\cdot\mathbf{n}\,dS = 0\,, \quad j = 1, \ldots, p_\Gamma\,, \tag{3}$$

where Γ^j, $j = 1, \ldots, p_\Gamma$, are the connected components of the boundary of Ω^C. The latter is denoted by $\Gamma := \partial\Omega^C$. Here and in the sequel we denote by $\mathbf{v}_L$ the restriction of a vector field $\mathbf{v}$ to Ω^L, $L = I, C$.

We introduce an artificial computational domain $\Omega \subset \mathbb{R}^3$, which is a box containing the conductors and its immediate neighborhood, big enough so that one can assume a zero field beyond. As before we write Ω^C for the conductor region and $\Omega^I := \Omega \setminus \overline{\Omega^C}$. On $\partial\Omega$ homogeneous boundary conditions for either $\mathbf{H}$ or $\mathbf{E}$ are imposed: throughout we will demand

$$\mathbf{H}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\,.$$

This implies another compatibility condition for $\mathbf{J}_{e,I}$, namely

$$\mathbf{J}_{e,I}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\,. \tag{4}$$

Obviously, we cannot expect a solution for $\mathbf{E}$ to be unique, because it can be altered by any gradient supported in Ω^I and will still satisfy the equations. However, imposing the constraints

$$\operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}_I) = 0 \text{ in } \Omega^I, \quad \int_{\Gamma^j} \boldsymbol{\epsilon}\mathbf{E}_I\cdot\mathbf{n}\,dS = 0,\ j = 1, \ldots, p_\Gamma - 1, \quad \boldsymbol{\epsilon}\mathbf{E}_I\cdot\mathbf{n} = 0 \text{ on } \partial\Omega, \tag{5}$$

(that are implied by (2)) will restore uniqueness of the solution for $\mathbf{E}$.

The complete eddy current model we consider in the sequel is of the form:

$$\begin{gathered} \mathbf{curl}\,\mathbf{H} = \mathbf{J}_e + \boldsymbol{\sigma}\mathbf{E}\,, \quad i\omega\boldsymbol{\mu}\mathbf{H} + \mathbf{curl}\,\mathbf{E} = \mathbf{0} \quad \text{in } \Omega\,, \\ \operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}_I) = 0 \quad \text{in } \Omega^I\,, \quad \int_{\Gamma^j} \boldsymbol{\epsilon}\mathbf{E}_I\cdot\mathbf{n}\,dS = 0\,, \quad j = 1, \ldots, p_\Gamma - 1\,, \\ \boldsymbol{\epsilon}\mathbf{E}\cdot\mathbf{n} = 0\,, \quad \mathbf{H}\times\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega\,. \end{gathered} \tag{6}$$

The existence and uniqueness of a solution of problem (6) has been proven in [4].

Dropping the displacement current converts Ampere's law into the purely algebraic constraint $\mathbf{curl}\,\mathbf{H} = \mathbf{J}_{e,I}$ in Ω^I. This raises problems not encountered with the full Maxwell's equations. This paper will be devoted to how to deal with these problems in the context of a variational formulation based on the magnetic field $\mathbf{H}$. We will focus on approaches that forgo the "direct option" to incorporate the constraint into the trial space. Instead it is enforced by means of augmented variational equations.

Adding extra equations may seem wasteful and, indeed, it is, because the resulting formulations will, after a finite element Galerkin discretization, feature many additional degrees of freedom. However, this is the price to pay for avoiding the cumbersome "topological preprocessing", that is the construction of cuts [15], that is indispensable in the case of the "direct option". Hence, these augmented formulations can become relevant for practical computations. Here we are going to present a couple of possibilities to take into account the seemingly simple constraint $\mathbf{curl}\,\mathbf{H} = \mathbf{J}_{e,I}$ in Ω^I. Each variant will come with its own issues of stability and uniqueness.

A brief outline of the paper is as follows: in the next section we introduce notations and function spaces needed for the remainder of the article. Then we review the well-known $\mathbf{H}$-based variational formulation of the eddy current problem. From these basic equations we derive augmented mixed formulations in the fourth section. In the fifth section their finite element Galerkin discretization will be discussed. Finally, in Sect. 6 we give a-priori error estimates.

2 Basic concepts

As usual, we indicate by $H^s(\Omega)$ or $H^s(\partial\Omega)$, $s \in \mathbb{R}$, the Sobolev space of order s of real or complex measurable functions defined on Ω or $\partial\Omega$, respectively. If $\Sigma \subset \partial\Omega$ we indicate with $H^1_{0,\Sigma}(\Omega)$ the subspace of $H^1(\Omega)$ constituted by those functions φ satisfying $\varphi_{|\Sigma} = 0$. As usual $H^1_0(\Omega) := H^1_{0,\partial\Omega}(\Omega)$.

The space $H(\mathbf{curl};\Omega)$ (respectively, $H(\mathrm{div};\Omega)$) indicates the set of the real or complex vector valued functions $\mathbf{v} \in (L^2(\Omega))^3$ such that $\mathbf{curl}\,\mathbf{v} \in (L^2(\Omega))^3$ (respectively, $\mathrm{div}\,\mathbf{v} \in L^2(\Omega)$). If $\Sigma \subset \partial\Omega$, by $H_{0,\Sigma}(\mathbf{curl};\Omega)$ we designate the subspace of $H(\mathbf{curl};\Omega)$ of those functions $\mathbf{v}$ satisfying $(\mathbf{v} \times \mathbf{n})_{|\Sigma} = \mathbf{0}$. We set $H_0(\mathbf{curl};\Omega) := H_{0,\partial\Omega}(\mathbf{curl};\Omega)$. $H^0(\mathbf{curl};\Omega)$ denotes the subspace of curl-free functions of $H(\mathbf{curl};\Omega)$ and $H^0_{0,\Sigma}(\mathbf{curl};\Omega) = H_{0,\Sigma}(\mathbf{curl};\Omega) \cap H^0(\mathbf{curl};\Omega)$. Analogously $H_{0,\Sigma}(\mathrm{div};\Omega)$ stands for the subspace of $H(\mathrm{div};\Omega)$ containing functions $\mathbf{v}$ satisfying $(\mathbf{v}\cdot\mathbf{n})_{|\Sigma} = 0$. As above, we set $H_0(\mathrm{div};\Omega) := H_{0,\partial\Omega}(\mathrm{div};\Omega)$. Moreover, $H^0(\mathrm{div};\Omega)$ denotes the subspace of -divergence-free functions of $H(\mathrm{div};\Omega)$ and $H^0_{0,\Sigma}(\mathrm{div};\Omega) := H_{0,\Sigma}(\mathrm{div};\Omega) \cap H^0(\mathrm{div};\Omega)$. Finally, $H^s(\mathbf{curl};\Omega)$ designates the space of vector functions $\mathbf{v} \in (H^s(\Omega))^3$ such that $\mathbf{curl}\,\mathbf{v} \in (H^s(\Omega))^3$.

Topology enters our considerations through the space of *harmonic vector fields*

$$\mathcal{H} := H^0_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \cap H^0_{0,\Gamma}(\mathrm{div};\Omega^I)\ . \tag{7}$$

Moreover, for the sake of brevity, we introduce the space of admissible electric fields

$$W^I := \{\mathbf{N}_I \in (L^2(\Omega^I))^3 \,|\, \mathbf{N}_I \text{ satisfies (5)}\} \,, \tag{8}$$

and the "space of unique vector potentials"

$$Y^I := H_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \cap H^0_{0,\Gamma}(\mathrm{div};\Omega^I) \cap \mathcal{H}^\perp \,. \tag{9}$$

They owe their name to following result, which will be usefull in the sequel. It is essentially contained in [1, 13]. For a constructive proof see [3].

Theorem 2.1 *For each* $\mathbf{v}_I \in W^I$ *there is a unique* $\mathbf{q}_I \in Y^I$ *such that* $\mathbf{curl}\,\mathbf{q}_I = \boldsymbol{\epsilon}\mathbf{v}_I$ *and*

$$\|\mathbf{q}_I\|_{L^2(\Omega^I)} \leq C_1 \|\boldsymbol{\epsilon}\mathbf{v}_I\|_{L^2(\Omega^I)} \,.$$

3 The H-based variational formulation

Basically, two different variational formulations of (6) exist, either based on the electric field $\mathbf{E}$ or the magnetic field $\mathbf{H}$ [8]. They correspond to the primal and dual formulation of second order elliptic prohlem. Yet, the algebraic constraint on $\mathbf{curl}\,\mathbf{H}$ manifests itself in a entirely different way in the two formulations. Therefore we restrict ourselves to the $\mathbf{H}$-based approach.

The generic form of the $\mathbf{H}$-based variational formulation involves the Hilbert space of *complex-valued* vector functions

$$V^0 := \{\mathbf{v} \in H_0(\mathbf{curl};\Omega) \,|\, \mathbf{curl}\,\mathbf{v}_I = \mathbf{0} \text{ in } \Omega^I\} \,,$$

endowed with the natural norm

$$||\mathbf{v}||^2_{V^0} := \int_\Omega |\mathbf{v}|^2 + \int_{\Omega^C} |\,\mathbf{curl}\,\mathbf{v}_C|^2 \,.$$

We will also need the affine space

$$V^{\mathbf{J}_{e,I}} := \{\mathbf{v} \in H_0(\mathbf{curl};\Omega) \,|\, \mathbf{curl}\,\mathbf{v}_I = \mathbf{J}_{e,I} \text{ in } \Omega^I\} = \mathbf{H}^* + V^0 \,,$$

where $\mathbf{H}^*$ is a function in $H_0(\mathbf{curl};\Omega)$ such that $\mathbf{curl}\,\mathbf{H}^*_I = \mathbf{J}_{e,I}$ in Ω^I. The magnetic field we are looking for belongs to $V^{\mathbf{J}_{e,I}}$. Moreover for each $\mathbf{v} \in V^0$

$$0 = \int_\Omega (i\omega\boldsymbol{\mu}\mathbf{H} + \mathbf{curl}\,\mathbf{E}) \cdot \overline{\mathbf{v}} = i\omega \int_\Omega \boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} + \int_{\Omega^C} \mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C.$$

Using the strong form of Ampere's law in the conductor, namely $\mathbf{E}_C = \boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C})$, we arrive at

$$0 = i\omega \int_\Omega \boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} + \int_{\Omega^C} \boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C}) \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C.$$

So, the magnetic field $\mathbf{H}$ solves the following problem:

$$\begin{cases} \text{Find } \mathbf{H} \in V^{\mathbf{J}_{e,I}} : \\ i\omega \int_\Omega \boldsymbol{\mu}\mathbf{H} \cdot \overline{\mathbf{v}} + \int_{\Omega^C} \boldsymbol{\sigma}^{-1} \, \mathbf{curl}\, \mathbf{H}_C \cdot \mathbf{curl}\, \overline{\mathbf{v}}_C = \int_{\Omega^C} \boldsymbol{\sigma}^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl}\, \overline{\mathbf{v}}_C \quad \forall\, \mathbf{v} \in V^0. \end{cases} \tag{10}$$

The existence and uniqueness of a solution of (10) follows from the Lax-Milgram lemma, since, under our assumptions on the material coefficients, the bilinear form is trivially coercive on V^0. Next, we have to recover the electric field in Ω. In Ω^C, from Ampere's law we have

$$\mathbf{E}_C = \boldsymbol{\sigma}^{-1}(\mathbf{curl}\, \mathbf{H}_C - \mathbf{J}_{e,C}) \ , \tag{11}$$

while in Ω^I there exists a unique $\mathbf{E}_I \in H(\mathbf{curl}; \Omega^I)$ such that

$$\mathbf{curl}\, \mathbf{E}_I = -i\omega \boldsymbol{\mu} \mathbf{H}_I \qquad \operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}_I) = 0 \quad \text{in } \Omega^I \ , \tag{12}$$

$$\boldsymbol{\epsilon}\mathbf{E}_I \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \ , \qquad \int_{\Gamma^j} \boldsymbol{\epsilon}\mathbf{E}_I \cdot \mathbf{n} \, dS = 0 \quad j = 1, \dots, p_\Gamma - 1 \ , \tag{13}$$

$$\mathbf{E}_I \times \mathbf{n}_I = -\mathbf{E}_C \times \mathbf{n}_C \quad \text{on } \Gamma \ . \tag{14}$$

Here, $\mathbf{n}_L$ denotes the unit outward normal vector on $\partial\Omega_L$, $L = I, C$. We refer to [4] for more details.

Then $(\mathbf{H}, \mathbf{E})$ with $\mathbf{H}$ solution of (10) and $\mathbf{E}$ defined as

$$\mathbf{E} = \begin{cases} \mathbf{E}_C & \text{in } \Omega^C \\ \mathbf{E}_I & \text{in } \Omega^I \end{cases}$$

is the unique solution of (6).

Remark 3.1. We note that a finite element method based on (10) would have to deal with the constrained space $V^{\mathbf{J}_{e,I}}$. The direct way to deal with the constraint in V^0 makes use of scalar magnetic potentials by representing

$$V^0{}_{|\Omega^I} = \mathbf{grad}\, H^1_{\partial\Omega}(\Omega^I) \oplus \mathcal{H} \ ,$$

(see [3, 7]). It would be a perfect solution, unless we had to construct a basis of $\mathcal{H}$ in order to continue with discretization. Such a basis is readily available, once we have "cuts" at our disposal, i.e. a collection of surface in Ω^I that cut any non-bounding cycle [9, 18]. Finding these cuts for arbitrary shape of Ω^C seems to be a challenging problem [15]. △

4 Mixed formulations

The main idea is to reformulate (10) as a saddle point problem in non-constrained vector spaces by introducing Lagrange multipliers.

Let us define the bilinear form in $H_0(\mathbf{curl};\Omega)$

$$a(\mathbf{w},\mathbf{v}) := i\omega \int_\Omega \boldsymbol{\mu}\mathbf{w}\cdot\overline{\mathbf{v}} + \int_{\Omega^C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{w}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C\ .$$

We can introduce a Lagrange multiplier $\mathbf{A}_I \in (L^2(\Omega^I))^3$ and consider the saddle point problem

$$\begin{cases} \text{Find } (\mathbf{H},\mathbf{A}_I) \in H_0(\mathbf{curl};\Omega)\times (L^2(\Omega^I))^3 : \\ a(\mathbf{H},\mathbf{v}) + \int_{\Omega^I} \mathbf{curl}\,\overline{\mathbf{v}}_I\cdot\mathbf{A}_I = \int_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}\cdot\mathbf{curl}\,\overline{\mathbf{v}}_C & \forall\,\mathbf{v}\in H_0(\mathbf{curl};\Omega) \\ \int_{\Omega^I} \mathbf{curl}\,\mathbf{H}_I\cdot\overline{\mathbf{N}}_I = \int_{\Omega^I}\mathbf{J}_{e,I}\cdot\overline{\mathbf{N}}_I & \forall\,\mathbf{N}_I\in (L^2(\Omega^I))^3\ . \end{cases} \tag{15}$$

This problem does not have a unique solution as it is possible to add any function of $H^0_{0,\Gamma}(\mathbf{curl};\Omega^I)$ to $\mathbf{A}_I$. However if $(\mathbf{H},\mathbf{A}_I)$ is a solution of (15) then $\mathbf{H}$ is the solution of (10), and $\mathbf{A}_I$ satisfies $\mathbf{curl}\,\mathbf{A}_I = -i\omega\boldsymbol{\mu}\mathbf{H}_I = \mathbf{curl}\,\mathbf{E}_I$ and $\mathbf{A}_I\times\mathbf{n}_I = \boldsymbol{\sigma}^{-1}(\mathbf{J}_{e,C} - \mathbf{curl}\ \mathbf{H}_C)\times\mathbf{n}_C = -\mathbf{E}_C\times\mathbf{n}_C = \mathbf{E}_I\times\mathbf{n}_I$ on Γ. Thus, in order to restore uniqueness of solution it is natural to look for $\mathbf{A}_I$ in the constrained space W^I defined in (8). Then it is obvious that $\mathbf{A}_I = \mathbf{E}_I$. From Theorem 2.1 it is easily verified that W^I is equal to the range space $\boldsymbol{\epsilon}^{-1}\,\mathbf{curl}\,H_{0,\partial\Omega}(\mathbf{curl};\Omega^I)$.

Thus, we consider the *two-field formulation*:

$$\begin{cases} \text{Find } (\mathbf{H},\mathbf{A}_I) \in H_0(\mathbf{curl};\Omega)\times W^I : \\ a(\mathbf{H},\mathbf{v}) + \int_{\Omega^I} \mathbf{curl}\,\overline{\mathbf{v}}_I\cdot\mathbf{A}_I = \int_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}\cdot\mathbf{curl}\,\overline{\mathbf{v}}_C & \forall\,\mathbf{v}\in H_0(\mathbf{curl};\Omega) \\ \int_{\Omega^I} \mathbf{curl}\,\mathbf{H}_I\cdot\overline{\mathbf{N}}_I = \int_{\Omega^I}\mathbf{J}_{e,I}\cdot\overline{\mathbf{N}}_I & \forall\,\mathbf{N}_I\in W^I\ . \end{cases} \tag{16}$$

Theorem 4.1 *A unique solution of* (16) *exists.*

Proof. We can appeal to Theorem 2.1 and the general theory of variational saddle point problems [10]. □

Again, the space W^I involves some constraints. So we introduce another Lagrange multiplier to impose these: we consider the space

$$H^1_*(\Omega^I) := \{\varphi\in H^1(\Omega^I)\mid \varphi_{|\Gamma^j} \text{ is constant } \forall\, j=1,\dots,p_\Gamma-1,\ \ \varphi_{|\Gamma^{p_\Gamma}}=0\},$$

and it is easily verified by integration by parts that $\mathbf{N}_I\in W^I$ if and only if $\mathbf{N}_I\in (L^2(\Omega^I))^3$ and $\int_{\Omega^I}\boldsymbol{\epsilon}\mathbf{N}_I\cdot\nabla\overline{\psi}_I = 0$ for all $\psi_I\in H^1_*(\Omega^I)$. Eventually we confront the following problem:

$$\begin{cases} \text{Find } (\mathbf{H},\mathbf{A}_I,\phi_I) \text{ in } H_0(\mathbf{curl};\Omega)\times (L^2(\Omega^I))^3\times H^1_*(\Omega^I) : \\ a(\mathbf{H},\mathbf{v}) + \int_{\Omega^I} \mathbf{curl}\,\overline{\mathbf{v}}_I\cdot\mathbf{A}_I = \int_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}\cdot\mathbf{curl}\,\overline{\mathbf{v}}_C & \forall\,\mathbf{v}\in H_0(\mathbf{curl};\Omega) \\ \int_{\Omega^I} \mathbf{curl}\,\mathbf{H}_I\cdot\overline{\mathbf{N}}_I + \int_{\Omega^I}\boldsymbol{\epsilon}\overline{\mathbf{N}}_I\cdot\nabla\phi_I = \int_{\Omega^I}\mathbf{J}_{e,I}\cdot\overline{\mathbf{N}}_I & \forall\,\mathbf{N}_I\in (L^2(\Omega^I))^3 \\ \int_{\Omega^I}\boldsymbol{\epsilon}\mathbf{A}_I\cdot\nabla\overline{\psi}_I = 0 & \forall\,\psi_I\in H^1_*(\Omega^I)\ . \end{cases} \tag{17}$$

We note that if $(\mathbf{H}, \mathbf{A}_I, \phi_I)$ is a solution of (17) then $\phi_I = 0$ (just taken $\mathbf{N}_I = \nabla\phi_I$ in (17)), and $(\mathbf{H}, \mathbf{A}_I)$ is solution of (16).

Introducing the bilinear forms

$$b(\cdot,\cdot) : H_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \times (L^2(\Omega^I))^3 \to \mathbb{C} , \quad b(\mathbf{v}_I, \mathbf{N}_I) := \int_{\Omega^I} \mathbf{curl}\, \mathbf{v}_I \cdot \overline{\mathbf{N}}_I ,$$

and

$$c(\cdot,\cdot) : (L^2(\Omega^I))^3 \times H^1_*(\Omega^I) \to \mathbb{C} , \quad c(\mathbf{N}_I, \psi_I) := \int_{\Omega^I} \boldsymbol{\epsilon}\mathbf{N}_I \cdot \nabla\overline{\psi}_I ,$$

and the linear operators

$$F(\mathbf{v}) := \int_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\, \overline{\mathbf{v}}_C , \quad \mathbf{v} \in H_0(\mathbf{curl};\Omega)$$

and

$$G(\mathbf{N}_I) := \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{N}}_I , \quad \mathbf{N}_I \in (L^2(\Omega^I))^3,$$

problem (17) can be rewritten as

$$\begin{cases} \text{Find } (\mathbf{H}, \mathbf{A}_I, \phi_I) \text{ in } H_0(\mathbf{curl};\Omega) \times (L^2(\Omega^I))^3 \times H^1_*(\Omega^I) : \\ a(\mathbf{H}, \mathbf{v}) + \overline{b(\mathbf{v}_I, \mathbf{A}_I)} = F(\mathbf{v}) \quad \forall\, \mathbf{v} \in H_0(\mathbf{curl};\Omega) \\ b(\mathbf{H}_I, \mathbf{N}_I) + \overline{c(\mathbf{N}_I, \phi_I)} = G(\mathbf{N}_I) \quad \forall\, \mathbf{N}_I \in (L^2(\Omega^I))^3 \\ c(\mathbf{A}_I, \psi_I) = 0 \quad \forall\, \psi_I \in H^1_*(\Omega^I) . \end{cases}$$

In order to proof that (17) has a unique solution, we can use the following result, which is Lemma 4.1 in [11] extended to complex Hilbert spaces.

Lemma 4.2 *Let X, Q, M be three complex Hilbert spaces and $a : X \times X \to \mathbb{C}$, $b : X \times Q \to \mathbb{C}$, $c : Q \times M \to \mathbb{C}$ be three continuous bilinear forms, i.e. there exist three positive constants c_1, c_2, c_3 such that $|a(\mathbf{v},\mathbf{w})| \le c_1\|\mathbf{v}\|_X\|\mathbf{w}\|_X$, $|b(\mathbf{v},\mathbf{N})| \le c_2\|\mathbf{v}\|_X\|\mathbf{N}\|_Q$, $|c(\mathbf{N},\psi)| \le c_3\|\mathbf{N}\|_Q\|\psi\|_M$ for all $\mathbf{v}$, $\mathbf{w} \in X$, $\mathbf{N} \in Q$ and $\psi \in M$. Given $f \in X'$, $g \in Q'$, $l \in M'$, let us consider the saddle point problem*

$$\begin{cases} \textit{Find } (\mathbf{H}, \mathbf{A}, \phi) \textit{ in } X \times Q \times M : \\ a(\mathbf{H}, \mathbf{v}) + \overline{b(\mathbf{v}, \mathbf{A})} = \langle f, \mathbf{v}\rangle \quad \forall\, \mathbf{v} \in X \\ b(\mathbf{H}, \mathbf{N}) + \overline{c(\mathbf{N}, \phi)} = \langle g, \mathbf{N}\rangle \quad \forall\, \mathbf{N} \in Q \\ c(\mathbf{A}, \psi) = \langle l, \psi\rangle \quad \forall\, \psi \in M . \end{cases} \tag{18}$$

Let $Q^0 \subset Q$ and $X^0 \subset X$ be two subspaces as follows:

$$Q^0 = \{\mathbf{N} \in Q \mid c(\mathbf{N}, \psi) = 0 \quad \forall\, \psi \in M\},$$

$$X^0 = \{\mathbf{v} \in X \mid b(\mathbf{v}, \mathbf{N}) = 0 \quad \forall\, \mathbf{N} \in Q^0\}.$$

Assume that $a(\cdot,\cdot)$ is X^0-coercive, i.e.,

$$|a(\mathbf{v}, \mathbf{v})| \geq \alpha \|\mathbf{v}\|_X^2 \quad \forall\, \mathbf{v} \in X^0, \tag{19}$$

and that the following inf–sup conditions hold

$$\inf_{\mathbf{N}\in Q^0} \sup_{\mathbf{v}\in X} \frac{|b(\mathbf{v}, \mathbf{N})|}{\|\mathbf{v}\|_X \|\mathbf{N}\|_Q} \geq \beta, \tag{20}$$

$$\inf_{\psi\in M} \sup_{\mathbf{N}\in Q} \frac{|c(\mathbf{N}, \psi)|}{\|\mathbf{N}\|_Q \|\psi\|_M} \geq \gamma, \tag{21}$$

for some positive constants α, β, γ. Then problem (18) has a unique solution.

Now we are in a position to prove the following result:

Theorem 4.3 *Problem (17) has a unique solution.*

Proof. In order to verify the assumptions of Lemma 4.2, first recall that the spaces W^I and V^0 can also be characterized as

$$W^I = \{\mathbf{N}_I \in (L^2(\Omega^I))^3 \mid \int_{\Omega^I} \boldsymbol{\epsilon}\mathbf{N}_I \cdot \nabla\overline{\psi}_I = 0 \quad \forall\, \psi_I \in H^1_*(\Omega^I)\}$$

and

$$V^0 = \{\mathbf{v} \in H_0(\mathbf{curl}; \Omega) \mid \int_{\Omega^I} \mathbf{curl}\,\mathbf{v}_I \cdot \overline{\mathbf{N}}_I = 0 \quad \forall\, \mathbf{N}_I \in W^I\}\ ,$$

(in the latter case, just take $\mathbf{N}_I = \boldsymbol{\epsilon}^{-1}\,\mathbf{curl}\,\mathbf{v}_I$).

Since the bilinear form $a(\cdot,\cdot)$ is coercive on the space V^0, we need only show that the two inf-sup conditions are satisfied, more precisely, that there exist two positive constants β and γ such that

$$\sup_{\mathbf{v}\in H_0(\mathbf{curl};\Omega)} \frac{\left|\int_{\Omega^I} \mathbf{curl}\,\mathbf{v}_I \cdot \overline{\mathbf{N}}_I\right|}{\|\mathbf{v}\|_{H(\mathbf{curl};\Omega)}} \geq \beta \|\mathbf{N}_I\|_{L^2(\Omega^I)} \tag{22}$$

for all $\mathbf{N}_I \in W^I$, and

$$\sup_{\mathbf{N}_I\in (L^2(\Omega^I))^3} \frac{\left|\int_{\Omega^I} \boldsymbol{\epsilon}\mathbf{N}_I \cdot \nabla\overline{\psi}_I\right|}{\|\mathbf{N}_I\|_{L^2(\Omega^I)}} \geq \gamma \|\psi_I\|_{H^1(\Omega^I)} \tag{23}$$

for all $\psi_I \in H^1_*(\Omega^I)$.

Poincare's inequality gives us a constant $C_2 > 0$ such that $\|\psi_I\|_{H^1(\Omega^I)} \leq C_2 \|\nabla \psi_I\|_{L^2(\Omega^I)}$ for all $\psi_I \in H^1_*(\Omega^I)$. Moreover, since $\boldsymbol{\epsilon}$ is assumed to be uniformly positive definite, there exist two positive constants $\boldsymbol{\epsilon}_*$ and $\boldsymbol{\epsilon}^*$ such that for all $\mathbf{N}_I \in (L^2(\Omega^I))^3$

$$\boldsymbol{\epsilon}_* \|\mathbf{N}_I\|^2_{L^2(\Omega^I)} \leq \int_{\Omega^I} \boldsymbol{\epsilon} \mathbf{N}_I \cdot \mathbf{N}_I \leq \boldsymbol{\epsilon}^* \|\mathbf{N}_I\|^2_{L^2(\Omega^I)}.$$

Hence, given $\psi_I \in H^1_*(\Omega^I)$ and choosing $\mathbf{N}_I = \nabla \psi_I$ we have

$$\sup_{\mathbf{N}_I \in (L^2(\Omega^I))^3} \frac{\left| \int_{\Omega^I} \boldsymbol{\epsilon} \mathbf{N}_I \cdot \nabla \overline{\psi}_I \right|}{\|\mathbf{N}_I\|_{L^2(\Omega^I)}} \geq \frac{\int_{\Omega^I} \boldsymbol{\epsilon} \nabla \psi_I \cdot \nabla \overline{\psi}_I}{\|\nabla \psi_I\|_{L^2(\Omega^I)}} \geq \boldsymbol{\epsilon}_* \|\nabla \psi_I\|_{L^2(\Omega^I)} \geq \frac{\boldsymbol{\epsilon}_*}{C_2} \|\psi_I\|_{H^1(\Omega^I)}.$$

Concerning (22), by Theorem 2.1 for all $\mathbf{N}_I \in W^I$ there exists $\mathbf{q}_I \in Y^I$ such that $\mathbf{N}_I = \boldsymbol{\epsilon}^{-1} \operatorname{\mathbf{curl}} \mathbf{q}_I$. Let $\mathbf{q} \in H(\operatorname{\mathbf{curl}}; \Omega)$ be a continuous extension of $\mathbf{q}_I$ into Ω^C; hence, $\|\mathbf{q}\|_{H(\operatorname{\mathbf{curl}};\Omega)} \leq C_3 \|\mathbf{q}_I\|_{H(\operatorname{\mathbf{curl}};\Omega^I)}$. By the stability estimate of Theorem 2.1 we can infer that

$$\|\mathbf{q}\|^2_{H(\operatorname{\mathbf{curl}};\Omega)} \leq C_3^2 \|\mathbf{q}_I\|^2_{H(\operatorname{\mathbf{curl}};\Omega^I)} \leq C_3^2 (1 + C_1^2) \| \operatorname{\mathbf{curl}} \mathbf{q}_I \|^2_{L^2(\Omega^I)}.$$

Thus,

$$\begin{aligned}
\sup_{\mathbf{v} \in H_0(\operatorname{\mathbf{curl}} \Omega)} \frac{\left| \int_{\Omega^I} \operatorname{\mathbf{curl}} \mathbf{v}_I \cdot \overline{\mathbf{N}}_I \right|}{\|\mathbf{v}\|_{H(\operatorname{\mathbf{curl}};\Omega)}} &\geq \frac{\left| \int_{\Omega^I} \operatorname{\mathbf{curl}} \mathbf{q}_I \cdot \overline{\mathbf{N}}_I \right|}{\|\mathbf{q}\|_{H(\operatorname{\mathbf{curl}};\Omega)}} \\
&\geq \frac{1}{(1 + C_1^2)^{1/2} C_3} \frac{\left| \int_{\Omega^I} \operatorname{\mathbf{curl}} \mathbf{q}_I \cdot \overline{\mathbf{N}}_I \right|}{\| \operatorname{\mathbf{curl}} \mathbf{q}_I \|_{L^2(\Omega^I)}} = \frac{1}{(1 + C_1^2)^{1/2} C_3} \frac{\int_{\Omega^I} \boldsymbol{\epsilon} \mathbf{N}_I \cdot \overline{\mathbf{N}}_I}{\|\boldsymbol{\epsilon} \mathbf{N}_I\|_{L^2(\Omega^I)}} \\
&\geq \frac{\boldsymbol{\epsilon}_*}{(1 + C_1^2)^{1/2} C_3 \boldsymbol{\epsilon}^*} \|\mathbf{N}_I\|_{L^2(\Omega^I)} \, .
\end{aligned}$$

□

5 Finite element discretization

We are aiming for Galerkin finite element discretization of both the two-field problem (16) and the three-field formulation (17). In both cases we want to verify the assumptions of the theory of discrete saddle point problem [10, Chap. 2].

We assume that Ω, Ω^C, Ω^I are Lipschitz polyhedra and consider a family of regular tetrahedral meshes $\{\mathcal{T}_h\}_h$ of Ω such that each element $K \in \mathcal{T}_h$ is contained either in $\overline{\Omega^C}$ or in $\overline{\Omega^I}$. We denote $\mathcal{T}_{C,h}$, $\mathcal{T}_{I,h}$ the restriction of $\mathcal{T}_h$ to Ω^C and Ω^I, respectively. The parameter h will also provide the meshwidth of $\mathcal{T}_h$.

We employ Nédélec curl-conforming edge elements of the lowest order to approximate the magnetic field: let V_h be the finite elements space defined by

$$V_h := \{ \mathbf{v}_h \in H(\operatorname{\mathbf{curl}}; \Omega) \mid \mathbf{v}_h(\mathbf{x})_{|K} = \mathbf{a}_K + \mathbf{b}_K \times \mathbf{x} \quad \forall \, K \in \mathcal{T}_h \} \, ,$$

where $\mathbf{a}_K$ and $\mathbf{b}_K$ are two constant vectors in $\mathbb{R}^3$. It is known that any function $\mathbf{v}_h \in V_h$ is uniquely determined by the following degrees of freedom [16, Sect. 3.2]

$$M_e(\mathbf{v}) = \{ \int_e \mathbf{v} \cdot \boldsymbol{\tau} \, ds \mid \ e \text{ is an edge of } \mathcal{T}_h \},$$

where $\boldsymbol{\tau}$ is the unit vector along the edge e. These edge moments make sense for any $\mathbf{v} \in (H^s(\Omega))^3$ with $\mathbf{curl}\,\mathbf{v} \in (L^p(\Omega))^3$ with $s > 1/2$ and $p > 2$ (see [6, Lemma 4.7] and [16, Lemma 3.13]). Moreover the following interpolation error estimate holds (see [2, 11] and [16, Theor. 3.14]).

Lemma 5.1 *Denoting by $\pi_h \mathbf{w} \in V_h$ the interpolant of $\mathbf{w}$, for $1/2 < s \leq 1$, we have*

$$\|\pi_h \mathbf{w} - \mathbf{w}\|_{L^2(K)} \leq C_4 \, h_K^s (\|\mathbf{w}_I\|_{H^s(K)} + \|\,\mathbf{curl}\,\mathbf{w}_I\|_{H^s(K)}) \quad \forall\, \mathbf{w} \in H^s(\mathbf{curl}; K),$$

where h_K is the diameter of $K \in \mathcal{T}_h$.

The homogeneoous boundary conditions on $\partial\Omega$ are incorporated by setting degrees of freedom on $\partial\Omega$ to zero. Thus we end up with the spaces

$$X_h := V_h \cap H_0(\mathbf{curl}; \Omega) \text{ and } X_h^I := \{ \mathbf{v}_{h|\Omega^I} \mid \mathbf{v}_h \in X_h \} \, .$$

For additional information about edge elements the reader is referred to [16, Chap. 3], [2], and [14, Chap. III, Sect. 5.3].

5.1 Two-field formulation

The challenge is the approximations of the constrained space W^I. However, we can take the cue from the representation in Theorem 2.1 and lift it into the discrete setting. More precisely, we choose

$$W_h^I := \boldsymbol{\epsilon}^{-1}\,\mathbf{curl}\, X_h^I$$

as trial space for W^I. Note that this is a conforming discretization in the sense that $W_h^I \subset W^I$. This results in the following discrete two-field problem.

$$\left\{ \begin{array}{lcll} \text{Find } (\mathbf{H}_h, \mathbf{A}_{I,h}) \text{ in } X_h \times W_h^I : & & & \\ a(\mathbf{H}_h, \mathbf{v}_h) + \int_{\Omega_I} \mathbf{curl}\,\overline{\mathbf{v}}_{I,h} \cdot \mathbf{A}_{I,h} & = & \int_{\Omega_C} \sigma^{-1} \mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{v}}_{C,h} & \forall\, \mathbf{v}_h \in X_h \, , \\ \int_{\Omega_I} \mathbf{curl}\,\mathbf{H}_{I,h} \cdot \overline{\mathbf{N}}_{I,h} & = & \int_{\Omega_I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{N}}_{I,h} & \forall\, \mathbf{N}_{I,h} \in W_h^I \, . \end{array} \right. \tag{24}$$

Theorem 5.2 *Problem* (24) *has a unique solution.*

Proof. As in the case of the continuous problem, it is straightforward that the bilinear form $a(\cdot,\cdot)$ is continuous in X_h and coercive in

$$X_h^0 := \{\mathbf{v}_h \in X_h \mid \int_{\Omega_I} \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \overline{\mathbf{N}}_{I,h} = 0 \;\forall\, \mathbf{N}_{I,h} \in W_h^I\} \,, \tag{25}$$

since in particular $\epsilon^{-1}\,\mathbf{curl}\,\mathbf{v}_{I,h} \in W_h^I$, so that $\mathbf{v}_h \in X_h^0$ implies $\mathbf{curl}\,\mathbf{v}_{h|\Omega^I} = \mathbf{0}$.

To prove a uniform discrete inf-sup-condition we rely on the following lemma. It is a variant of a discrete Poincaré-Friedrichs inequality, see [16, Theorem 4.7] for a proof.

Lemma 5.3 *Let* $X_h^{I,0} := \{\mathbf{v}_{I,h} \in X_h^I \mid \mathbf{curl}\,\mathbf{v}_{I,h} = \mathbf{0}\}$ *and*

$$(X_h^{I,0})^\perp := \{\mathbf{p}_{I,h} \in X_h^I \mid \int_{\Omega_I} \mathbf{p}_{I,h} \cdot \overline{\mathbf{v}}_{I,h} = 0 \;\forall\, \mathbf{v}_{I,h} \in X_h^{I,0}\} \,.$$

There exists a positive constant C_5*, independent of* h*, such that for all* $\mathbf{p}_{I,h} \in (X_h^{I,0})^\perp$

$$\|\mathbf{p}_{I,h}\|_{L^2(\Omega_I)} \leq C_5 \|\,\mathbf{curl}\,\mathbf{p}_{I,h}\|_{L^2(\Omega_I)} \,.$$

By the definition of W_h^I, for each $\mathbf{N}_{I,h} \in W_h^I$ there exists $\widehat{\mathbf{q}}_{I,h} \in X_h^I$ such that $\mathbf{N}_{I,h} = \epsilon^{-1}\,\mathbf{curl}\,\widehat{\mathbf{q}}_{I,h}$. By projecting on $(X_h^{I,0})^\perp$, we find a unique $\mathbf{q}_{I,h} \in (X_h^{I,0})^\perp$ with the same property. Call $\mathbf{q}_h$ some uniform discrete extension of $\mathbf{q}_{I,h}$ to Ω^C (the existence of such an extension has been proved in [2]). Then $\mathbf{q}_h \in X_h$ and

$$\|\mathbf{q}_h\|_{H(\mathbf{curl};\Omega)} \leq C_6 \|\mathbf{q}_{I,h}\|_{H(\mathbf{curl};\Omega_I)} \leq C_6(1+C_5^2)^{1/2} \|\,\mathbf{curl}\,\mathbf{q}_{I,h}\|_{L^2(\Omega_I)} \,.$$

Since the constant $C_6(1+C_5^2)^{1/2}$ is independent of h, $\mathbf{q}_h$ is a suitable candidate in the discrete inf-sup condition:

$$\sup_{\mathbf{v}_h \in X_h} \frac{|\int_{\Omega_I} \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \overline{\mathbf{N}}_{I,h}|}{\|\mathbf{v}_h\|_{H_0(\mathbf{curl};\Omega)}} \geq \frac{|\int_{\Omega_I} \mathbf{curl}\,\mathbf{q}_h \cdot \overline{\mathbf{N}}_{I,h}|}{\|\mathbf{q}_h\|_{H_0(\mathbf{curl};\Omega)}} \geq \frac{\boldsymbol{\epsilon}_*}{C_6(1+C_5^2)^{1/2}\boldsymbol{\epsilon}^*} \|\mathbf{N}_{I,h}\|_{L^2(\Omega_I)} \,. \tag{26}$$

All assumptions of [10, Chap. II, Theor. 1.1] are satisfied. □

Remark 5.1. The only way to implement the space W_h^I is to rely on its very definition, that is we obtain its elements as $\boldsymbol{\epsilon}^{-1}\,\mathbf{curl}$ of edge element functions. Yet, these will no longer be unique. The bottom line is that in a practical implementation of the two-field method we will face a singular system of linear equations. As its kernel is well separated, conjugate gradient type iterative solvers will perform well. △

5.2 Three-field formulation

Apart from $\mathbf{H}$ we have to approximate $\mathbf{A}_I$ (namely, $\mathbf{E}_I$) and ϕ_I in (17). To discretize $\mathbf{A}_I \in L^2(\Omega^I)$ we choose piecewise constant vector functions in the space

$$Q_h^I := \{\mathbf{N}_{I,h} \in (L^2(\Omega^I))^3 \mid \mathbf{N}_{I,h|K} \in (P_0)^3 \;\forall\, K \in \mathcal{T}_{I,h}\} \,.$$

In order to approximate the Lagrangian multiplier $\phi_I \in H^1_*(\Omega^I)$ it would be natural to rely on piecewise linear Lagrangian finite elements. However, it turns out that this space is too small to guarantee uniform stability of the discretized mixed formulation. We have to switch to a larger space for the approximation of the Lagrangian multiplier; it will be the nonconforming Crouzeix-Raviart elements, defined as follows: let P_k denote the standard space of polynomials of total degree less than or equal to k and

$$\begin{aligned} U_h^I \;:=\; & \{\psi_{I,h} \in L^2(\Omega^I) \mid \psi_{I,h|K} \in P_1,\ \forall\, K \in \mathcal{T}_{I,h} \text{ and } \psi_{I,h} \text{ is continuous at the} \\ & \text{centroid of any face } f \text{ common to two elements in } \mathcal{T}_h\}. \end{aligned}$$

Then the discrete ϕ_I will belong to

$$\begin{aligned} M_h^I \;:=\; & \{\psi_{I,h} \in U_h^I \mid \psi_{I,h}(\mathbf{p}) \text{ is equal for all midpoints } \mathbf{p} \text{ of faces of } \Gamma^j, \\ & j = 1, \dots, p_\Gamma - 1, \text{ and } \psi_{I,h}(\mathbf{p}) = 0 \text{ for all midpoints } \mathbf{p} \text{ of faces of } \Gamma^{p_\Gamma}\}\,. \end{aligned}$$

Note that, since functions in U_h^I are no longer continuous, they are no longer in $H^1(\Omega^I)$. Therefore we must define a modified bilinear form $c_h : (L^2(\Omega^I))^3 \times (H^1_*(\Omega^I) + M_h^I) \to \mathbb{C}$ and a norm on $H^1_*(\Omega^I) + M_h^I$. For each $\psi_I \in H^1_*(\Omega^I) + M_h^I$ we denote $\widetilde{\nabla}\psi_I$ the function in $(L^2(\Omega^I))^3$ defined as

$$(\widetilde{\nabla}\psi_I)_{|K} := \nabla(\psi_{I|K}) \quad \forall\, K \in \mathcal{T}_{I,h}.$$

Note that if $\psi_I \in H^1_*(\Omega^I)$, then $\widetilde{\nabla}\psi_I = \nabla\psi_I$. Similarly, we define the bilinear form

$$c_h(\mathbf{N}_I, \psi_I) := \sum_K \int_K \boldsymbol{\epsilon}\mathbf{N}_I \cdot \nabla\psi_I = \int_{\Omega^I} \boldsymbol{\epsilon}\mathbf{N}_I \cdot \widetilde{\nabla}\psi_I \quad \forall\, \mathbf{N}_I \in (L^2(\Omega^I))^3,\ \psi_I \in H^1_*(\Omega^I) + M_h^I$$

and the norm in $H^1_*(\Omega^I) + M_h^I$

$$\|\psi_I\|_h^2 := \sum_K \int_K |\nabla\psi_I|^2 = \|\widetilde{\nabla}\psi_I\|^2_{L^2(\Omega^I)}\,.$$

Then, the finite elements approximation of (17) can be formulated as follows:

$$\left\{ \begin{array}{lcll} \text{Find } (\mathbf{H}_h, \mathbf{A}_{I,h}, \phi_{I,h}) \text{ in } X_h \times Q_h^I \times M_h^I : & & & \\ a(\mathbf{H}_h, \mathbf{v}_h) + \overline{b(\mathbf{v}_{I,h}, \mathbf{A}_{I,h})} & = & F(\mathbf{v}_h) & \forall\, \mathbf{v}_h \in X_h \\ b(\mathbf{H}_{I,h}, \mathbf{N}_{I,h}) + \overline{c_h(\mathbf{N}_{I,h}, \phi_{I,h})} & = & G(\mathbf{N}_{I,h}) & \forall\, \mathbf{N}_{I,h} \in Q_h^I \\ c_h(\mathbf{A}_{I,h}, \psi_{I,h}) & = & 0 & \forall\, \psi_{I,h} \in M_h^I\,. \end{array} \right. \tag{27}$$

To show that this problem has a unique solution we need the following lemma, (see [17]).

Lemma 5.4 *We have the $L^2(\Omega^I)$-orthogonal decomposition $Q_h^I = \mathbf{curl}\, X_h^I \oplus \widetilde{\nabla} M_h^I$.*

Proof. The proof has two parts. In the first part we show that for all $\mathbf{v}_{I,h} \in X_h^I$ and $\psi_{I,h} \in M_h^I$ we have the orthogonality $\int_{\Omega^I} \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \widetilde{\nabla}\psi_{I,h} = 0$. In the second part we establish that $\dim(Q_h^I) = \dim(\mathbf{curl}\,X_h^I) + \dim(\widetilde{\nabla} M_h^I)$.

For any $\mathbf{v}_{I,h} \in X_h^I$ and $\psi_{I,h} \in M_h^I$ integration by parts yields

$$\begin{aligned}
\int_{\Omega^I} \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \widetilde{\nabla}\psi_{I,h} &= \sum_K \int_K \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \nabla\psi_{I,h} \\
&= \sum_K \int_{\partial K} \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \mathbf{n}\,\psi_{I,h} \\
&= \sum_{f\in\mathcal{F}_{\rm int}} \int_f \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \mathbf{n}\,[\psi_{I,h}]_f + \sum_{f\in\mathcal{F}_{\partial\Omega}} \int_f \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \mathbf{n}\,\psi_{I,h} \\
&\qquad + \sum_{j=1}^{p_\Gamma} \sum_{f\in\mathcal{F}_{\Gamma^j}} \int_f \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \mathbf{n}\,\psi_{I,h}\ ,
\end{aligned}$$

where $\mathcal{F}_{\rm int}$ is the set of internal faces of the triangulation $\mathcal{T}_{I,h}$, $\mathcal{F}_{\partial\Omega}$ and $\mathcal{F}_{\Gamma^j}$ denote the set of faces of $\mathcal{T}_{I,h}$ on $\partial\Omega$ and Γ^j, respectively, and $[\psi_{I,h}]_f$ denotes the jump of $\psi_{I,h}$ across the face f. Note that, for all $f \in \mathcal{F}_{\rm int}$, $(\mathbf{curl}\,\mathbf{v}_{I,h}\cdot\mathbf{n})_{|f}$ is constant and $\int_f[\psi_{I,h}]_f = 0$ since $[\psi_{I,h}]_f$ is a linear function and it is equal zero in the centroid of f. Moreover $(\mathbf{curl}\,\mathbf{v}_{I,h}\cdot\mathbf{n})_{|f} = 0$ for all $f \in \partial\Omega$, and, using that $\psi_{h|\Gamma^j}$ is a constant ψ_j for all $j = 1,\dots,p_\Gamma$, we have $\sum_{f\in\mathcal{F}_{\Gamma^j}} \int_f \mathbf{curl}\,\mathbf{v}_{I,h}\cdot\mathbf{n}\,\psi_{I,h} = \psi_j \int_{\Gamma^j} \mathbf{curl}\,\mathbf{v}_{I,h}\cdot\mathbf{n} = 0$, hence

$$\int_{\Omega^I} \mathbf{curl}\,\mathbf{v}_{I,h} \cdot \widetilde{\nabla}\psi_{I,h} = 0.$$

Let us introduce the Raviart-Thomas finite element space [10, Chap. III]

$$RT_h := \{\mathbf{v}_h \in H(\mathrm{div};\Omega^I) \mid \mathbf{v}_h(\mathbf{x})_{|K} = \mathbf{a}_K + b_K\,\mathbf{x} \quad \forall\, K \in \mathcal{T}_{I,h}\},$$

where $\mathbf{a}_K$ is a constant vector and b_K is a real number, and the subspaces

$$RT_{0,\partial\Omega} := RT_h \cap H_{0,\partial\Omega}(\mathrm{div},\Omega^I), \quad RT^0_{0,\partial\Omega} := RT_h \cap H^0_{0,\partial\Omega}(\mathrm{div},\Omega^I).$$

By arguments from discrete cohomology, it can be proven (see [9]) that

$$\dim(\mathbf{curl}\,X_h^I) = \dim(RT^0_{0,\partial\Omega}(\Omega^I)) - (p_\Gamma - 1).$$

Let us denote by $\#K$ the number of tetrahedra of $\mathcal{T}_{I,h}$, by $\#\mathcal{F}$ the total number of faces of $\mathcal{T}_{I,h}$ and by $\#\mathcal{F}_{\partial\Omega}$, $\#\mathcal{F}_\Gamma$, the number of faces of $\mathcal{T}_{I,h}$ on $\partial\Omega$ and by Γ respectively. It is not difficult to prove that:

$$\begin{aligned}
&\dim(RT^0_{0,\partial\Omega}) = \dim(RT_{0,\partial\Omega}(\Omega^I)) - \dim(\mathrm{div}(RT_{0,\partial\Omega}(\Omega^I))) = (\#\mathcal{F} - \#\mathcal{F}_{\partial\Omega}) - \#K,\\
&\dim(M_h^I) = (\#\mathcal{F} - \#\mathcal{F}_\Gamma) + (p_\Gamma - 1) = \dim(\widetilde{\nabla}(M_h^I)),\\
&\dim(Q_h^I) = 3\#K.
\end{aligned}$$

Since $4\#K = 2\#\mathcal{F} - (\#\mathcal{F}_{\partial\Omega} + \#\mathcal{F}_\Gamma)$ then

$$\begin{aligned}
&\dim(\mathbf{curl}\,X_h^I) + \dim(\widetilde{\nabla} M_h^I)\\
&\quad = [(\#\mathcal{F} - \#\mathcal{F}_{\partial\Omega}) - \#K - (p_\Gamma - 1)] + [(\#\mathcal{F} - \#\mathcal{F}_\Gamma) + (p_\Gamma - 1)]\\
&\quad = 2\#\mathcal{F} - (\#\mathcal{F}_{\partial\Omega} + \#\mathcal{F}_\Gamma) - \#K\\
&\quad = 4\#K - \#K\\
&\quad = \dim(Q_h^I).
\end{aligned}$$

Since, trivially, $\mathbf{curl}\, X_h^I \subset Q_h^I$ and $\widetilde{\nabla} M_h^I \subset Q_h^I$, the proof is finished. □

Using Lemma 5.3, we can now prove the main result of this section.

Theorem 5.5 *If we assume that $\boldsymbol{\epsilon}$ is piecewise constant in Ω^I, then Problem (27) has a unique solution.*

Proof.

Conditions (19) and (20) follow as in the proof ot Theorem 5.2, provided that we show that the space

$$\{\mathbf{v}_h \in X_h \mid b(\mathbf{v}_{I,h}, \mathbf{N}_{I,h}) = 0 \quad \forall\, \mathbf{N}_{I,h} \in Q_h^{I,0}\} ,$$

where $Q_h^{I,0} \subset Q_h^I$ is defined as follows

$$Q_h^{I,0} := \{\mathbf{N}_{I,h} \in Q_h^I \mid c_h(\mathbf{N}_{I,h}, \psi_{I,h}) = 0 \quad \forall\, \psi_{I,h} \in M_h^I\},$$

coincides with the space X_h^0 defined in (25). In fact, it is enough to prove that $Q_h^{I,0} = W_h^I$. Since $\boldsymbol{\epsilon}$ is piecewise constant, for each $\mathbf{N}_{I,h} \in Q_h^{I,0}$ we have $\boldsymbol{\epsilon}\mathbf{N}_{I,h} \in Q_h^I$. Therefore, using Lemma 5.4, we obtain that $\boldsymbol{\epsilon}\mathbf{N}_{I,h} \in \mathbf{curl}\, X_h^I$, hence $Q_h^{I,0} \subset \boldsymbol{\epsilon}^{-1}\, \mathbf{curl}\, X_h^I$. The viceverse is straightforward, proceeding as in the proof of Lemma 5.4.

Concerning the inf–sup condition (21) note that for all $\psi_{I,h} \in M_h^I$ one has $\widetilde{\nabla}\psi_{I,h} \in Q_h^I$, hence from the definition of the norm $\|\cdot\|_h$

$$\sup_{\mathbf{N}_{I,h}\in Q_h^I} \frac{|c_h(\mathbf{N}_{I,h}, \psi_{I,h})|}{\|\mathbf{N}_{I,h}\|_{L^2(\Omega^I)}\|\psi_{I,h}\|_h} \geq \frac{|c_h(\widetilde{\nabla}\psi_{I,h}, \psi_{I,h})|}{\|\widetilde{\nabla}\psi_{I,h}\|_{L^2(\Omega^I)}\|\psi_{I,h}\|_h} = \frac{\int_{\Omega^I} \boldsymbol{\epsilon}\widetilde{\nabla}\psi_{I,h}\cdot\widetilde{\nabla}\psi_{I,h}}{\|\widetilde{\nabla}\psi_{I,h}\|^2_{L^2(\Omega^I)}} \geq \boldsymbol{\epsilon}_*. \qquad (28)$$

□

Remark 5.2. Note that $\mathbf{J}_{e,I} = \mathbf{curl}\, \mathbf{K}_{e,I}$ for some $\mathbf{K}_{e,I} \in H_{0,\partial\Omega}(\mathbf{curl}; \Omega^I)$. If $\pi_h \mathbf{K}_{e,I}$ is well–defined, we can define $G_h(\mathbf{N}_I) := \int_{\Omega^I} \mathbf{curl}(\pi_h \mathbf{K}_{e,I}) \cdot \mathbf{N}_I$. If in problem (27) we replace G with G_h it is easily showed that the new $\phi_{I,h}$ is equal to zero. △

6 Error estimates

Given the discrete inf-sup-conditions established in Sect. 5.1, the quasi-optimality of the discrete solution of the two-field problem is standard [10, Chap. 2]. Here, we are only concerned with the discrete three-field problem (27).

We denote by c_1 and c_2 the continuity constants of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ respectively, by α the coercivity constant of $a(\cdot,\cdot)$ in V^0 and by β and γ two positive constants, independent of h, such that

$$\inf_{\mathbf{N}_{I,h}\in Q_h^{I,0}} \sup_{\mathbf{v}_h\in X_h} \frac{|b(\mathbf{v}_{I,h}, \mathbf{N}_{I,h})|}{\|\mathbf{v}_h\|_{H(\mathbf{curl},\Omega)}\|\mathbf{N}_{I,h}\|_{L^2(\Omega^I)}} \geq \beta, \qquad (29)$$

and

$$\inf_{\psi_{I,h}\in M_h^I}\sup_{\mathbf{N}_{I,h}\in Q_h}\frac{|c_h(\mathbf{N}_{I,h},\psi_{I,h})|}{\|\phi_{I,h}\|_h\|\mathbf{N}_{I,h}\|_{L^2(\Omega^I)}}\geq\gamma. \tag{30}$$

¿From (26) and (28) we can take $\beta = \frac{\epsilon_*}{C_6(1+C_5^2)^{1/2}\epsilon^*}$ and $\gamma = \epsilon_*$.

Theorem 6.1 *Let* $(\mathbf{H},\mathbf{A}_I,\phi_I) \in H_0(\mathbf{curl};\Omega)\times(L^2(\Omega^I))^3\times H^1_*(\Omega^I)$ *be the solution of Problem (17) and* $(\mathbf{H}_h,\mathbf{A}_{I,h},\phi_{I,h})\in X_h\times Q_h^I\times M_h^I$ *the solution of Problem* (27)*. Then the following error estimates hold:*

$$\|\mathbf{H}-\mathbf{H}_h\|_{H(\mathbf{curl};\Omega)}\leq\big(1+\frac{c_1}{\alpha}\big)\big(1+\frac{c_2}{\beta}\big)\inf_{\mathbf{v}_h\in X_h}\|\mathbf{H}-\mathbf{v}_h\|_{H(\mathbf{curl};\Omega)}\ , \tag{31}$$

$$\|\mathbf{A}_I-\mathbf{A}_{I,h}\|_{L^2(\Omega^I)}\leq\big(1+\frac{c_2}{\beta}\big)\inf_{\mathbf{N}_{I,h}\in Q_h^{I,0}}\|\mathbf{A}_I-\mathbf{N}_{I,h}\|_{L^2(\Omega^I)}+\frac{c_1}{\beta}\|\mathbf{H}-\mathbf{H}_h\|_{H(\mathbf{curl};\Omega)}\ , \tag{32}$$

$$\|\phi_I-\phi_{I,h}\|_h=\|\phi_{I,h}\|_h\leq\frac{c_2}{\gamma}\|\mathbf{H}-\mathbf{H}_h\|_{H(\mathbf{curl};\Omega)}\ . \tag{33}$$

Proof. The proof follows the lines of the proofs in [10, Ch. 2]. For all $\mathbf{v}_h^*$, $\mathbf{v}_h\in X_h$ and $\mathbf{N}_{I,h}\in Q_h^I$

$$\begin{aligned}a(\mathbf{H}_h-\mathbf{v}_h^*,\mathbf{v}_h)+\overline{b(\mathbf{v}_{I,h},\mathbf{A}_{I,h}-\mathbf{N}_{I,h})} &= F(\mathbf{v}_h)-a(\mathbf{v}_h^*,\mathbf{v}_h)-\overline{b(\mathbf{v}_{I,h},\mathbf{N}_{I,h})}\\ &= a(\mathbf{H}-\mathbf{v}_h^*,\mathbf{v}_h)+\overline{b(\mathbf{v}_{I,h},\mathbf{A}_I-\mathbf{N}_{I,h})}\ .\end{aligned}$$

Note that if $\mathbf{v}_h\in X_h^0$ then $\mathbf{curl}\,\mathbf{v}_{I,h}=\mathbf{0}$ in Ω^I, therefore $a(\mathbf{H}_h-\mathbf{v}_h^*,\mathbf{v}_h)=a(\mathbf{H}-\mathbf{v}_h^*,\mathbf{v}_h)$. If we take $\mathbf{v}_h^*\in X_h^G:=\{\mathbf{v}_h\in X_h\mid b(\mathbf{v}_{I,h},\mathbf{N}_{I,h})=G(\mathbf{N}_{I,h})\ \forall\,\mathbf{N}_{I,h}\in Q_h^{I,0}\}$, then $\mathbf{H}_h-\mathbf{v}_h^*\in X_h^0$ and we find

$$a(\mathbf{H}_h-\mathbf{v}_h^*,\mathbf{H}_h-\mathbf{v}_h^*=a(\mathbf{H}-\mathbf{v}_h^*,\mathbf{H}_h-\mathbf{v}_h^*).$$

Since $X_h^0\subset V^0$, from coerciveness we conclude

$$\begin{aligned}\|\mathbf{H}-\mathbf{H}_h\|_{H(\mathbf{curl};\Omega)} &\leq \|\mathbf{H}-\mathbf{v}_h^*\|_{H(\mathbf{curl};\Omega)}+\|\mathbf{H}_h-\mathbf{v}_h^*\|_{H(\mathbf{curl};\Omega)}\\ &\leq \big(1+\frac{c_1}{\alpha}\big)\|\mathbf{H}-\mathbf{v}_h^*\|_{H(\mathbf{curl};\Omega)}\quad\forall\,\mathbf{v}_h^*\in X_h^G.\end{aligned} \tag{34}$$

Moreover, from the inf–sup condition (29), for all $\mathbf{v}_h\in X_h$ there exists a unique $\mathbf{z}_h\in(X_h^0)^\perp$ such that $b(\mathbf{z}_{I,h},\mathbf{N}_{I,h})=b(\mathbf{H}_I-\mathbf{v}_{I,h},\mathbf{N}_{I,h})$ for all $\mathbf{N}_{I,h}\in Q_h^{I,0}$ and

$$\|\mathbf{z}_h\|_{H(\mathbf{curl};\Omega)}\leq\frac{c_2}{\beta}\|\mathbf{H}-\mathbf{v}_h\|_{H(\mathbf{curl};\Omega)}.$$

Setting $\mathbf{v}_h^*:=\mathbf{z}_h+\mathbf{v}_h$, for all $\mathbf{N}_{I,h}\in Q_h^{I,0}$ we have

$$b(\mathbf{v}_{I,h}^*,\mathbf{N}_{I,h})=b(\mathbf{H}_I,\mathbf{N}_{I,h})=b(\mathbf{H}_I,\mathbf{N}_{I,h})+\overline{c(\mathbf{N}_{I,h},\phi_I)}=G(\mathbf{N}_{I,h})\ ,$$

hence $\mathbf{v}_h^* \in X_h^G$. Furthermore,

$$\|\mathbf{H} - \mathbf{v}_h^*\|_{H(\mathbf{curl};\Omega)} \le \|\mathbf{H} - \mathbf{v}_h\|_{H(\mathbf{curl};\Omega)} + \|\mathbf{z}_h\|_{H(\mathbf{curl};\Omega)} \le \big(1 + \frac{c_2}{\beta}\big)\|\mathbf{H} - \mathbf{v}_h\|_{H(\mathbf{curl};\Omega)} \ ,$$

and (31) follows from (34).

To obtain (32) we use the inf–sup condition (29). For each $\mathbf{N}_{I,h} \in Q_h^{I,0}$ we find

$$\|\mathbf{A}_{I,h} - \mathbf{N}_{I,h}\|_{L^2(\Omega^I)} \le \frac{1}{\beta} \sup_{\mathbf{v}_h \in X_h} \frac{|b(\mathbf{v}_{I,h}, \mathbf{A}_{I,h} - \mathbf{N}_{I,h})|}{\|\mathbf{v}_h\|_{H(\mathbf{curl},\Omega)}}.$$

On the other hand

$$b(\mathbf{v}_{I,h}, \mathbf{A}_{I,h} - \mathbf{N}_{I,h}) = \overline{F(\mathbf{v}_h)} - \overline{a(\mathbf{H}_h, \mathbf{v}_h)} - b(\mathbf{v}_{I,h}, \mathbf{N}_{I,h}) = \overline{a(\mathbf{H} - \mathbf{H}_h, \mathbf{v}_h)} + b(\mathbf{v}_{I,h}, \mathbf{A}_I - \mathbf{N}_{I,h}),$$

then

$$\|\mathbf{A}_{I,h} - \mathbf{N}_{I,h}\|_{L^2(\Omega^I)} \le \frac{c_1}{\beta}\|\mathbf{H} - \mathbf{H}_h\|_{H(\mathbf{curl};\Omega)} + \frac{c_2}{\beta}\|\mathbf{A}_I - \mathbf{N}_{I,h}\|_{L^2(\Omega^I)},$$

which yields (32).

To obtain (33) we use the inf–sup condition (30) that in particular gives

$$\|\phi_{I,h}\|_h \le \frac{1}{\gamma} \sup_{\mathbf{N}_{I,h} \in Q_h} \frac{|c_h(\mathbf{N}_{I,h}, \phi_{I,h})|}{\|\mathbf{N}_{I,h}\|_{L^2(\Omega^I)}}.$$

On the other hand

$$\begin{aligned} c_h(\mathbf{N}_{I,h}, \phi_{I,h}) &= \overline{G(\mathbf{N}_{I,h})} - \overline{b(\mathbf{H}_{I,h}, \mathbf{N}_{I,h})} \\ &= \overline{b(\mathbf{H}_I, \mathbf{N}_{I,h})} + c(\mathbf{N}_{I,h}, \phi_I) - \overline{b(\mathbf{H}_{I,h}, \mathbf{N}_{I,h})} \\ &= \overline{b(\mathbf{H}_I - \mathbf{H}_{I,h}, \mathbf{N}_{I,h})} \ , \end{aligned}$$

then

$$\|\phi_{I,h}\|_h \le \frac{c_2}{\gamma}\|\mathbf{H} - \mathbf{H}_h\|_{H(\mathbf{curl},\Omega)}.$$

□

Remark 6.1. Note that $Q_h^{I,0} = \boldsymbol{\epsilon}^{-1}\,\mathbf{curl}\,X_h^I$ and that there exists $\mathbf{q}_I \in H_{0,\partial\Omega}(\mathbf{curl};\Omega^I)$ such that $\boldsymbol{\epsilon}\mathbf{A}_I = \mathbf{curl}\,\mathbf{q}_I$. Hence

$$\begin{aligned} \inf_{\mathbf{N}_{I,h} \in Q_h^{I,0}} \|\mathbf{A}_I - \mathbf{N}_{I,h}\|_{L^2(\Omega^I)} &= \inf_{\mathbf{q}_{I,h} \in X_h^I} \|\boldsymbol{\epsilon}^{-1}(\mathbf{curl}\,\mathbf{q}_I - \mathbf{curl}\,\mathbf{q}_{I,h})\|_{L^2(\Omega^I)} \\ &\le C \inf_{\mathbf{q}_{I,h} \in X_h^I} \|\mathbf{q}_I - \mathbf{q}_{I,h}\|_{H(\mathbf{curl};\Omega^I)} \ . \end{aligned}$$

△

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