A unified FEM–BEM approach for electro–magnetostatics and time-harmonic eddy-current problems

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Abstract. In this paper we consider a unified approach for solving the electro-magnetostatics problem and the eddy current problem in terms of suitable potentials. A new variational formulation is proposed, in which standard results of potential theory are used to reduce the problem in the external domain to an integral equation on the boundary of the conductor. The existence and uniqueness of the solution is proved, by showing that the associated sesquilinear form is coercive. A numerical approximation scheme, based on nodal finite elements in the conductor and boundary elements on its boundary, is devised and proved to be convergent. It is also shown that the solution of the time-harmonic eddy current problem tends to the solution of the electro-magnetostatics problem as the frequency tends to 0. The same convergence holds, uniformly with respect to the mesh size, for the finite element solutions.

Keywords: eddy-currents, electro-magnetostatics, vector and scalar potentials, well-posedness, finite elements, boundary elements.

1 Introduction

The Maxwell system of electromagnetism reads:

$$\begin{cases} \dfrac{\partial \mathcal{D}}{\partial t} + \mathcal{J} = \operatorname{curl} \mathcal{H} & \text{Maxwell–Ampère equation} \\ \dfrac{\partial \mathcal{B}}{\partial t} + \operatorname{curl} \mathcal{E} = 0 & \text{Faraday equation} \\ \operatorname{div} \mathcal{B} = 0 & \text{Gauss magnetic equation ,} \end{cases} \tag{1}$$

where $\mathcal{E}$ and $\mathcal{H}$ are the electric and magnetic field, $\mathcal{D}$ and $\mathcal{B}$ the electric and magnetic induction, respectively, and $\mathcal{J}$ is the total electric current density.

For linear materials, the constitutive relations $\mathcal{B} = \mu\,\mathcal{H}$ (where μ is the magnetic permeability) and $\mathcal{D} = \varepsilon\,\mathcal{E}$ (where ε is the electric permittivity) are assumed to hold, as well as the (generalized) Ohm's law $\mathcal{J} = \sigma\,\mathcal{E} + \mathcal{J}_e$ (where σ is the electric conductivity and $\mathcal{J}_e$ is the applied current density).

In this paper we study the Maxwell equations in the static case (the electric and magnetic inductions are time-independent) and in the time-harmonic eddy-current case (the fields depend periodically on time and the displacement current term $\frac{\partial \mathcal{D}}{\partial t}$ is neglected). In a unified presentation, this means that the electric field $\mathcal{E}$, the magnetic field $\mathcal{H}$ and the applied current density $\mathcal{J}_e$ are assumed to be of the form

$$\begin{aligned} \mathcal{E}(t,\mathbf{x}) &= \operatorname{Re}[\mathbf{E}(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{H}(t,\mathbf{x}) &= \operatorname{Re}[\mathbf{H}(\mathbf{x}) \exp(i\omega t)] \\ \mathcal{J}_e(t,\mathbf{x}) &= \operatorname{Re}[\mathbf{J}_e(\mathbf{x}) \exp(i\omega t)] \;, \end{aligned} \tag{2}$$

where $\omega \in \mathbb{R}$ is a given angular frequency, satisfying $\omega = 0$ in the static case and $\omega \neq 0$ in the eddy-current case.

Aim of this unified presentation is to furnish a method that can be used as a direct solver for some inverse problems of electroencephalography (EEG) and magnetoencephalography (MEG). In this respect, though in many papers devoted to these topics only the static case is considered ([34], [23]), recently some researches have focused on the time-harmonic case, that is a more precise model for describing the electric and magnetic activities in the brain (see [2]). Clearly, the static case is much easier to solve, as, due to the irrotationality condition, one can reduce the problem to the only determination of a scalar potential for the electric field in Ω_C (a suitable Neumann condition on Γ is the correct boundary condition to add). However, in no way that simple approach can be extended to the time-harmonic case, as irrotationality no more holds.

Another feature that is important when considering an inverse problem is that the direct solver should have a (relatively) small computational cost: we are therefore interested in presenting a method that is simple in the choice of the finite elements used for solving the problem inside the human head, and that is as cheap as possible in the external part. In this respect, a natural choice is to employ boundary elements for reducing the problem in the external air to a problem on the surface of the head.

We are thus proposing a formulation that naturally leads to a coupled FEM–BEM algorithm, and that, moreover, is stable with respect to the frequency ω: namely, the solution of the eddy-current problem converges to the solution of the static problem as ω tends to 0. This happens for both the exact solution and the discrete solution, therefore the same numerical scheme can be applied without computational problems for small value of the frequency, and even for $\omega = 0$.

Let us mention that the idea of coupling a variational approach in one region with a potential approach in another region has been proposed first by engineers (e.g., Zienkiewicz, Kelly and Bettess [35]). The mathematical analysis of this procedure has been performed for many problems, starting from the pioneering works of Brezzi, Johnson and Nédélec ([10], [25]) devoted to the Laplace operator. An important improvement is due to the work of Costabel ([13], [14]), that shows how to resort to a symmetric (or else to a positive) problem. Extensions to other problems are in [17], [19], [20], [21], [12], [11], [6] and [4], [5].

For the eddy-current problem, the first FEM–BEM couplings have been proposed by Bossavit and Vérité [9] (for the magnetic field, and using the Steklov–Poincaré operator) and Mayergoyz, Chari and Konrad [29] (for the electric field, and using special basis functions near Γ). A more recent result, for axisymmetric problems associated to the modelling of induction furnaces, is due to Bermúdez, Gómez, Muñiz and Salgado [7]. Symmetric formulations à la Costabel are due to Hiptmair [24] (unknowns: $\mathbf{E}_C$ in Ω_C, $\mathbf{H} \times \mathbf{n}$ on Γ) and Meddahi and Selgas [31] (unknowns: $\mathbf{H}_C$ in Ω_C, $\mu\mathbf{H} \cdot \mathbf{n}$ on Γ).

Finally, only for magnetostatics, an approch in terms of magnetic vector potentials has been proposed by Kuhn, Langer and Schöberl [26] and Kuhn and Steinbach [27]. With respect to the choice of potentials, our approach is close to these last ones.

Let us consider a bounded simply-connected open set $\Omega_C \subset \mathbf{R}^3$, with boundary Γ (Ω_C represents the human head). The unit outward normal vector on Γ will be denoted by $\mathbf{n}$.

We assume that the electric conductivity σ and the magnetic permeability μ are uniformly positive definite symmetric matrices in $\overline{\Omega_C}$, with bounded entries. The electric conductivity σ and the applied current density $\mathbf{J}_e$ are vanishing in $\Omega_I := \mathbb{R}^3 \setminus \overline{\Omega_C}$. Moreover, the magnetic permeability μ and the electric permittivity ε are assumed to be a positive constant in Ω_I, say $\mu_0 > 0$ and $\varepsilon_0 > 0$.

As it is well-known, in the present situation the eddy-current problem in terms of the magnetic field $\mathbf{H}$ and the electric field $\mathbf{E}_C$ reads:

$$\begin{cases} \operatorname{curl} \mathbf{E}_C + i\omega\mu_C \mathbf{H}_C = \mathbf{0} & \text{in } \Omega_C \\ \operatorname{curl} \mathbf{H}_C - \sigma \mathbf{E}_C = \mathbf{J}_e & \text{in } \Omega_C \\ \operatorname{curl} \mathbf{H}_I = \mathbf{0} & \text{in } \Omega_I \\ \operatorname{div}(\mu_0 \mathbf{H}_I) = 0 & \text{in } \Omega_I \\ \mathbf{H}_C \times \mathbf{n} - \mathbf{H}_I \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \\ \mu_C \mathbf{H}_C \cdot \mathbf{n} - \mu_0 \mathbf{H}_I \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ \mathbf{H}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \ , \end{cases} \tag{3}$$

where we have set $\mathbf{E}_C := \mathbf{E}_{|\Omega_C}$ (and similarly for Ω_I and any other restriction of function). If needed, the electric field $\mathbf{E}_I$ can be computed after having determined $\mathbf{H}_I$ and $\mathbf{E}_C$ in (3), by solving

$$\begin{cases} \operatorname{curl} \mathbf{E}_I = -i\omega\mu_0 \mathbf{H}_I & \text{in } \Omega_I \\ \operatorname{div}(\varepsilon_0 \mathbf{E}_I) = 0 & \text{in } \Omega_I \\ \mathbf{E}_I \times \mathbf{n} = \mathbf{E}_C \times \mathbf{n} & \text{on } \Gamma \\ \mathbf{E}_I(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \ . \end{cases} \tag{4}$$

In the following, we are going to consider problem (3) in terms of the Coulomb gauged vector potential formulation in Ω_C, while a magnetic scalar potential will be used in Ω_I. A successive step will reduce the problem in Ω_I to a problem on Γ, by using classical arguments of potential theory.

We will prove that this last formulation is well-posed, namely, there exists a solution for it and this solution is unique. The main point will be the proof that the associated bilinear form is coercive, so that we can apply the Lax–Milgram lemma. A further consequence will be the stability of the solution with respect to the frequency, as ω goes to 0.

A numerical algorithm based on nodal finite elements in Ω_C and standard boundary elements on Γ will be then proposed and analyzed, showing in particular that, under suitable assumptions, the numerical solution converges to the exact one as the mesh size h tends to 0, and, again, that stability in ω holds, uniformly with respect to h. In this respect, as it is well-known for nodal finite element approximation, convergence is not assured when the conductor Ω_C is a non-convex polyhedral domain; for the sake of completeness, this particular case is considered in the Appendix.

This paper is organized as follows. In Section 2 we present the strong formulation in terms of the potentials $(\mathbf{A}_C, V_C) - \psi_I$. The reduced problem, based on the trace on Γ of the scalar potential ψ_I, is derived in Section 3, where the correspondent weak formulation is also obtained. Section 4 is devoted to the existence and uniqueness theorem. The stability of the method as the frequency ω goes to 0 is proved in Section 5, whereas in Section 6 the numerical approximation is presented and analyzed.

2 The $(\mathbf{A}_C, V_C) - \psi_I$ formulation

We are looking for a magnetic vector potential $\mathbf{A}_C$, a scalar electric potential V_C and a scalar magnetic potential ψ_I such that

$$\mu_C \mathbf{H}_C = \operatorname{curl} \mathbf{A}_C \quad , \quad \mathbf{E}_C = -i\omega \mathbf{A}_C - \operatorname{grad} V_C \quad , \quad \mathbf{H}_I = \operatorname{grad} \psi_I \ . \tag{5}$$

In this way one has $\operatorname{curl} \mathbf{E}_C = -i\omega \operatorname{curl} \mathbf{A}_C = -i\omega\mu_C \mathbf{H}_C$, and therefore the Faraday equation in Ω_C is satisfied. Note that, in particular, when $\omega = 0$ one finds $\mathbf{E}_C = -\operatorname{grad} V_C$, therefore for the static case the usual formulation in terms of a scalar electric potential is recovered.

In order to have a unique vector potential $\mathbf{A}_C$, it is necessary to impose some gauge conditions: here we are considering the Coulomb gauge $\operatorname{div} \mathbf{A}_C = 0$ in Ω_C, with $\mathbf{A}_C \cdot \mathbf{n} = 0$ on Γ.

Since we would like to find a unique solution $(\mathbf{A}_C, V_C) - \psi_I$, we have also to impose a suitable condition to V_C, that is determined up to an additive constant; for instance, we require that

$$\int_{\Omega_C} V_C = 0 \ .$$

In conclusion, we are left with the problem

$$\begin{cases} \operatorname{curl}(\mu_C^{-1} \operatorname{curl} \mathbf{A}_C) & \\ \qquad\qquad + i\omega\sigma \mathbf{A}_C + \sigma \operatorname{grad} V_C = \mathbf{J}_e & \text{in } \Omega_C \\ \Delta \psi_I = 0 & \text{in } \Omega_I \\ \operatorname{div} \mathbf{A}_C = 0 & \text{in } \Omega_C \\ \mathbf{A}_C \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ (\mu_C^{-1} \operatorname{curl} \mathbf{A}_C) \times \mathbf{n} - \operatorname{grad} \psi_I \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \\ \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} - \mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ |\psi_I(\mathbf{x})| + |\operatorname{grad} \psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \\ \int_{\Omega_C} V_C = 0 \ . & \end{cases} \tag{6}$$

As it is well known (see, e.g., Morisue [32]), the Coulomb gauge condition $\operatorname{div} \mathbf{A}_C = 0$ in Ω_C can be incorporated in the Ampère equation. Introducing the constant $\mu_* > 0$, representing a suitable average in Ω_C of the entries of the matrix μ_C, one considers

$$
\begin{cases}
\operatorname{curl}(\mu_C^{-1} \operatorname{curl} \mathbf{A}_C) - \mu_*^{-1} \operatorname{grad} \operatorname{div} \mathbf{A}_C & \\
\qquad\qquad + i\omega\sigma \mathbf{A}_C + \sigma \operatorname{grad} V_C = \mathbf{J}_e & \text{in } \Omega_C \\
\Delta \psi_I = 0 & \text{in } \Omega_I \\
\operatorname{div}(i\omega\sigma \mathbf{A}_C + \sigma \operatorname{grad} V_C) = \operatorname{div} \mathbf{J}_e & \text{in } \Omega_C \\
(i\omega\sigma \mathbf{A}_C + \sigma \operatorname{grad} V_C) \cdot \mathbf{n} = \mathbf{J}_e \cdot \mathbf{n} & \text{on } \Gamma \\
\mathbf{A}_C \cdot \mathbf{n} = 0 & \text{on } \Gamma \\
(\mu_C^{-1} \operatorname{curl} \mathbf{A}_C) \times \mathbf{n} - \operatorname{grad} \psi_I \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \\
\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} - \mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n} = 0 & \text{on } \Gamma \\
|\psi_I(\mathbf{x})| + |\operatorname{grad} \psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \\
\int_{\Omega_C} V_C = 0 \ , &
\end{cases}
\tag{7}
$$

the two additional equations appearing in (7) being necessary as the modification in the Ampére equation does not assure now that the electric field $\mathbf{E}_C = -i\omega \mathbf{A}_C - \operatorname{grad} V_C$ satisfies the necessary conditions $\operatorname{div}(\sigma \mathbf{E}_C) = -\operatorname{div} \mathbf{J}_e$ in Ω_C and $\sigma \mathbf{E}_C \cdot \mathbf{n} = -\mathbf{J}_e \cdot \mathbf{n}$ on Γ.

3 The $(\mathbf{A}_C, V_C) - \psi_\Gamma$ weak formulation

To reduce the number of unknowns, we want to transform the problem for ψ_I to a problem on the interface Γ. This is possible using some known results of potential theory.

Let us introduce on $\Gamma \cup \Omega_I$ the single layer and double layer potentials

$$
\mathcal{S}_{\mathcal{L}}(\lambda)(\mathbf{x}) := \int_\Gamma \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \lambda(\mathbf{y}) dS_y \tag{8}
$$

$$
\mathcal{D}_{\mathcal{L}}(\eta)(\mathbf{x}) := \int_\Gamma \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|^3} \eta(\mathbf{y}) dS_y \tag{9}
$$

and on Γ the hypersingular integral operator

$$
\mathcal{H}(\eta)(\mathbf{x}) := -\operatorname{grad} \left(\int_\Gamma \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|^3} \eta(\mathbf{y}) dS_y \right) \cdot \mathbf{n}(\mathbf{x}) \ . \tag{10}
$$

Since the adjoint operator of $\mathcal{D}_{\mathcal{L}}(\eta)$ reads

$$
\mathcal{D}_{\mathcal{L}}{}'(\lambda)(\mathbf{x}) = \int_\Gamma \frac{(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}(\mathbf{x})}{4\pi|\mathbf{x} - \mathbf{y}|^3} \lambda(\mathbf{y}) dS_y \ , \tag{11}
$$

due to the matching condition

$$
\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} - \mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n} = 0 \ \text{ on } \Gamma
$$

from potential theory it is well-known that the trace $\psi_\Gamma := \psi_{I|\Gamma}$ satisfies

$$
\frac{1}{2}\psi_\Gamma - \mathcal{D}_{\mathcal{L}}(\psi_\Gamma) + \frac{1}{\mu_0}\mathcal{S}_{\mathcal{L}}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) = 0 \ \text{ on } \Gamma \tag{12}
$$

$$
\frac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} + \mathcal{D}_{\mathcal{L}}{}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) + \mu_0 \mathcal{H}(\psi_\Gamma) = 0 \ \text{ on } \Gamma \tag{13}
$$

(see, e.g., McLean [30]).

As a second step, we can devise a weak form of the $(\mathbf{A}_C, V_C) - \psi_\Gamma$ formulation. In fact, a standard integration by parts yields

$$
\int_\Gamma \mathbf{n} \times \operatorname{grad} \psi_I \cdot \overline{\mathbf{w}_C} = -\int_\Gamma \psi_\Gamma \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} \ .
$$

Moreover, multiplying $(7)_1$, (13) and $(7)_3$ by suitable test functions $(\mathbf{w}_C, \eta, Q_C)$, integrating in Ω_C and Γ, and integrating by parts, from the other matching condition

$$\mathbf{n} \times \mu_C^{-1} \operatorname{curl} \mathbf{A}_C - \mathbf{n} \times \operatorname{grad} \psi_I = \mathbf{0} \text{ on } \Gamma$$

and the interface equation (12) we end up with the following weak problem

$$\begin{aligned}
&\textstyle\int_{\Omega_C} (\mu_C^{-1} \operatorname{curl} \mathbf{A}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + \mu_*^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div} \overline{\mathbf{w}_C} + i\omega\sigma \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \sigma \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad + \textstyle\int_\Gamma [-\frac{1}{2}\psi_\Gamma - \mathcal{D}_\mathcal{L}(\psi_\Gamma) + \frac{1}{\mu_0} \mathcal{S}_\mathcal{L}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n})] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} = \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{w}_C} \\
&\textstyle\int_\Gamma [\frac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} + \mathcal{D}_\mathcal{L}{}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) + \mu_0 \mathcal{H}(\psi_\Gamma)] \overline{\eta} = 0 \\
&\textstyle\int_{\Omega_C} (i\omega\sigma \mathbf{A}_C \cdot \operatorname{grad} \overline{Q_C} + \sigma \operatorname{grad} V_C \cdot \operatorname{grad} \overline{Q_C}) = \int_{\Omega_C} \mathbf{J}_e \cdot \operatorname{grad} \overline{Q_C} \ .
\end{aligned}$$

We know that the hypersingular operator is coercive in the constrained space

$$H_\sharp^{1/2}(\Gamma) := \left\{ \eta \in H^{1/2}(\Gamma) \mid \int_\Gamma \eta = 0 \right\} ,$$

and that $\mathcal{D}_\mathcal{L}(1) = -\frac{1}{2}$ and $\mathcal{H}(1) = 0$ (see, e.g., McLean [30]). Therefore, it is convenient to rewrite the preceding problem replacing ψ_Γ with its projection on $H_\sharp^{1/2}(\Gamma)$, namely, $q := \psi_\Gamma - \psi_\sharp$, where the constant $\psi_\sharp$ is the mean value of ψ_Γ, i.e., $\psi_\sharp := [\text{meas}\,(\Gamma)]^{-1} \int_\Gamma \psi_\Gamma$.

In conclusion, introducing the spaces

$$W := \{\mathbf{w}_C \in H(\operatorname{curl}; \Omega_C) \mid \operatorname{div} \mathbf{w}_C \in L^2(\Omega_C) \ , \ \mathbf{w}_C \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$$

$$H_\sharp^1(\Omega_C) := \left\{ Q_C \in H^1(\Omega_C) \mid \int_{\Omega_C} Q_C = 0 \right\} ,$$

we are looking for the solution of the following coupled problem:

find $(\mathbf{A}_C, V_C, q) \in W \times H_\sharp^1(\Omega_C) \times H_\sharp^{1/2}(\Gamma)$ such that

$$\begin{aligned}
&\textstyle\int_{\Omega_C} (\mu_C^{-1} \operatorname{curl} \mathbf{A}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + \mu_*^{-1} \operatorname{div} \mathbf{A}_C \operatorname{div} \overline{\mathbf{w}_C} + i\omega\sigma \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \sigma \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad + \textstyle\int_\Gamma [-\frac{1}{2} q - \mathcal{D}_\mathcal{L}(q) + \frac{1}{\mu_0} \mathcal{S}_\mathcal{L}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n})] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} = \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{w}_C} \\
&\textstyle\int_\Gamma [\frac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} + \mathcal{D}_\mathcal{L}{}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) + \mu_0 \mathcal{H}(q)] \overline{\eta} = 0 \\
&\textstyle\int_{\Omega_C} (i\omega\sigma \mathbf{A}_C \cdot \operatorname{grad} \overline{Q_C} + \sigma \operatorname{grad} V_C \cdot \operatorname{grad} \overline{Q_C}) = \int_{\Omega_C} \mathbf{J}_e \cdot \operatorname{grad} \overline{Q_C}
\end{aligned} \tag{14}$$

for all $(\mathbf{w}_C, Q_C, \eta) \in W \times H_\sharp^1(\Omega_C) \times H_\sharp^{1/2}(\Gamma)$.

We note that, for the ease of notation, in (14) we have used the integration symbol on Γ instead of the pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$; the same notation will be used in the sequel.

4 Existence and uniqueness of the weak and strong solutions

First of all, we want to prove that from a solution to (14) we can construct a solution to the strong problem (6).

Lemma 4.1 *Suppose that $(\mathbf{A}_C, V_C, q)$ is a solution to (14). Then* $\operatorname{div} \mathbf{A}_C = 0$ *in* Ω_C.

Proof. Take $g \in L^2(\Omega_C)$ with $\int_{\Omega_C} g = 0$, and choose $\mathbf{w}_C = \operatorname{grad} \phi_C$, where ϕ_C is the solution to the Neumann problem

$$\begin{cases} \Delta \phi_C = g & \text{in } \Omega_C \\ \operatorname{grad} \phi_C \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ \int_{\Omega_C} \phi_C = 0 \ . \end{cases}$$

Clearly, we have $\mathbf{w}_C \in W$ (and $\phi_C \in H^1_\sharp(\Omega_C)$). Using this test function in (14), we find $\int_{\Omega_C} (\operatorname{div} \mathbf{A}_C) \overline{g} = 0$, therefore $\operatorname{div} \mathbf{A}_C$ is equal to a constant in Ω_C, and this constant must be 0 since $\int_{\Omega_C} \operatorname{div} \mathbf{A}_C = 0$ by the boundary condition $\mathbf{A}_C \cdot \mathbf{n} = 0$ on Γ. □

Concerning the interface equations (12) and (13) we have

Lemma 4.2 *Suppose that $(\mathbf{A}_C, V_C, q)$ is a solution to (14). Then*

$$\frac{1}{2} q - \mathcal{D}_{\mathcal{L}}(q) + \frac{1}{\mu_0} \mathcal{S}_{\mathcal{L}}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) = c_0 \quad \text{on } \Gamma \tag{15}$$

$$\frac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} + \mathcal{D}_{\mathcal{L}}{}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) + \mu_0 \mathcal{H}(q) = 0 \quad \text{on } \Gamma \ , \tag{16}$$

for a suitable constant c_0.

Proof. Since $\int_\Gamma \mathcal{H}(\eta) = 0$ for each $\eta \in H^{1/2}(\Gamma)$ (see, e.g., Nédélec [33], Theorem 3.3.2) and $\mathcal{D}_{\mathcal{L}}(1) = -\frac{1}{2}$, we have

$$\begin{aligned} &\int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} + \mathcal{D}_{\mathcal{L}}{}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) + \mu_0 \mathcal{H}(q)] \\ &\quad = \int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} + \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} \, \mathcal{D}_{\mathcal{L}}(1) + \mu_0 \mathcal{H}(q)] = 0 \ . \end{aligned}$$

Therefore equation $(14)_2$ is satisfied not only for all $\eta \in H^{1/2}_\sharp(\Gamma)$, but also for all $\eta \in H^{1/2}(\Gamma)$, and equation (16) follows at once.

Consequently, it is well known from potential theory that we also obtain (15). □

We end up our argument by showing that

Lemma 4.3 *Suppose that $(\mathbf{A}_C, V_C, q)$ is a solution to (14). In the domain Ω_I define the function $\psi_I := \mathcal{D}_{\mathcal{L}}(q) - \frac{1}{\mu_0} \mathcal{S}_{\mathcal{L}}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n})$. Then*

$$\begin{aligned} &\int_{\Omega_C} (\mu_C^{-1} \operatorname{curl} \mathbf{A}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + i\omega\sigma \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \sigma \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\ &\quad + \int_\Gamma \mathbf{n} \times \operatorname{grad} \psi_I \cdot \overline{\mathbf{w}_C} = \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{w}_C} \end{aligned} \tag{17}$$

for all $\mathbf{w}_C \in W$. Therefore,

$$\begin{cases} \operatorname{curl}(\mu_C^{-1} \operatorname{curl} \mathbf{A}_C) + i\omega\sigma \mathbf{A}_C + \sigma \operatorname{grad} V_C = \mathbf{J}_e & \text{in } \Omega_C \\ \Delta \psi_I = 0 & \text{in } \Omega_I \\ (\mu_C^{-1} \operatorname{curl} \mathbf{A}_C) \times \mathbf{n} - \operatorname{grad} \psi_I \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \\ \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} - \mu_0 \operatorname{grad} \psi_I \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ |\psi_I(\mathbf{x})| + |\operatorname{grad} \psi_I(\mathbf{x})| = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty \ . \end{cases} \tag{18}$$

Proof. Well-known results of potential theory give that ψ_I is a harmonic function with $|\psi_I(\mathbf{x})|$ and $|\operatorname{grad} \psi_I(\mathbf{x})|$ decaying at infinity as $O(|\mathbf{x}|^{-1})$. Moreover, ψ_I satisfies the trace relations

$$\psi_{I|\Gamma} = \frac{1}{2} q + \mathcal{D}_{\mathcal{L}}(q) - \frac{1}{\mu_0} \mathcal{S}_{\mathcal{L}}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) \tag{19}$$

and

$$\operatorname{grad} \psi_I \cdot \mathbf{n} = -\mathcal{H}(q) - \mu_0^{-1} \Big[-\frac{1}{2} \operatorname{curl} \mathbf{A}_C \cdot \mathbf{n} + \mathcal{D}_{\mathcal{L}}{}'(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) \Big] \tag{20}$$

(see, e.g., McLean [30]).

From (16) and (20) we see that the interface condition $(18)_4$ is satisfied. Moreover, from $(14)_1$ and (19) we find that

$$
\begin{aligned}
&\textstyle\int_{\Omega_C} (\mu_C^{-1} \operatorname{curl} \mathbf{A}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + i\omega\sigma \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \sigma \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\
&\quad \textstyle - \int_\Gamma \psi_{I|\Gamma} \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} = \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{w}_C} \ .
\end{aligned}
$$

Since we have $-\int_\Gamma \psi_{I|\Gamma} \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} = \int_\Gamma \mathbf{n} \times \operatorname{grad} \psi_I \cdot \overline{\mathbf{w}_C}$, equation (17) clearly holds.

Now integration by parts assures that $(18)_1$, as well as the interface condition $(18)_3$, are satisfied. □

Remark 4.1. The function q determined in (14) is not the trace on Γ of the harmonic scalar potential ψ_I, namely, what we have called ψ_Γ. Indeed, from (15) and (19) we see that $\psi_\Gamma = q - c_0$. If needed, the constant c_0 is easily computed from q and $\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}$, as from (15) we have

$$
c_0 = (\operatorname{meas} \Gamma)^{-1} \int_\Gamma \Big(\frac{1}{2} q - \mathcal{D}_\mathcal{L}(q) + \frac{1}{\mu_0} \mathcal{S}_\mathcal{L}(\operatorname{curl} \mathbf{A}_C \cdot \mathbf{n}) \Big) \ .
$$

We also know that $-c_0$ is the mean value of ψ_Γ on Γ. △

Now we are going to show that there exists a unique solution to (14).

Theorem 4.4 *The sesquilinear form $\mathcal{A}(\cdot,\cdot)$ associated to (14) is coercive in $W \times H^1_\sharp(\Omega_C) \times H^{1/2}_\sharp(\Gamma)$, namely, there exists a constant $\kappa_0 > 0$ (which is independent of ω) such that for each $(\mathbf{w}_C, Q_C, \eta) \in W \times H^1_\sharp(\Omega_C) \times H^{1/2}_\sharp(\Gamma)$ one has*

$$
\begin{aligned}
|\mathcal{A}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)]| \geq \kappa_0 \Big(&\textstyle\int_{\Omega_C} (|\mathbf{w}_C|^2 + |\operatorname{curl} \mathbf{w}_C|^2 + |\operatorname{div} \mathbf{w}_C|^2) + ||\eta||^2_{1/2,\Gamma} \Big) \\
&\textstyle + \kappa_0 \chi_0 \int_{\Omega_C} (|Q_C|^2 + |\operatorname{grad} Q_C|^2) \ ,
\end{aligned}
\tag{21}
$$

where the constant $\chi_0 > 0$ is equal to $|\omega|^{-1}$ in the case $0 < |\omega| < 1$, is equal to ω^{-2} in the case $|\omega| \geq 1$ and is equal to 1 in the case $\omega = 0$.

As a consequence, for each $\mathbf{J}_e \in (L^2(\Omega_C))^3$, existence and uniqueness of the solution to (14) follow from the Lax–Milgram lemma.

Proof. First of all, let us recall that the operators $\mathcal{S}_\mathcal{L}$ and $\mathcal{H}$ are continuous from $H^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ and from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$, respectively, and satisfy

$$
\int_\Gamma \mathcal{S}_\mathcal{L}(\lambda)\, \overline{\lambda} \geq \kappa_1 ||\lambda||^2_{-1/2,\Gamma}
$$

$$
\int_\Gamma \mathcal{H}(\eta)\, \overline{\eta} \geq \kappa_2 ||\eta||^2_{1/2,\Gamma}
$$

for each $\lambda \in H^{-1/2}(\Gamma)$ and $\eta \in H^{1/2}_\sharp(\Gamma)$, and moreover that the operator $\mathcal{D}_\mathcal{L}$ is continuous from $H^{1/2}(\Gamma)$ into itself (see, e.g., McLean [30]).

The proof of the coerciveness is somehow different in the cases $\omega \neq 0$ and $\omega = 0$. We start with $\omega \neq 0$, and multiply the third equation in (14) by $i\omega^{-1}$. The sesquilinear form associated to the weak problem in this case satisfies

$$
\begin{aligned}
&\mathcal{A}_{(\omega \neq 0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)] \\
&\quad \textstyle = \int_{\Omega_C} (\mu_C^{-1} \operatorname{curl} \mathbf{w}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + \mu_*^{-1} |\operatorname{div} \mathbf{w}_C|^2 + i\omega\sigma \mathbf{w}_C \cdot \overline{\mathbf{w}_C} \\
&\qquad\qquad + \sigma \operatorname{grad} Q_C \cdot \overline{\mathbf{w}_C} - \sigma \mathbf{w}_C \cdot \operatorname{grad} \overline{Q_C} + i\omega^{-1} \sigma \operatorname{grad} Q_C \cdot \operatorname{grad} \overline{Q_C}) \\
&\qquad \textstyle + \int_\Gamma [-\frac{1}{2}\eta - \mathcal{D}_\mathcal{L}(\eta)] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} + \int_\Gamma [\frac{1}{2} \operatorname{curl} \mathbf{w}_C \cdot \mathbf{n} + \mathcal{D}_\mathcal{L}{}'(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n})] \overline{\eta} \\
&\qquad \textstyle + \int_\Gamma [\frac{1}{\mu_0} \mathcal{S}_\mathcal{L}(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}) \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} + \mu_0 \mathcal{H}(\eta)\, \overline{\eta}] \ .
\end{aligned}
$$

Since

$$\sigma \,\mathrm{grad}\, Q_C \cdot \overline{\mathbf{w}_C} - \sigma \mathbf{w}_C \cdot \mathrm{grad}\, \overline{Q_C} = 2\, i\, \mathrm{Im}\, (\sigma \,\mathrm{grad}\, Q_C \cdot \overline{\mathbf{w}_C})$$

and

$$\int_\Gamma \mathcal{D}_{\mathcal{L}}{}'(\mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n})]\overline{\eta} = \int_\Gamma \mathcal{D}_{\mathcal{L}}(\overline{\eta})\, \mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n} \ ,$$

$$[-\frac{1}{2}\eta - \mathcal{D}_{\mathcal{L}}(\eta)]\, \mathrm{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n} + \mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n}\, [\frac{1}{2}\overline{\eta} + \mathcal{D}_{\mathcal{L}}(\overline{\eta})] = -2\, i\, \mathrm{Im}\, \Big([\frac{1}{2}\eta + \mathcal{D}_{\mathcal{L}}(\eta)]\, \mathrm{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n} \Big) \ ,$$

we have

$$\begin{aligned} &\mathrm{Re}\, \mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)] = \textstyle\int_{\Omega_C} (\mu^{-1}\, \mathrm{curl}\, \mathbf{w}_C \cdot \mathrm{curl}\, \overline{\mathbf{w}_C} + \mu_*^{-1} |\, \mathrm{div}\, \mathbf{w}_C|^2) \\ &\quad + \textstyle\int_\Gamma [\frac{1}{\mu_0} \mathcal{S}_{\mathcal{L}}(\mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n})\ \mathrm{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n} + \mu_0 \mathcal{H}(\eta)\, \overline{\eta}] \end{aligned}$$

$$\begin{aligned} &\mathrm{Im}\, \mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)] = \textstyle\int_{\Omega_C} (\omega\sigma \mathbf{w}_C \cdot \overline{\mathbf{w}_C} + \omega^{-1}\sigma \,\mathrm{grad}\, Q_C \cdot \mathrm{grad}\, \overline{Q_C}) \\ &\quad +2\, \mathrm{Im}\, \textstyle\int_{\Omega_C} \sigma \,\mathrm{grad}\, Q_C \cdot \overline{\mathbf{w}_C} - 2\, \mathrm{Im}\, \int_\Gamma [\frac{1}{2}\eta + \mathcal{D}_{\mathcal{L}}(\eta)]\, \mathrm{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n} \ . \end{aligned}$$

Hence,

$$\begin{aligned} &\mathrm{Re}\, \mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)] \\ &\quad \geq \kappa_3 \Big(\textstyle\int_{\Omega_C} (|\, \mathrm{curl}\, \mathbf{w}_C|^2 + |\, \mathrm{div}\, \mathbf{w}_C|^2) + ||\, \mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n}||^2_{-1/2,\Gamma} + ||\eta||^2_{1/2,\Gamma} \Big) \ . \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\Big(2\, \mathrm{Im}\, \textstyle\int_\Gamma [\frac{1}{2}\eta + \mathcal{D}_{\mathcal{L}}(\eta)]\, \mathrm{curl}\, \overline{\mathbf{w}_C} \cdot \mathbf{n} \Big)^2 \leq C_1\, ||\eta||^2_{1/2,\Gamma}\, ||\, \mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n}||^2_{-1/2,\Gamma} \\ &\quad \leq \tfrac{C_1}{2} ||\eta||^4_{1/2,\Gamma} + \tfrac{C_1}{2} ||\, \mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n}||^4_{-1/2,\Gamma} \ , \end{aligned}$$

and also, for each $\alpha > 0$,

$$\begin{aligned} &\Big(2\, \mathrm{Im}\, \textstyle\int_{\Omega_C} \sigma \,\mathrm{grad}\, Q_C \cdot \overline{\mathbf{w}_C} \Big)^2 \leq C_2 \Big(\int_{\Omega_C} |\, \mathrm{grad}\, Q_C|^2 \Big) \Big(\int_{\Omega_C} |\mathbf{w}_C|^2 \Big) \\ &\quad \leq \alpha \Big(\textstyle\int_{\Omega_C} |\, \mathrm{grad}\, Q_C|^2 \Big)^2 + C_3 \alpha^{-1} \Big(\int_{\Omega_C} |\mathbf{w}_C|^2 \Big)^2 \ . \end{aligned}$$

On the other hand, given a couple of real numbers a and b we have

$$a^2 = (a+b-b)^2 \leq 2(a+b)^2 + 2b^2 \ ,$$

therefore

$$(a+b)^2 \geq \frac{1}{2}a^2 - b^2 \ ,$$

and finally

$$(a+b)^2 \geq 2\tau (a+b)^2 \geq \tau a^2 - 2\tau b^2$$

for each $0 < \tau \leq 1/2$. Hence, for a suitable constant $C_4 > 0$, we find

$$\begin{aligned} &(\mathrm{Im}\, \mathcal{A}_{(\omega\neq 0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)])^2 \\ &\quad \geq \tau\omega^2 \kappa_4 \Big(\textstyle\int_{\Omega_C} |\mathbf{w}_C|^2 \Big)^2 + \tau\omega^{-2}\kappa_4 \Big(\int_{\Omega_C} |\, \mathrm{grad}\, Q_C|^2 \Big)^2 \\ &\qquad -4\tau\alpha \Big(\textstyle\int_{\Omega_C} |\, \mathrm{grad}\, Q_C|^2 \Big)^2 - C_4 \tau \alpha^{-1} \Big(\int_{\Omega_C} |\mathbf{w}_C|^2 \Big)^2 \\ &\qquad -C_4 \tau ||\eta||^4_{1/2,\Gamma} - C_4 \tau ||\, \mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n}||^4_{-1/2,\Gamma} \\ &\quad \geq \tau\omega^{-2}\kappa_4 \Big(\textstyle\int_{\Omega_C} |\, \mathrm{grad}\, Q_C|^2 \Big)^2 \\ &\qquad -4\tau\alpha \Big(\textstyle\int_{\Omega_C} |\, \mathrm{grad}\, Q_C|^2 \Big)^2 - C_4 \tau \alpha^{-1} \Big(\int_{\Omega_C} |\mathbf{w}_C|^2 \Big)^2 \\ &\qquad -C_4 \tau ||\eta||^4_{1/2,\Gamma} - C_4 \tau ||\, \mathrm{curl}\, \mathbf{w}_C \cdot \mathbf{n}||^4_{-1/2,\Gamma} \ . \end{aligned}$$

Let us recall now the following Poincaré's type inequalities (see, for instance, Dautray and Lions [18], Volume 2, Chapter IV, Section 7, Proposition 2; Girault and Raviart [22], Chapter I, Lemma 3.6): there exist constants $\kappa_5 > 0$ and $\kappa_6 > 0$ such that

$$\int_{\Omega_C} |\operatorname{grad} Q_C|^2 \geq \kappa_5 \int_{\Omega_C} (|\operatorname{grad} Q_C|^2 + |Q_C|^2) \quad \text{for all } Q_C \in H^1_\sharp(\Omega_C)$$

and

$$\int_{\Omega_C} (|\operatorname{curl} \mathbf{w}_C|^2 + |\operatorname{div} \mathbf{w}_C|^2) \geq \kappa_6 \int_{\Omega_C} (|\operatorname{curl} \mathbf{w}_C|^2 + |\operatorname{div} \mathbf{w}_C|^2 + |\mathbf{w}_C|^2) \quad \text{for all } \mathbf{w}_C \in W \ .$$

Coerciveness follows, for the case $\omega \neq 0$, by choosing α so small that $\omega^{-2}\kappa_4 - 4\alpha > 0$, and then τ small enough to have $\kappa_3^2 - C_4\tau > 0$ and $\kappa_3^2\kappa_6^2 - C_4\alpha^{-1}\tau > 0$. In particular, we have $\alpha = O(\omega^{-2})$, and $\tau = O(1)$ for $0 < |\omega| < 1$ and $\tau = O(\omega^{-2})$ for $|\omega| \geq 1$. Thus the constant κ_0 in (21) can be clearly chosen independent of ω, and the constant χ_0 is $O(|\omega|^{-1})$ for $0 < |\omega| < 1$ and $O(\omega^{-2})$ for $|\omega| \geq 1$.

In the case $\omega = 0$, we multiply the third equation in (14) by $\beta > 0$ (to be chosen in the sequel). The sesquilinear form associated to the weak problem in this case satisfies

$$\begin{aligned}
&\mathcal{A}_{(\omega=0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)] \\
&\quad = \int_{\Omega_C} (\mu_C^{-1} \operatorname{curl} \mathbf{w}_C \cdot \operatorname{curl} \overline{\mathbf{w}_C} + \mu_*^{-1} |\operatorname{div} \mathbf{w}_C|^2 \\
&\qquad\qquad + \sigma \operatorname{grad} Q_C \cdot \overline{\mathbf{w}_C} + \beta\sigma \operatorname{grad} Q_C \cdot \operatorname{grad} \overline{Q_C}) \\
&\qquad + \int_\Gamma [-\tfrac{1}{2}\eta - \mathcal{D}_\mathcal{L}(\eta)] \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} + \int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{w}_C \cdot \mathbf{n} + \mathcal{D}_\mathcal{L}{}'(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n})]\overline{\eta} \\
&\qquad + \int_\Gamma [\tfrac{1}{\mu_0} \mathcal{S}_\mathcal{L}(\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}) \operatorname{curl} \overline{\mathbf{w}_C} \cdot \mathbf{n} + \mu_0 \mathcal{H}(\eta)\, \overline{\eta}] \ .
\end{aligned}$$

We split $\int_{\Omega_C} \sigma \operatorname{grad} Q_C \cdot \overline{\mathbf{w}_C}$ into its real and imaginary part, and, for each $\alpha > 0$ and a suitable constant $C_5 > 0$, we end up with

$$\begin{aligned}
&\Big(\operatorname{Re} \mathcal{A}_{(\omega=0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)]\Big)^2 \\
&\quad \geq \kappa_7^2 \Big(\int_{\Omega_C} (|\operatorname{curl} \mathbf{w}_C|^2 + |\operatorname{div} \mathbf{w}_C|^2 + \beta |\operatorname{grad} Q_C|^2) + ||\operatorname{curl} \mathbf{w}_C \cdot \mathbf{n}||^2_{-1/2,\Gamma} + ||\eta||^2_{1/2,\Gamma} \Big)^2 \\
&\qquad - C_5\alpha^{-1} \Big(\int_{\Omega_C} |\operatorname{grad} Q_C|^2 \Big)^2 - \alpha \Big(\int_{\Omega_C} |\mathbf{w}_C|^2 \Big)^2
\end{aligned}$$

and $(\operatorname{Im} \mathcal{A}_{(\omega=0)}[(\mathbf{w}_C, Q_C, \eta), (\mathbf{w}_C, Q_C, \eta)])^2 \geq 0$, thus the thesis follows by choosing α so small that $\kappa_7^2\kappa_6^2 - \alpha > 0$, and then β large enough to have $\kappa_7^2\beta^2 - C_5\alpha^{-1} > 0$. □

5 Stability as ω goes to 0

We are now interested in showing that the solution to problem (14) is stable with respect to ω, namely, if we denote by $(\mathbf{A}_C^\omega, V_C^\omega, q^\omega)$ the solution to (14) correspondent to the frequency ω, we have $(\mathbf{A}_C^\omega, V_C^\omega, q^\omega) \to (\mathbf{A}_C^0, V_C^0, q^0)$ as $\omega \to 0$.

Theorem 5.1 *For $0 < |\omega| < 1$, the solutions to (14) satisfy*

$$\begin{aligned}
&\int_{\Omega_C} (|\mathbf{A}_C^\omega - \mathbf{A}_C^0|^2 + |\operatorname{curl} \mathbf{A}_C^\omega - \operatorname{curl} \mathbf{A}_C^0|^2) \leq C\,\omega^2 \\
&\int_{\Omega_C} (|V_C^\omega - V_C^0|^2 + |\operatorname{grad} V_C^\omega - \operatorname{grad} V_C^0|^2) \leq C\,\omega^2 \\
&||q^\omega - q^0||^2_{1/2,\Gamma} \leq C\,\omega^2 \ .
\end{aligned} \tag{22}$$

Proof. By linearity, the difference $(\mathbf{Z}_C, N_C, p) := (\mathbf{A}_C^\omega, V_C^\omega, q^\omega) - (\mathbf{A}_C^0, V_C^0, q^0)$ satisfies

$$\begin{aligned}
&\int_{\Omega_C}(\mu_C^{-1}\operatorname{curl}\mathbf{Z}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C}+\mu_*^{-1}\operatorname{div}\mathbf{Z}_C\operatorname{div}\overline{\mathbf{w}_C}+i\omega\sigma\mathbf{Z}_C\cdot\overline{\mathbf{w}_C}+\sigma\operatorname{grad}N_C\cdot\overline{\mathbf{w}_C})\\
&\quad+\int_\Gamma[-\tfrac12 p-\mathcal{D}_\mathcal{L}(p)+\tfrac{1}{\mu_0}\mathcal{S}_\mathcal{L}(\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n})]\operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}=-\int_{\Omega_C}i\omega\sigma\mathbf{A}_C^0\cdot\overline{\mathbf{w}_C}\\
&\int_\Gamma[\tfrac12\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n}+\mathcal{D}_\mathcal{L}{}'(\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n})+\mu_0\mathcal{H}(p)]\overline{\eta}=0\\
&\int_{\Omega_C}(-\sigma\mathbf{Z}_C\cdot\operatorname{grad}\overline{Q_C}+i\omega^{-1}\sigma\operatorname{grad}N_C\cdot\operatorname{grad}\overline{Q_C})=\int_{\Omega_C}\sigma\mathbf{Z}_C^0\cdot\operatorname{grad}\overline{Q_C}
\end{aligned}\tag{23}$$

(here, we have $\operatorname{div}\mathbf{Z}_C = 0$ by Lemma 4.1; however, we prefer to write everything in terms of the sesquilinear form $\mathcal{A}_{(\omega\neq0)}(\cdot,\cdot)$).

Therefore, from the coerciveness of the $\mathcal{A}_{(\omega\neq0)}(\cdot,\cdot)$ and taking into account that $0 < |\omega| < 1$, from (21) we obtain at once that

$$\begin{aligned}
&\int_{\Omega_C}(|\mathbf{Z}_C|^2+|\operatorname{curl}\mathbf{Z}_C|^2+|\operatorname{div}\mathbf{Z}_C|^2)+||p||^2_{1/2,\Gamma}+\chi_0\int_{\Omega_C}(|N_C|^2+|\operatorname{grad}N_C|^2)\\
&\quad\le\kappa_0^{-1}c_1\Big[|\omega|\big(\int_{\Omega_C}|\mathbf{A}_C^0|^2\big)^{1/2}\big(\int_{\Omega_C}|\mathbf{Z}_C|^2\big)^{1/2}+\big(\int_{\Omega_C}|\mathbf{Z}_C^0|^2\big)^{1/2}\big(\int_{\Omega_C}|\operatorname{grad}N_C|^2\big)^{1/2}\Big]\\
&\quad\le\kappa_0^{-1}c_2|\omega|^2\alpha_1^{-1}\int_{\Omega_C}|\mathbf{A}_C^0|^2+\kappa_0^{-1}c_2\alpha_2^{-1}\int_{\Omega_C}|\mathbf{Z}_C^0|^2+\alpha_1\int_{\Omega_C}|\mathbf{Z}_C|^2+\alpha_2\int_{\Omega_C}|\operatorname{grad}N_C|^2
\end{aligned}\tag{24}$$

for each $\alpha_1 > 0$ and $\alpha_2 > 0$. Choosing $\alpha_1 = 1/2$ and $\alpha_2 = \chi_0/2 = O(|\omega|^{-1})$ (see Theorem 4.4), we have that the left hand side in (24) is $O(|\omega|)$. In particular,

$$\int_{\Omega_C}(|N_C|^2+|\operatorname{grad}N_C|^2)=\chi_0^{-1}O(|\omega|)=O(\omega^2)$$

and

$$\int_{\Omega_C}(|\mathbf{Z}_C|^2+|\operatorname{curl}\mathbf{Z}_C|^2+|\operatorname{div}\mathbf{Z}_C|^2)+||p||^2_{1/2,\Gamma}=O(|\omega|)\ .$$

Rewriting the first two equations in (23) as

$$\begin{aligned}
&\int_{\Omega_C}(\mu_C^{-1}\operatorname{curl}\mathbf{Z}_C\cdot\operatorname{curl}\overline{\mathbf{w}_C}+\mu_*^{-1}\operatorname{div}\mathbf{Z}_C\operatorname{div}\overline{\mathbf{w}_C})\\
&\qquad+\int_\Gamma[-\tfrac12 p-\mathcal{D}_\mathcal{L}(p)+\tfrac{1}{\mu_0}\mathcal{S}_\mathcal{L}(\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n})]\operatorname{curl}\overline{\mathbf{w}_C}\cdot\mathbf{n}\\
&\quad=-\int_{\Omega_C}(i\omega\sigma\mathbf{Z}_C\cdot\overline{\mathbf{w}_C}+\sigma\operatorname{grad}N_C\cdot\overline{\mathbf{w}_C})-\int_{\Omega_C}i\omega\sigma\mathbf{A}_C^0\cdot\overline{\mathbf{w}_C}\\
&\int_\Gamma[\tfrac12\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n}+\mathcal{D}_\mathcal{L}{}'(\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n})+\mu_0\mathcal{H}(p)]\overline{\eta}=0\ ,
\end{aligned}\tag{25}$$

and proceeding as in the proof of Theorem 4.4, we obtain that the sesquilinear form at the left hand side is coercive (with coerciveness constant $K_0 > 0$ independent of ω). Thus we have

$$\begin{aligned}
&\int_{\Omega_C}(|\mathbf{Z}_C|^2+|\operatorname{curl}\mathbf{Z}_C|^2+|\operatorname{div}\mathbf{Z}_C|^2)+||\operatorname{curl}\mathbf{Z}_C\cdot\mathbf{n}||^2_{-1/2,\Gamma}+||p||^2_{1/2,\Gamma}\\
&\quad\le K_0^{-1}c_3\Big[|\omega|\int_{\Omega_C}|\mathbf{Z}_C|^2+\big(\int_{\Omega_C}|\operatorname{grad}N_C|^2\big)^{1/2}\big(\int_{\Omega_C}|\mathbf{Z}_C|^2\big)^{1/2}\\
&\qquad+|\omega|\big(\int_{\Omega_C}|\mathbf{A}_C^0|^2\big)^{1/2}\big(\int_{\Omega_C}|\mathbf{Z}_C|^2\big)^{1/2}\Big]\\
&\quad=O(\omega^2)+O(|\omega|)\big(\int_{\Omega_C}|\mathbf{Z}_C|^2\big)^{1/2}\le O(\omega^2)+\tfrac12\int_{\Omega_C}|\mathbf{Z}_C|^2\ .
\end{aligned}$$

The result thus follows. □

We note that the asymptotic behaviour determined here above is in agreement with that obtained, by formal asymptotic expansion, in Ammari, Buffa and Nédélec [3] for the electric and the magnetic fields.

6 Numerical approximation

In this section we deal with the numerical approximation of problem (14). In the sequel we assume that Ω_C is a Lipschitz polyhedral domain, and that $\mathcal{T}_{C,h}$ and $\mathcal{T}_{\Gamma,h}$ are two regular families of triangulations of Ω_C and Γ, respectively. For the sake of simplicity, we suppose that each element K of $\mathcal{T}_{C,h}$ is a tetrahedron; however, the results below also hold for hexahedral elements. Let us note that the mesh induced on Γ by $\mathcal{T}_{C,h}$ is not needed to coincide with $\mathcal{T}_{\Gamma,h}$.

Let $\mathbf{P}_r$, $r \geq 1$, be the space of polynomials of degree less than or equal to r. We will employ the discrete spaces given by nodal finite elements:

$$W_h^r := \{\mathbf{w}_{C,h} \in (C^0(\overline{\Omega_C}))^3 \mid \mathbf{w}_{C,h|K} \in (\mathbf{P}_r)^3 \ \forall\, K \in \mathcal{T}_h \ , \ \mathbf{w}_{C,h} \cdot \mathbf{n} = 0 \text{ on } \Gamma\} \ ,$$

$$X_h^s := \{Q_{C,h} \in C^0(\overline{\Omega_C}) \mid Q_{C,h|K} \in \mathbf{P}_s \ \forall\, K \in \mathcal{T}_{C,h} \ , \ \textstyle\int_{\Omega_C} Q_{C,h} = 0\} \ ,$$

and

$$Y_{\Gamma,h}^t := \{\eta_h \in C^0(\overline{\Gamma}) \mid \eta_{h|T} \in \mathbf{P}_t \ \forall\, T \in \mathcal{T}_{\Gamma,h} \ , \ \textstyle\int_\Gamma \eta_h = 0\} \ ,$$

Clearly, for each $r \geq 1$, $s \geq 1$ and $t \geq 1$ we have $W_h^r \subset W$, $X_h^s \subset H_\sharp^1(\Omega_C)$ and $Y_{\Gamma,h}^t \subset H_\sharp^{1/2}(\Gamma)$, therefore we are considering a conforming finite element approximation.

The discrete problem is given by

$$\begin{aligned}
&\text{find } (\mathbf{A}_{C,h}, V_{C,h}, q_h) \in W_h^r \times X_h^s \times Y_{\Gamma,h}^t \text{ such that}\\
&\textstyle\int_{\Omega_C} (\mu_C^{-1} \operatorname{curl} \mathbf{A}_{C,h} \cdot \operatorname{curl} \overline{\mathbf{w}_{C,h}} + \mu_*^{-1} \operatorname{div} \mathbf{A}_{C,h} \operatorname{div} \overline{\mathbf{w}_{C,h}}\\
&\qquad + i\omega\sigma \mathbf{A}_{C,h} \cdot \overline{\mathbf{w}_{C,h}} + \sigma \operatorname{grad} V_{C,h} \cdot \overline{\mathbf{w}_{C,h}})\\
&\quad + \textstyle\int_\Gamma [-\frac{1}{2} q_h - \mathcal{D}_\mathcal{L}(q_h) + \frac{1}{\mu_0} \mathcal{S}_\mathcal{L}(\operatorname{curl} \mathbf{A}_{C,h} \cdot \mathbf{n})] \operatorname{curl} \overline{\mathbf{w}_{C,h}} \cdot \mathbf{n} = \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{w}_{C,h}}\\
&\textstyle\int_\Gamma [\frac{1}{2} \operatorname{curl} \mathbf{A}_{C,h} \cdot \mathbf{n} + \mathcal{D}_\mathcal{L}{}'(\operatorname{curl} \mathbf{A}_{C,h} \cdot \mathbf{n}) + \mu_0 \mathcal{H}(q_h)] \overline{\eta_h} = 0\\
&\textstyle\int_{\Omega_C} (i\omega\sigma \mathbf{A}_{C,h} \cdot \operatorname{grad} \overline{Q_{C,h}} + \sigma \operatorname{grad} V_{C,h} \cdot \operatorname{grad} \overline{Q_{C,h}}) = \int_{\Omega_C} \mathbf{J}_e \cdot \operatorname{grad} \overline{Q_{C,h}}\\
&\text{for all } (\mathbf{w}_{C,h}, Q_{C,h}, \eta_h) \in W_h^r \times X_h^s \times Y_{\Gamma,h}^t \ .
\end{aligned} \tag{26}$$

Existence and uniqueness of the discrete solution follow by the Lax–Milgram lemma, applied in $W_h^r \times X_h^s \times Y_{\Gamma,h}^t$.

We also have

Theorem 6.1 *Assume that Ω_C is a convex polyhedron, or else that the solution $(\mathbf{A}_C, V_C, q)$ is smooth enough. Then the discrete solution $(\mathbf{A}_{C,h}, V_{C,h}, q_h)$ converges in $W \times H_\sharp^1(\Omega_C) \times H_\sharp^{1/2}(\Gamma)$ to the exact solution $(\mathbf{A}_C, V_C, q)$.*

Proof. Let us start noting that, as proved in Lemma 4.2, $(\mathbf{A}_C, V_C, q)$ and $(\mathbf{A}_{C,h}, V_{C,h}, q_h)$ are solutions to problems (14) and (26), respectively, also for all test functions Q_C and η, $Q_{C,h}$ and η_h with non-vanishing mean value, so that finite element interpolants can be used as test functions.

Then, if the solution $(\mathbf{A}_C, V_C, q)$ is smooth enough, the result follows by applying Céa lemma and standard interpolation results.

If the domain Ω_C is convex, it is known (see [16]) that smooth functions with vanishing normal component are dense in W, and the same arguments can be applied. □

Remark 6.1. If Ω_C is a non-convex polyhedral domain, it can happen that the solution $\mathbf{A}_C$ is non-smooth, namely, not even an element of $(H^1(\Omega_C))^3$ (see [15]). Therefore in that case a convergence result cannot hold, as the finite element space W_h^r is a closed proper subspace of $(H^1(\Omega_C))^3$.

For non-convex domains, an alternative approch is presented in the Appendix. △

Concerning the behaviour with respect to the frequency ω, in the discrete case we can repeat the proof of Theorem 5.1 and obtain (with obvious notation):

Theorem 6.2 *For $0 < |\omega| < 1$, the solutions to (26) satisfy*

$$\begin{aligned}
&\int_{\Omega_C} (|\mathbf{A}^\omega_{C,h} - \mathbf{A}^0_{C,h}|^2 + |\operatorname{curl} \mathbf{A}^\omega_{C,h} - \operatorname{curl} \mathbf{A}^0_{C,h}|^2 + |\operatorname{div} \mathbf{A}^\omega_{C,h} - \operatorname{div} \mathbf{A}^0_{C,h}|^2) \leq C\,\omega^2 \\
&\int_{\Omega_C} (|V^\omega_{C,h} - V^0_{C,h}|^2 + |\operatorname{grad} V^\omega_{C,h} - \operatorname{grad} V^0_{C,h}|^2) \leq C\,\omega^2 \\
&||q^\omega_h - q^0_h||^2_{1/2,\Gamma} \leq C\,\omega^2 \ ,
\end{aligned} \tag{27}$$

where the constant $C > 0$ does not depend on h.

An important point of the above result is that the behaviour in ω is uniform with respect to h; it is not evident that this is true for other finite element approximation schemes, as it is not always possible to show that the associated sesquilinear form is coercive uniformly with respect to ω (for our approach, this has been proved in Theorem 4.4).

This uniform behaviour is a desirable feature if one wants to test the sensitivity of the electromagnetic model with respect to ω: in particular, how far one can push the static approximation (i.e., the choice $\omega = 0$) in real life problems.

7 Appendix

As we noted in Remark 6.1, if the conductor Ω_C is a polyhedral non-convex set it can happen that the convergence of the finite element approximation does not hold. Therefore, it is suitable to follow an alternative approach.

We start by recalling that, when the conductor has a complex geometry, it is usual to enclose it into a "simpler" set, and in this new region to look for a vector potential of the magnetic induction. When the enlarged set is simply-connected, this approach is generally called the $(\mathbf{A}, V) - \mathbf{A} - \psi$ formulation (see Leonard and Rodger [28], Bíró and Preis [8]); the analysis of this method has been performed by Acevedo and Rodríguez [1].

In our case, we assume that the conductor $\overline{\Omega_C}$ is included into a *polyhedral convex* bounded open set Ω_A, as small as possible. Setting now $\Omega_I := \mathbb{R}^3 \setminus \overline{\Omega_A}$, $\Gamma_A := \partial\Omega_A$, and

$$W_A := \{\mathbf{w} \in H(\operatorname{curl}; \Omega_A) \,|\, \operatorname{div} \mathbf{w} \in L^2(\Omega_A) \ , \ \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \Gamma_A\} \ ,$$

the weak formulation reads:

find $(\mathbf{A}, V_C, q) \in W_A \times H^1_\sharp(\Omega_C) \times H^{1/2}_\sharp(\Gamma_A)$ such that

$$\begin{aligned}
&\int_{\Omega_A} (\mu^{-1} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \overline{\mathbf{w}} + \mu_*^{-1} \operatorname{div} \mathbf{A} \operatorname{div} \overline{\mathbf{w}}) + \int_{\Omega_C} (i\omega\sigma \mathbf{A}_C \cdot \overline{\mathbf{w}_C} + \sigma \operatorname{grad} V_C \cdot \overline{\mathbf{w}_C}) \\
&\qquad + \int_\Gamma [-\tfrac{1}{2} q - \mathcal{D}_\mathcal{L}(q) + \tfrac{1}{\mu_0} \mathcal{S}_\mathcal{L}(\operatorname{curl} \mathbf{A} \cdot \mathbf{n})] \operatorname{curl} \overline{\mathbf{w}} \cdot \mathbf{n} = \int_{\Omega_C} \mathbf{J}_e \cdot \overline{\mathbf{w}_C} \\
&\int_\Gamma [\tfrac{1}{2} \operatorname{curl} \mathbf{A} \cdot \mathbf{n} + \mathcal{D}_\mathcal{L}{}'(\operatorname{curl} \mathbf{A} \cdot \mathbf{n}) + \mu_0 \mathcal{H}(q)] \overline{\eta} = 0 \\
&\int_{\Omega_C} (i\omega\sigma \mathbf{A}_C \cdot \operatorname{grad} \overline{Q_C} + \sigma \operatorname{grad} V_C \cdot \operatorname{grad} \overline{Q_C}) = \int_{\Omega_C} \mathbf{J}_e \cdot \operatorname{grad} \overline{Q_C}
\end{aligned} \tag{28}$$

for all $(\mathbf{w}, Q_C, \eta) \in W_A \times H^1_\sharp(\Omega_C) \times H^{1/2}_\sharp(\Gamma_A)$.

The results presented in Section 4, as well as those in Sections 5 and 6, can be easily obtained also for this formulation, with essentially the same proofs. In particular, the finite element approximation scheme converges, as stated in Theorem 6.1, since the domain Ω_A is convex.

The determination of a precise order of convergence needs the knowledge of the regularity of the solution: as usual, if $(\mathbf{A}, V_C, q) \in H^{k+1}(\Omega_A) \times H^{k+1}(\Omega_C) \times H^{k+1/2}(\Gamma_A)$, where the integer $k \geq 1$ is equal to $r = s = t$, the degree of polynomial approximation, we have

$$\Big(\int_{\Omega_A} (|\mathbf{A} - \mathbf{A}_h|^2 + |\operatorname{curl}(\mathbf{A} - \mathbf{A}_h)|^2 + |\operatorname{div}(\mathbf{A} - \mathbf{A}_h)|^2) + \int_{\Omega_C} (|V_C - V_{C,h}|^2 + |\operatorname{grad}(V_C - V_{C,h})|^2) + ||q - q_h||^2_{1/2,\Gamma_A} \Big)^{1/2} \leq C h^k \ .$$

On the other hand, a typical assumption for the conductivity σ in the head is that it is a piecewise smooth (but not globally continuous) positive definite symmetric matrix. In this case, it is not clear if the solution is regular as required above. In general, one could expect that the solution belongs to $H^{1+\gamma}(\Omega_A) \times H^{1+\gamma}(\Omega_C) \times H^{1/2+\gamma}(\Gamma_A)$ for some γ with $0 < \gamma < 1/2$; however, we have not a proof of this result.

It is worthy to note that the same difficulty arises if one assumes $\omega = 0$, namely, one just considers the electrostatics problem. In this case one has to approximate the solution V_C to

$$\begin{cases} \operatorname{div}(\sigma \operatorname{grad} V_C) = \operatorname{div} \mathbf{J}_e & \text{in } \Omega_C \\ \sigma \operatorname{grad} V_C \cdot \mathbf{n} = \mathbf{J}_e \cdot \mathbf{n} & \text{on } \Gamma \\ \int_{\Omega_C} V_C = 0 \ , \end{cases}$$

and the regularity of V_C is not easily determined for a piecewise smooth positive definite symmetric matrix σ. Therefore, also in this case the rate of convergence of a finite element approximation scheme is not easily determined.

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