A formulation of the eddy-current problem in the presence of electric ports

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Abstract

The time-harmonic eddy current problem with either voltage or current intensity excitation is considered. We propose and analyze a new finite element approximation of the problem, based on a weak formulation where the main unknowns are the electric field in the conductor, a scalar magnetic potential in the insulator and, for the voltage excitation problem, the current intensity. The finite element approximation uses edge elements for the electric field and nodal elements for the scalar magnetic potential, and an optimal error estimate is proved. Some numerical results illustrating the performance of the method are also presented.

1 Introduction

This paper deals with a new formulation and finite element approximation of the time-harmonic eddy current model in a bounded domain with non-local boundary conditions. This problem arises when the full field equations are coupled with circuits. On the common interface between the two models, the boundary data for the domain where the eddy current model is considered are either input current intensities or voltages. (See, e.g., [11], [15], [18], [19].)

The computational domain will be a simply-connected bounded open set $\Omega \subset \mathbb{R}^3$, with a connected and Lipschitz boundary $\partial\Omega$. It is split into two Lipschitz subdomains, a conducting region Ω_C and a non-conducting region $\Omega_D = \Omega \setminus \overline{\Omega}_C$; the latter is assumed to be non-empty and connected. The conducting region Ω_C is assumed to be simply-connected, and it is not strictly contained in Ω, i.e., $\partial\Omega \cap \partial\Omega_C \neq \emptyset$. (For a more general geometrical situation, see Remark 3.2.) We shall denote the interface between the two regions by Γ, and the different parts of the boundary $\partial\Omega$ by $\Gamma_C = \partial\Omega \cap \partial\Omega_C$ and $\Gamma_D = \partial\Omega \cap \partial\Omega_D$. Moreover, we will suppose that $\Gamma_C = \Gamma_E \cup \Gamma_J$, where Γ_E and Γ_J are two disjoint and connected surfaces on Γ_C ('electric ports'). Therefore, with these notations, we have $\partial\Omega_C = \Gamma_E \cup \Gamma_J \cup \Gamma$, $\partial\Omega_D = \Gamma_D \cup \Gamma$ (see Figure 1).

The equations of the eddy-current problem consist of Faraday's law

$$\mathbf{curl}\,\mathbf{E} = -i\omega\boldsymbol{\mu}\mathbf{H} \quad \text{in } \Omega\,, \tag{1}$$

and Ampère's law

$$\mathbf{curl}\,\mathbf{H} = \boldsymbol{\sigma}\mathbf{E} + \mathbf{J} \quad \text{in } \Omega\,, \tag{2}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric and the magnetic field, respectively, $\mathbf{J}$ is a generator current and $\omega \neq 0$ is a given angular frequency. The magnetic permeability $\boldsymbol{\mu}$ is assumed to be a symmetric tensor, uniformly positive definite in Ω. Concerning the electric conductivity $\boldsymbol{\sigma}$, the same

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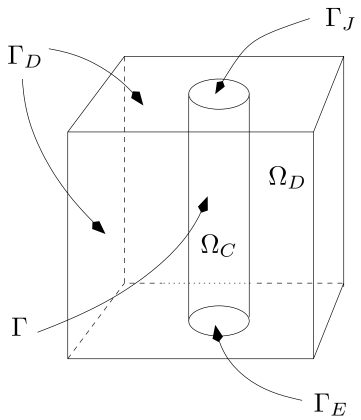

Figure 1: The computational domain

assumption holds for $\boldsymbol{\sigma}_{|\Omega_C}$, while $\boldsymbol{\sigma}_{|\Omega_D} \equiv \mathbf{0}$ as Ω_D is a non-conducting medium. Equations (1)–(2) do not completely determine the electric field in Ω_D and it is necessary to demand the condition

$$\operatorname{div}(\boldsymbol{\varepsilon}\mathbf{E}_{|\Omega_D}) = 0, \tag{3}$$

where $\boldsymbol{\varepsilon}$ is the electric permeability, assumed to be a symmetric tensor, uniformly positive definite in Ω_D.

Concerning the boundary conditions, we want to model the electromagnetic fields in the case of an electric current passing along the 'cylinder' Ω_C, and impose this electric current as a certain given intensity on Γ_J, or as a potential difference between Γ_E and Γ_J. So, following [11] we impose the following boundary conditions

$$\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{4}$$
$$\mathbf{E}_C \times \mathbf{n}_C = \mathbf{0} \quad \text{on} \quad \Gamma_C = \Gamma_E \cup \Gamma_J, \tag{5}$$
$$\boldsymbol{\varepsilon}\mathbf{E}_D \cdot \mathbf{n}_D = 0 \quad \text{on} \quad \Gamma_D, \tag{6}$$

where $\mathbf{E}_S$ and $\mathbf{H}_S$ denote $\mathbf{E}_{|\Omega_S}$ and $\mathbf{H}_{|\Omega_S}$ respectively, $S = C, D$, and $\mathbf{n}_C$ and $\mathbf{n}_D$ denote the unit outward normal vectors to Ω_C and Ω_D, respectively. When considering the boundary of the whole domain Ω the unit outward normal vector is denoted by $\mathbf{n}$.

Moreover we impose either the current intensity traversing Γ_J

$$\int_{\Gamma_J} \mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{n}_C = I, \tag{7}$$

or a potential difference. In this respect, since $\boldsymbol{\mu}\mathbf{H} \cdot \mathbf{n} = 0$ on $\partial\Omega$, then $\operatorname{div}_\tau(\mathbf{E} \times \mathbf{n}) = 0$ and, since $\partial\Omega$ is simply connected, there exists a surface potential v such that $\mathbf{E} \times \mathbf{n} = \operatorname{grad} v \times \mathbf{n}$ on $\partial\Omega$. Due to (5) the function v must be constant on each connected component Γ_E and Γ_J. Moreover, since v is defined up to a constant, we can take it equal to zero on Γ_E. The voltage $V \in \mathbb{C}$ will be the constant value on Γ_J of the surface electric potential v that is null on Γ_E:

$$\mathbf{E} \times \mathbf{n} = \operatorname{grad} v \times \mathbf{n} \quad \text{on } \partial\Omega \text{ with } v_{|\Gamma_J} = V \text{ and } v_{|\Gamma_E} = 0\,. \tag{8}$$

Remark 1.1 The set of boundary conditions (4)–(6) allows us to assign either the current intensity or the voltage. This is not the case for other boundary conditions such as

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega \tag{9}$$

or

$$\begin{aligned} \mathbf{E}_C \times \mathbf{n}_C = \mathbf{0} \quad &\text{on} \quad \Gamma_C = \Gamma_E \cup \Gamma_J\,, \\ \boldsymbol{\varepsilon}\mathbf{E}_D \cdot \mathbf{n}_D = 0 \quad &\text{on} \quad \Gamma_D\,, \\ \mathbf{H}_D \times \mathbf{n}_D = \mathbf{0} \quad &\text{on} \quad \Gamma_D\,. \end{aligned} \tag{10}$$

In fact, the solution of the eddy current problem (1)–(3) with boundary conditions (9) or (10) is unique (see [5]); if $\mathbf{J} = \mathbf{0}$ it is the null solution. □

System (1)–(7), and its finite element approximation has been studied in [9]. The problem is formulated in terms of the magnetic field and the input current intensity is imposed by means of Lagrange multipliers. In [10] and in [20] the problem is described in terms of a current vector potential and a magnetic scalar potential, using the so-called $\mathbf{T} - \mathbf{T}_0 - \phi$ formulation. We want also to mention the paper [8], where both problems of voltage and current excitation have been studied in terms of the electric field, but in a computational domain which reduces to the only conductor Ω_C.

This paper deals with a new finite element approximation of system (1)–(6) either with assigned current intensity or assigned voltage. A weak formulation of the problem (1)–(6) is given considering as main unknowns the electric field in the conductor and the magnetic field in the insulator. The latter is decomposed as the sum of the gradient of a function in $H^1(\Omega_D)$ plus a harmonic field. When the input current intensity is given, this harmonic field is univocally determined, hence the unknowns of the problem reduce to the electric field in the conductor and a scalar magnetic potential in the insulator. On the other hand, when the voltage is given the unknowns of the problem are the electric field in the conductor, a scalar magnetic potential in the insulator and the current intensity. For the finite element approximation, the harmonic field is replaced by the generalized gradient of a piecewise linear function that has a jump of height 1 across a particular surface in Ω_D.

The plan of the paper is as follows: Section 2 is devoted to notation and to recall the orthogonal decomposition of $(L^2(\Omega_D))^3$ that is a key point for the formulation of the problem in the insulator. In Section 3 we obtain the weak formulation of the voltage excitation problem and the current excitation problem. In Section 4 we prove the existence and uniqueness of solution of both problems. In Section 5, we introduce the finite element discretization and obtain the error estimates. Finally, in Section 6 we report some numerical results for two different problems: a test case with a known analytical solution and an application to a metallurgical furnace.

2 Notation and preliminaries

As usual, we denote by $H^s(\Omega)$, $s \geq 0$, the Sobolev space of (classes of equivalence) of real or complex functions belonging to $L^2(\Omega)$ together with all their distributional derivatives of order less than or equal to s. It is well known that the trace space of $H^1(\Omega)$ over $\partial\Omega$ is the Sobolev space $H^{1/2}(\partial\Omega)$. The space $H^{-1/2}(\partial\Omega)$ denotes the dual space of $H^{1/2}(\partial\Omega)$.

The space $\boldsymbol{H}(\mathbf{curl}\,;\Omega)$ (respectively $\boldsymbol{H}(\mathrm{div}\,;\Omega)$) indicates the set of real or complex functions $\mathbf{v} \in (L^2(\Omega))^3$ such that $\mathbf{curl}\,\mathbf{v} \in (L^2(\Omega))^3$ (respectively $\mathrm{div}\,\mathbf{v} \in L^2(\Omega)$). By $\boldsymbol{H}^0(\mathbf{curl}\,;\Omega)$ we denote the set of functions belonging to $\boldsymbol{H}(\mathbf{curl}\,;\Omega)$ with vanishing curl in Ω. Given a certain subset $\Lambda \subset \partial\Omega$, we denote by $\boldsymbol{H}_{0,\Lambda}(\mathbf{curl}\,;\Omega)$ the space of functions $\mathbf{v} \in \boldsymbol{H}(\mathbf{curl}\,;\Omega)$ with vanishing tangential trace on Λ, namely, $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Λ. In particular $\boldsymbol{H}_0(\mathbf{curl}\,;\Omega) := \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl}\,;\Omega)$. Similarly we denote by $\boldsymbol{H}_{0,\Lambda}(\mathrm{div}\,;\Omega)$ the space of functions $\mathbf{v} \in \boldsymbol{H}(\mathrm{div}\,;\Omega)$ such that $\mathbf{v} \cdot \mathbf{n}$ is null on Λ; in particular $\boldsymbol{H}_0(\mathrm{div}\,;\Omega) = \boldsymbol{H}_{0,\partial\Omega}(\mathrm{div}\,;\Omega)$.

We recall the trace space for $\boldsymbol{H}(\mathbf{curl}\,;\Omega)$:

$$\boldsymbol{H}^{-1/2}(\mathrm{div}_{\tau}; \partial\Omega) := \{\mathbf{v} \times \mathbf{n} \mid \mathbf{v} \in \boldsymbol{H}(\mathbf{curl}\,;\Omega)\}$$

(see, e.g., [14], [1], [12], [13]). For easy of reading, in the sequel we always express duality pairings by (surface) integrals. In particular given $\mathbf{v}_C \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$ and $\mathbf{w}_D \in \boldsymbol{H}(\mathbf{curl}\,;\Omega_D)$

$$\int_{\Gamma} \mathbf{v}_C \times \mathbf{n}_C \cdot \overline{\mathbf{w}}_D := \int_{\Omega_C} (\mathbf{v}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}} - \mathbf{curl}\,\mathbf{v}_C \cdot \overline{\mathbf{w}}) \,, \tag{11}$$

where $\mathbf{w}$ is any continuous extension of the trace of $\mathbf{w}_D$, defined on $\partial\Omega_D$, to $\mathbb{R}^3 \backslash \overline{\Omega}_D$. We notice that the right hand side of (11) does not depend on the extension $\mathbf{w}$ considered, since, given any other extension, $\mathbf{w}_*$, we have $(\mathbf{w}-\mathbf{w}_*)_{|\partial\Omega_C} \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}\,;\Omega_C)$, and thus using Proposition 3.5 in [16], we know that

$$\int_{\Omega_C} (\mathbf{v}_C \cdot \mathbf{curl}\,(\overline{\mathbf{w}-\mathbf{w}_*}) - \mathbf{curl}\,\mathbf{v}_C \cdot (\overline{\mathbf{w}-\mathbf{w}_*})) = 0\,.$$

We introduce the following space of Neumann harmonic fields:

$$\mathcal{H}_{\boldsymbol{\mu}}(\Omega_D) := \{\mathbf{v}_D \in (L^2(\Omega_D))^3 |\, \mathbf{curl}\,\mathbf{v}_D = \mathbf{0},\, \mathrm{div}\,(\boldsymbol{\mu}\mathbf{v}_D) = 0,\, \boldsymbol{\mu}\mathbf{v}_D \cdot \mathbf{n}_D = 0 \text{ on } \partial\Omega_D\}\,.$$

Since the conductor Ω_C touches the boundary of the computational domain in the two contacts, the non-conducting region Ω_D is not simply connected; its first Betti number, that coincides with the dimension of the space $\mathcal{H}_{\boldsymbol{\mu}}(\Omega_D)$ (see, e.g., [6]), is equal to one. Moreover, there exists one 'cutting' surface Σ (the interior of a compact and connected Lipschitz manifold $\overline{\Sigma}$, with boundary $\partial\Sigma$) such that $\Sigma \subset \Omega_D$, $\partial\Sigma \subset \partial\Omega_D$ and the open set $\Omega_D \backslash \Sigma$ is simply connected. Let z denote the solution of the following elliptic problem:

$$\begin{cases} \mathrm{div}\,(\boldsymbol{\mu}\,\mathrm{grad}\,z) = 0 & \text{in } \Omega_D\backslash\Sigma\,, \\ \boldsymbol{\mu}\,\mathrm{grad}\,z \cdot \mathbf{n}_D = 0 & \text{on } \partial\Omega_D \setminus \partial\Sigma\,, \\ [z]_\Sigma = 1\,, & \\ [\boldsymbol{\mu}\,\mathrm{grad}\,z \cdot \mathbf{n}_D]_\Sigma = 0\,, & \end{cases} \tag{12}$$

where $[z]_\Sigma$ denotes the jump of z across Σ. Then $\boldsymbol{\varrho}_D := \widetilde{\mathrm{grad}}\,z$ is a (basis) function of $\mathcal{H}_{\boldsymbol{\mu}}(\Omega_D)$, $\widetilde{\mathrm{grad}}\,z$ denoting the extension to Ω_D of $\mathrm{grad}\,z$ computed in $\Omega_D\backslash\Sigma$. Moreover, we can assume that $\boldsymbol{\varrho}_D$ is chosen such that $\int_{\partial\Gamma_J} \boldsymbol{\varrho}_D \cdot \mathbf{t} = 1$, where $\mathbf{t}$ is the tangential vector counterclockwise oriented with respect to $\mathbf{n}_C$ on Γ_J.

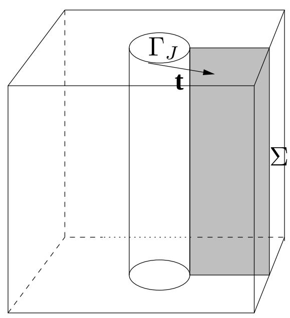

Figure 2: The cutting surface

Any given vector function $\mathbf{v}_D \in (L^2(\Omega_D))^3$ can be decomposed into the following sum (see, e.g., [3]):

$$\mathbf{v}_D = \boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{q}_D + \mathrm{grad}\,\psi_D + \alpha\boldsymbol{\varrho}_D\,,$$

and this decomposition is $L^2(\boldsymbol{\mu};\Omega_D)$-orthogonal, namely,

$$\int_{\Omega_D} \boldsymbol{\mu}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{q}_D)\cdot \mathrm{grad}\,\psi_D = 0\ ,\ \int_{\Omega_D} \boldsymbol{\mu}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{q}_D)\cdot \boldsymbol{\varrho}_D = 0\ ,\ \int_{\Omega_D} \boldsymbol{\mu}\,\mathrm{grad}\,\psi_D \cdot \boldsymbol{\varrho}_D = 0\,.$$

Here $\mathbf{q}_D \in \boldsymbol{H}(\mathbf{curl}\,;\Omega_D)$ is the solution of

$$\begin{cases} \mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{q}_D) = \mathbf{curl}\,\mathbf{v}_D & \text{in } \Omega_D\,, \\ \mathrm{div}\,\mathbf{q}_D = 0 & \text{in } \Omega_D\,, \\ \mathbf{q}_D \times \mathbf{n}_D = 0 & \text{on } \partial\Omega_D\,, \end{cases}$$

(notice that this problem has a unique solution since $\partial\Omega_D$ is connected) and $\psi_D \in H^1(\Omega_D)/\mathbb{C}$ is the solution of

$$\begin{cases} \operatorname{div}(\boldsymbol{\mu}\operatorname{grad}\psi_D) = \operatorname{div}(\boldsymbol{\mu}\,\mathbf{v}_D) & \text{in } \Omega_D\,, \\ \boldsymbol{\mu}\operatorname{grad}\psi_D\cdot\mathbf{n}_D = \boldsymbol{\mu}\,\mathbf{v}_D\cdot\mathbf{n}_D & \text{on } \partial\Omega_D\,. \end{cases}$$

If $\mathbf{curl}\,\mathbf{v}_D = \mathbf{0}$ then $\mathbf{q}_D = \mathbf{0}$, hence

$$\mathbf{v}_D = \operatorname{grad}\psi_D + \alpha\boldsymbol{\varrho}_D\,, \tag{13}$$

and $\int_{\partial\Gamma_J}\mathbf{v}_D\cdot\mathbf{t} = \alpha$.

3 Coupled $\mathbf{E}_C/\mathbf{H}_D$ formulation

Our aim is to introduce and analyze a weak formulation of system (1)–(6) with assigned current intensity or voltage, where the main unknowns are the electric field in the conductor $\mathbf{E}_C$ and the magnetic field in the insulator $\mathbf{H}_D$. We assume that $\mathbf{J} \in (L^2(\Omega))^3$ and, for the sake of simplicity, in the sequel we also assume that $\mathbf{J}_{|\Omega_D} = \mathbf{0}$. This means that $\mathbf{H}_D = \operatorname{grad}\psi_D + K\boldsymbol{\varrho}_D$ with $\psi_D \in H^1(\Omega_D)$ and $K \in \mathbb{C}$.

Remark 3.1 Notice that from Stokes Theorem

$$I = \int_{\Gamma_J}\mathbf{curl}\,\mathbf{H}_C\cdot\mathbf{n}_C = \int_{\partial\Gamma_J}\mathbf{H}_C\cdot\mathbf{t} = \int_{\partial\Gamma_J}\mathbf{H}_D\cdot\mathbf{t} = K\,.$$

This means that, when the current intensity is assigned, the main unknowns in our formulation are in fact $\mathbf{E}_C$ and the magnetic scalar potential ψ_D. □

Computing the magnetic field from Faraday's equation (1) and inserting it in Ampère's law (2), we obtain

$$\mathbf{curl}\,(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C) + i\omega\boldsymbol{\sigma}\mathbf{E}_C = -i\omega\mathbf{J}_C.$$

For each $\mathbf{w}_C \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$, by integration by parts one finds

$$\int_{\Omega_C}\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{w}}_C - \int_{\Gamma}\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\times\mathbf{n}_C\cdot\overline{\mathbf{w}}_C = -i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_C\,.$$

From Faraday's equation and the matching condition

$$\mathbf{H}_C\times\mathbf{n}_C + \mathbf{H}_D\times\mathbf{n}_D = \mathbf{0} \quad \text{on } \Gamma$$

one has that

$$\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\times\mathbf{n}_C = i\omega\mathbf{H}_D\times\mathbf{n}_D \quad \text{on } \Gamma\,,$$

therefore,

$$\int_{\Omega_C}\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{w}}_C - i\omega\int_{\Gamma}\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathbf{H}_D = -i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_C\,. \tag{14}$$

On the other hand, multiplying Faraday's equation by a test function $\mathbf{v}_D = \operatorname{grad}\phi_D$ with $\phi_D \in H^1(\Omega_D)$, by integration by parts one has

$$i\omega\int_{\Omega_D}\boldsymbol{\mu}\mathbf{H}_D\cdot\operatorname{grad}\overline{\phi}_D = -\int_{\Omega_D}\mathbf{curl}\,\mathbf{E}_D\cdot\operatorname{grad}\overline{\phi}_D = \int_{\partial\Omega_D}\mathbf{E}_D\times\mathbf{n}_D\cdot\operatorname{grad}\overline{\phi}_D\,.$$

Denoting by ϕ any extension of ϕ_D in $H^1(\Omega)$, we have

$$\begin{aligned} &\textstyle\int_{\partial\Omega_D}\mathbf{E}_D\times\mathbf{n}_D\cdot\operatorname{grad}\overline{\phi}_D \\ &\quad = \textstyle\int_{\partial\Omega}\mathbf{E}\times\mathbf{n}\cdot\operatorname{grad}\overline{\phi} + \int_{\Gamma}\mathbf{E}_D\times\mathbf{n}_D\cdot\operatorname{grad}\overline{\phi}_D - \int_{\Gamma_C}\mathbf{E}_C\times\mathbf{n}_C\cdot\operatorname{grad}\overline{\phi} \\ &\quad = \textstyle -\int_{\Gamma}\mathbf{E}_C\times\mathbf{n}_C\cdot\operatorname{grad}\overline{\phi}_D \end{aligned}$$

because $\mathrm{div}_\tau(\mathbf{E}\times\mathbf{n}) = 0$ on $\partial\Omega$ and $\mathbf{E}_C \times \mathbf{n}_C = \mathbf{0}$ on Γ_C. Hence

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \mathrm{grad}\,\overline{\phi}_D = -\int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \mathrm{grad}\,\overline{\phi}_D\,. \tag{15}$$

In a similar way, taking as test function $\boldsymbol{\varrho}_D$ one obtains

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \boldsymbol{\varrho}_D = -\int_{\Omega_D} \mathbf{curl}\,\mathbf{E}_D \cdot \boldsymbol{\varrho}_D = \int_{\partial\Omega_D} \mathbf{E}_D \times \mathbf{n}_D \cdot \boldsymbol{\varrho}_D\,.$$

Denoting by $\boldsymbol{\varrho}$ any extension of $\boldsymbol{\varrho}_D$ in $\boldsymbol{H}(\mathbf{curl}\,;\Omega)$, we have

$$\int_{\partial\Omega_D} \mathbf{E}_D \times \mathbf{n}_D \cdot \boldsymbol{\varrho}_D = \int_{\partial\Omega} \mathbf{E}\times\mathbf{n}\cdot\boldsymbol{\varrho} + \int_\Gamma \mathbf{E}_D \times \mathbf{n}_D \cdot \boldsymbol{\varrho}_D\,.$$

Using that $\mathbf{E}\times\mathbf{n} = \mathrm{grad}\,v \times \mathbf{n}$ on $\partial\Omega$ we have

$$\int_{\partial\Omega} \mathbf{E}\times\mathbf{n}\cdot\boldsymbol{\varrho} = \int_{\partial\Omega} \mathrm{grad}\,v\times\mathbf{n}\cdot\boldsymbol{\varrho} = -\int_{\partial\Omega} \boldsymbol{\varrho}\times\mathbf{n}\cdot\mathrm{grad}\,v = \int_{\partial\Omega} \mathbf{curl}\,\boldsymbol{\varrho}\cdot\mathbf{n}\,v\,.$$

Since $\mathbf{curl}\,\boldsymbol{\varrho} = \mathbf{0}$ in Ω_D, $v = V$ on Γ_J and $v = 0$ on Γ_E we obtain, using Stokes' Theorem on Γ_J, that

$$\int_{\partial\Omega} \mathbf{curl}\,\boldsymbol{\varrho}\cdot\mathbf{n}\,v = V\int_{\Gamma_J} \mathbf{curl}\,\boldsymbol{\varrho}\cdot\mathbf{n}_C = V\int_{\partial\Gamma_J} \boldsymbol{\varrho}_D\cdot\mathbf{t} = V\,.$$

Hence

$$i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \boldsymbol{\varrho}_D = V - \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_D\,. \tag{16}$$

As we noticed before, $\mathbf{H}_D \in \boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$ can be decomposed as $\mathbf{H}_D = \mathrm{grad}\,\psi_D + I\boldsymbol{\varrho}_D$ where $\psi_D \in H^1(\Omega_D)$ and $I \in \mathbb{C}$ is the current intensity. Moreover, as we have already remarked, this decomposition of $\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$ is $L^2(\boldsymbol{\mu};\Omega_D)$-orthogonal in the sense that

$$\int_{\Omega_D} \boldsymbol{\mu}(\mathrm{grad}\,\varphi_D + K\boldsymbol{\varrho}_D)\cdot(\mathrm{grad}\,\overline{\phi}_D + \overline{Q}\boldsymbol{\varrho}_D) = \int_{\Omega_D} \boldsymbol{\mu}\,\mathrm{grad}\,\varphi_D\cdot\mathrm{grad}\,\overline{\phi}_D + K\overline{Q}\int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D$$

for all φ_D and $\phi_D \in H^1(\Omega_D)$ and K and $Q \in \mathbb{C}$. Hence, from (14), (15) and (16), multiplying these two last equations by $-i\omega$, we have that $\mathbf{E}_C$ and $\mathbf{H}_D = \mathrm{grad}\,\psi_D + I\boldsymbol{\varrho}_D$ are such that for each $\mathbf{w}_C \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$ and for each $\phi_D \in H^1(\Omega_D)$ and $Q \in \mathbb{C}$ it holds

$$\begin{array}{lcl} \int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{w}}_C) & & \\ \qquad -i\omega\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathrm{grad}\,\psi_D - i\omega I\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D & = & -i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_C \\ -i\omega\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\mathrm{grad}\,\overline{\phi}_D + \omega^2\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\psi_D\cdot\mathrm{grad}\,\overline{\phi}_D & = & 0 \\ -i\omega\overline{Q}\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D + \omega^2 I\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D & = & -i\omega V\overline{Q}\,. \end{array} \tag{17}$$

When the voltage V is given and the current intensity I is unknown, these three equations determine $\mathbf{E}_C$, ψ_D and I. On the other hand, when the current intensity I is given, the first two equations are enough to determine the two unknowns of the problem $\mathbf{E}_C$ and ψ_D. The voltage V can be computed using the third equation.

In conclusion we have the following formulations:

Voltage excitation problem

$$\left\{\begin{array}{lcl} \text{Find } (\mathbf{E}_C,\psi_D,I) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\times\mathbb{C}\ : & & \\ \int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{w}}_C) & & \\ \qquad -i\omega\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathrm{grad}\,\psi_D - i\omega I\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D & = & -i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_C \\ -i\omega\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\mathrm{grad}\,\overline{\phi}_D + \omega^2\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\psi_D\cdot\mathrm{grad}\,\overline{\phi}_D & = & 0 \\ -i\omega\overline{Q}\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D + \omega^2 I\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D & = & -i\omega V\overline{Q} \\ \text{for all } (\mathbf{w}_C,\phi_D,Q) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\times\mathbb{C}\,. & & \end{array}\right. \tag{18}$$

Current excitation problem

$$
\begin{cases}
\text{Find } (\mathbf{E}_C,\psi_D)\in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\ : \\[2mm]
\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C+i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{w}}_C)-i\omega\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\operatorname{grad}\psi_D \\
\qquad\qquad\qquad\qquad = -i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_C+i\omega I\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D \\[2mm]
-i\omega\int_\Gamma\mathbf{E}_C\times\mathbf{n}_C\cdot\operatorname{grad}\overline{\phi}_D+\omega^2\int_{\Omega_D}\boldsymbol{\mu}\operatorname{grad}\psi_D\cdot\operatorname{grad}\overline{\phi}_D=0 \\[2mm]
\text{for all } (\mathbf{w}_C,\phi_D)\in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\,.
\end{cases}
\tag{19}
$$

If $(\mathbf{E}_C,\psi_D)$ is the solution of the current excitation problem then

$$
V=\int_\Gamma\mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D+i\omega I\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D\,. \tag{20}
$$

Remark 3.2 These two formulations can be adapted in an easy way to the case of a connected but not simply-connected conductor Ω_C with two electric ports $\partial\Omega_C\cap\partial\Omega=\Gamma_E\cup\Gamma_J$. In this

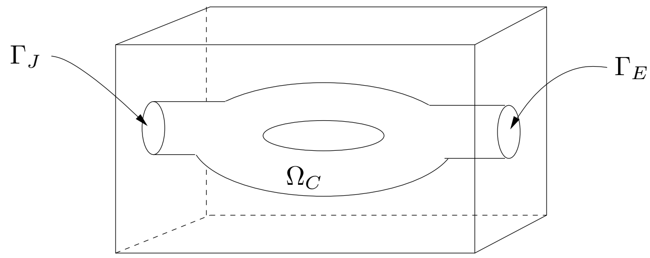

Figure 3: A non simply-connected conductor

case the space $\mathcal{H}_{\boldsymbol{\mu}}(\Omega_D)$ has dimension $p:=\beta_1(\Omega_D)>1$, where $\beta_1(\Omega_D)$ stands for the first Betti number of Ω_D, or, equivalently, for the dimension of the first homology group, a topological invariant measuring the number of non-bounding homologically independent cycles in Ω_D. Recall that in Ω_D it is possible to find p mutually disjoint, orientable two-dimensional surfaces such that $\Omega_D\setminus(\Sigma_1\cup\cdots\cup\Sigma_p)$ has trivial first homology group. Denote by z_k a function in $H^1(\Omega_D\setminus\Sigma_k)$ constructed as in (12). Let us set $\boldsymbol{\varrho}_{D,k}:=\widetilde{\operatorname{grad}}\,z_k$. Then $\{\boldsymbol{\varrho}_{D,1},\dots,\boldsymbol{\varrho}_{D,p}\}$ is a basis of $\mathcal{H}_{\boldsymbol{\mu}}(\Omega_D)$. Moreover, $\gamma_1,\dots\gamma_p$, the non-bounding homologically independent cycles in Ω_D, can be choosen such that $\gamma_k=\partial\Gamma^*_{C,k}$, where $\Gamma^*_{C,k}$ is an orientable two-dimensional surface contained in $\overline{\Omega}_C$, and $\int_{\gamma_l}\boldsymbol{\varrho}_{D,k}\cdot\mathbf{t}=\delta_{kl}$, $k,l\in\{1,\dots,p\}$.

In this more general geometrical situation the $L^2(\boldsymbol{\mu};\Omega_D)$-orthogonal decomposition of the space $\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$ still holds: any function $\mathbf{v}_D\in\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$ can be decomposed univocally as $\mathbf{v}_D=\operatorname{grad}\phi_D+\boldsymbol{\xi}_D$ with $\phi_D\in H^1(\Omega_D)/\mathbb{C}$ and $\boldsymbol{\xi}_D\in\mathcal{H}_{\boldsymbol{\mu}}(\Omega_D)$. In particular $\mathbf{H}_D=\operatorname{grad}\psi_D+\sum_{k=1}^p K_k\boldsymbol{\varrho}_{D,k}$ and, as done in Remark 3.1, from Stokes Theorem

$$
K_k=\int_{\gamma_k}\mathbf{H}_D\cdot\mathbf{t}=\int_{\gamma_k}\mathbf{H}_C\cdot\mathbf{t}=\int_{\Gamma^*_{C,k}}\mathbf{curl}\,\mathbf{H}_C\cdot\mathbf{n}_*=:I_k\,,
$$

where $\mathbf{n}_*$ is the unit vector normal to $\Gamma^*_{C,k}$ such that $\mathbf{t}$ is counterclockwise oriented with respect to $\mathbf{n}_*$ on $\Gamma^*_{C,k}$. Multiplying Faraday's equation by the function $\boldsymbol{\varrho}_{D,l}$ and proceeding as in the case of a simply-connected conductor, we obtain

$$
i\omega\int_{\Omega_D}\boldsymbol{\mu}\mathbf{H}_D\cdot\boldsymbol{\varrho}_{D,l}=V\int_{\partial\Gamma_J}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}-\int_\Gamma\mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_{D,l}\,.
$$

So $\mathbf{E}_C$ and $\mathbf{H}_D = \operatorname{grad}\psi_D + \sum_{k=1}^p I_k \boldsymbol{\varrho}_{D,k}$ are such that for each $\mathbf{w}_C \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$, $\phi_D \in H^1(\Omega_D)$ and $\mathbf{Q} \in \mathbb{C}^p$ it holds

$$
\begin{aligned}
&\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C\cdot\overline{\mathbf{w}}_C)\\
&\qquad -i\omega\int_\Gamma \overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\operatorname{grad}\psi_D - i\omega\sum_{k=1}^p I_k\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_{D,k} = -i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_C\\
&-i\omega\int_\Gamma \mathbf{E}_C\times\mathbf{n}_C\cdot\operatorname{grad}\overline{\phi}_D + \omega^2\int_{\Omega_D}\boldsymbol{\mu}\operatorname{grad}\psi_D\cdot\operatorname{grad}\overline{\phi}_D = 0\\
&-i\omega\overline{Q}_l\int_\Gamma\mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_{D,l} + \omega^2\overline{Q}_l\sum_{k=1}^p I_k\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_{D,k}\cdot\boldsymbol{\varrho}_{D,l}\\
&\qquad\qquad = -i\omega\overline{Q}_l V\int_{\partial\Gamma_J}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}\,,\ \forall\, l=1,\dots,p\,.
\end{aligned}
\tag{21}
$$

Clearly, if Ω_C is a non-connected set, for each connected component one has to reply the procedure described above. If Ω_C has q connected components $\Omega_{C,j}$, $j=1,\dots,q$, each one with two electric ports, then there are q different voltages V_j. In fact, on $\partial\Omega$ we have $\mathbf{E}\times\mathbf{n} = \operatorname{grad} v\times\mathbf{n}$, and, setting $\partial\Omega_{C,j}\cap\partial\Omega = \Gamma_{J,j}\cup\Gamma_{E,j}$, with $\Gamma_{J,j}$ and $\Gamma_{E,j}$ disjoint and connected surfaces, we have $v_{|\Gamma_{J,j}} = V_j^1$ and $v_{|\Gamma_{E,j}} = V_j^0$, where V_j^1 and V_j^0 are complex constants; then the voltages are defined as $V_j = V_j^1 - V_j^0$.

Multiplying Faraday's equation by $\boldsymbol{\varrho}_{D,l}$, a basis function of the space $\mathcal{H}_{\boldsymbol{\mu}}(\Omega_D)$, by integration by parts one has

$$
i\omega\int_{\Omega_D}\boldsymbol{\mu}\mathbf{H}_D\cdot\boldsymbol{\varrho}_{D,l} = \int_{\partial\Omega_D}\mathbf{E}_D\times\mathbf{n}_D\cdot\boldsymbol{\varrho}_{D,l} = \int_{\partial\Omega}\mathbf{E}\times\mathbf{n}\cdot\boldsymbol{\varrho}_l + \int_\Gamma\mathbf{E}_D\times\mathbf{n}_D\cdot\boldsymbol{\varrho}_{D,l}\,,
$$

where $\boldsymbol{\varrho}_l$ is any extension of $\boldsymbol{\varrho}_{D,l}$ in $\boldsymbol{H}(\mathbf{curl}\,;\Omega)$. Moreover

$$
\begin{aligned}
\int_{\partial\Omega}\mathbf{E}\times\mathbf{n}\cdot\boldsymbol{\varrho}_l = \int_{\partial\Omega}\mathbf{curl}\,\boldsymbol{\varrho}_l\cdot\mathbf{n}\,v &= \sum_{j=1}^q\Big(V_j^1\int_{\Gamma_{J,j}}\mathbf{curl}\,\boldsymbol{\varrho}_l\cdot\mathbf{n}_C + V_j^0\int_{\Gamma_{E,j}}\mathbf{curl}\,\boldsymbol{\varrho}_l\cdot\mathbf{n}_C\Big)\\
&= \sum_{j=1}^q\Big(V_j^1\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t} + V_j^0\int_{\partial\Gamma_{E,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}\Big)\\
&= \sum_{j=1}^q V_j\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}\,,
\end{aligned}
$$

since, denoting by $\Gamma_j = \partial\Omega_{C,j}\setminus(\Gamma_{J,j}\cup\Gamma_{E,j})$, from Stokes Theorem we have

$$
\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t} + \int_{\partial\Gamma_{E,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t} = \int_{\partial\Gamma_j}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t} = \int_{\Gamma_j}\mathbf{curl}\,\boldsymbol{\varrho}_{D,l}\cdot\mathbf{n}_C = 0\,.
$$

So, the third equation in (21) becomes

$$
-i\omega\overline{Q}_l\int_\Gamma\mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_{D,l} + \omega^2\overline{Q}_l\sum_{k=1}^p I_k\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_{D,k}\cdot\boldsymbol{\varrho}_{D,l} = -i\omega\overline{Q}_l\sum_{j=1}^q V_j\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l}\cdot\mathbf{t}
$$

for each $l=1,\dots,p$.

In the voltage excitation problem the q voltages are given, and therefore the unknowns of the problem are the electric field in the conductor, the function ψ_D appearing in the $L^2(\boldsymbol{\mu};\Omega_D)$-orthogonal decomposition of $\mathbf{H}_D$ and the p intensities, whereas in the current intensity problem the p current intensities are given and the unknowns of the problem are the electric field in the conductor and the function ψ_D. The q voltages can be then computed in the following way: for each $j=1,\dots,q$, let $\boldsymbol{\varrho}_{D,l(j)}$ be a basis function of $\mathcal{H}_{\boldsymbol{\mu}}(\Omega_D)$ corresponding to a non-bounding cycle $\gamma_{l(j)} = \partial\Gamma^*_{C,l(j)}$ such that $\Gamma^*_{C,l(j)}\subset\overline{\Omega}_{C,j}$. Then

$$
V_j = \left(\int_{\partial\Gamma_{J,j}}\boldsymbol{\varrho}_{D,l(j)}\cdot\mathbf{t}\right)^{-1}\left(\int_\Gamma\mathbf{E}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_{D,l(j)} + i\omega\sum_{k=1}^p I_k\int_{\Omega_D}\boldsymbol{\mu}\,\boldsymbol{\varrho}_{D,k}\cdot\boldsymbol{\varrho}_{D,l(j)}\right),
$$

and this value depends on j but not on the choice of $l(j)$.

For the sake of simplicity in the following we limit ourselves to the case of a simply-connected conductor. □

4 Existence and uniqueness of the solution

Let us define in $\boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_D) \times \boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$ the sesquilinear form

$$\begin{aligned}\mathcal{A}\big((\mathbf{v}_C,\mathbf{u}_D),(\mathbf{w}_C,\mathbf{z}_D)\big) := \textstyle\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{v}_C\cdot\overline{\mathbf{w}}_C)\\ +\omega^2\textstyle\int_{\Omega_D}\boldsymbol{\mu}\mathbf{u}_D\cdot\overline{\mathbf{z}}_D - i\omega\big[\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathbf{u}_D + \int_\Gamma\mathbf{v}_C\times\mathbf{n}_C\cdot\overline{\mathbf{z}}_D\big]\,.\end{aligned}$$

and the antilinear functionals

$$\begin{array}{rcl} F(\mathbf{w}_C) & := & -i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_C \\ L_V(\mathbf{z}_D) & := & -i\omega c_0 V\int_{\Omega_D}\boldsymbol{\mu}\boldsymbol{\varrho}_D\cdot\overline{\mathbf{z}}_D \\ L_I(\mathbf{w}_C) & := & i\omega I\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\varrho}_D\,, \end{array}$$

where V and I are given complex constants and $c_0 = (\int_{\Omega_D}\boldsymbol{\mu}\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D)^{-1}$. Recall that if $\mathbf{z}_D \in \boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$ it can be univocally decomposed as $\mathbf{z}_D = \operatorname{grad}\phi_D + Q\boldsymbol{\varrho}_D$ with $\phi_D \in H^1(\Omega_D)/\mathbb{C}$ and $Q \in \mathbb{C}$. Then $L_V(\mathbf{z}_D) = -i\omega V\overline{Q}$.

It is easy to see, using the $L^2(\boldsymbol{\mu};\Omega_D)$-orthogonal decomposition of $\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$ presented in (13), that problem (18) is equivalent to the following one

$$\begin{cases}\text{Find } (\mathbf{E}_C,\mathbf{H}_D) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)\ : \\ \mathcal{A}\big((\mathbf{E}_C,\mathbf{H}_D),(\mathbf{w}_C,\mathbf{z}_D)\big) = F(\mathbf{w}_C) + L_V(\mathbf{z}_D) \\ \text{for all } (\mathbf{w}_C,\mathbf{z}_D) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)\,,\end{cases} \tag{22}$$

whereas problem (19) is equivalent to

$$\begin{cases}\text{Find } (\mathbf{E}_C,\psi_D) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\ : \\ \mathcal{A}\big((\mathbf{E}_C,\operatorname{grad}\psi_D),(\mathbf{w}_C,\operatorname{grad}\phi_D)\big) = F(\mathbf{w}_C) + L_I(\mathbf{w}_C) \\ \text{for all } (\mathbf{w}_C,\phi_D) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\,.\end{cases} \tag{23}$$

The antilinear functionals $F(\mathbf{w}_C)$ and $L_I(\mathbf{w}_C)$ are clearly continuous in $\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)$, whereas $L_V(\mathbf{z}_D)$ is continuous in $\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$ (see (11)). Hence the existence and uniqueness of the solution of these two problems follows from Lax-Milgram lemma once we prove that the sesquilinear form $\mathcal{A}(\cdot,\cdot)$ is coercive. This has been proved in [4]. For the sake of completeness, here below we present the proof.

Proposition 4.1 *The sesquilinear form $\mathcal{A}(\cdot,\cdot)$ is coercive on $\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)\times\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$*

Proof. We have

$$\begin{aligned}|\mathcal{A}((\mathbf{w}_C,\mathbf{z}_D),(\mathbf{w}_C,\mathbf{z}_D))|^2 &= (\textstyle\int_{\Omega_C}\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{w}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C + \omega^2\int_{\Omega_D}\boldsymbol{\mu}\mathbf{z}_D\cdot\overline{\mathbf{z}}_D)^2 \\ &\quad +\omega^2(\textstyle\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{w}_C\cdot\overline{\mathbf{w}}_C - 2\mathrm{Re}\int_\Gamma\mathbf{w}_C\times\mathbf{n}\cdot\overline{\mathbf{z}}_D)^2\,.\end{aligned}$$

Taking into account that $\mathbf{curl}\,\mathbf{z}_D = \mathbf{0}$ in Ω_D, from the continuity estimate

$$2\left|\int_\Gamma\mathbf{w}_C\times\mathbf{n}\cdot\overline{\mathbf{z}}_D\right| \le k_0\left(\int_{\Omega_D}|\mathbf{z}_D|^2\right)^{1/2}\left(\int_{\Omega_C}(|\mathbf{w}_C|^2 + |\mathbf{curl}\,\mathbf{w}_C|^2)\right)^{1/2}$$

and the inequality $(A+B)^2 \ge A^2/2 - B^2$ we find

$$\begin{aligned}&(\textstyle\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{w}_C\cdot\overline{\mathbf{w}}_C - 2\mathrm{Re}\int_\Gamma\mathbf{w}_C\times\mathbf{n}\cdot\overline{\mathbf{z}}_D)^2 \\ &\quad\ge \tfrac{1}{2}(\textstyle\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{w}_C\cdot\overline{\mathbf{w}}_C)^2 - k_0^2\left(\int_{\Omega_D}|\mathbf{z}_D|^2\right)\left(\int_{\Omega_C}(|\mathbf{w}_C|^2 + |\mathbf{curl}\,\mathbf{w}_C|^2)\right) \\ &\quad\ge \tfrac{1}{2}(\textstyle\int_{\Omega_C}\boldsymbol{\sigma}\mathbf{w}_C\cdot\overline{\mathbf{w}}_C)^2 - \delta^{-1}\tfrac{1}{2}k_0^2(\int_{\Omega_D}|\mathbf{z}_D|^2)^2 \\ &\qquad -\delta k_0^2\left(\textstyle\int_{\Omega_C}|\mathbf{w}_C|^2\right)^2 - \delta k_0^2\left(\int_{\Omega_C}|\mathbf{curl}\,\mathbf{w}_C|^2\right)^2\,,\end{aligned}$$

for each $\delta > 0$. Finally, for each $0 < \gamma \leq 1/2$ we also have

$$\omega^2 \left(\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_D \right)^2 \\ \geq 2\gamma\omega^2 \left(\int_{\Omega_C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_D \right)^2 ,$$

so that

$$|\mathcal{A}((\mathbf{w}_C, \mathbf{z}_D), (\mathbf{w}_C, \mathbf{z}_D))|^2 \geq (\nu_*^2 - 2\gamma\omega^2\delta k_0^2) \left(\int_{\Omega_C} |\mathbf{curl}\, \mathbf{w}_C|^2 \right)^2 \\ +(\omega^4\mu_*^2 - \gamma\omega^2\delta^{-1}k_0^2) \left(\int_{\Omega_D} |\mathbf{z}_D|^2 \right)^2 + \gamma\omega^2(\sigma_*^2 - 2\delta k_0^2) \left(\int_{\Omega_C} |\mathbf{w}_C|^2 \right)^2$$

for some positive constants ν_*, μ_* and σ_*. The proof of the coerciveness of $\mathcal{A}(\cdot, \cdot)$ follows by taking at first δ small enough and then γ small enough. □

Once we have obtained $\mathbf{E}_C$ and $\mathbf{H}_D$, the magnetic field $\mathbf{H}_C$ can be obtained directly from Faraday's law by setting

$$\mathbf{H}_C = (-i\omega\boldsymbol{\mu})^{-1}\mathbf{curl}\, \mathbf{E}_C ,$$

while $\mathbf{E}_D$ is the solution of the following problem:

$$\begin{cases} \mathbf{curl}\, \mathbf{E}_D = -i\omega\boldsymbol{\mu}\mathbf{H}_D & \text{in } \Omega_D , \\ \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{E}_D) = 0 & \text{in } \Omega_D , \\ \mathbf{E}_D \times \mathbf{n}_D = \mathbf{E}_C \times \mathbf{n}_D & \text{on } \Gamma , \\ \boldsymbol{\varepsilon}\mathbf{E}_D \cdot \mathbf{n}_D = 0 & \text{on } \Gamma_D . \end{cases} \tag{24}$$

Proposition 4.2 *System (24) has a solution, and it is unique.*

Proof. Concerning the uniqueness, we notice that the space

$$\mathcal{H} := \{\mathbf{v}_D \in (L^2(\Omega_D))^3 |\, \mathbf{curl}\, \mathbf{v}_D = \mathbf{0},\, \mathrm{div}\,(\boldsymbol{\varepsilon}\mathbf{v}_D) = 0,\, \boldsymbol{\varepsilon}\mathbf{v}_D \cdot \mathbf{n}_D = 0 \text{ on } \Gamma_D,\, \mathbf{v}_D \times \mathbf{n}_D = \mathbf{0} \text{ on } \Gamma\}$$

is trivial in the considered geometrical situation. In fact, given $\mathbf{v}_D \in \mathcal{H}$, one has $\mathbf{curl}\, \mathbf{v}_D = \mathbf{0}$ in $\Omega_D \backslash \Sigma$, that is a simply connected subset. Hence there exists $\psi_* \in H^1(\Omega_D \backslash \Sigma)$ such that $\mathrm{grad}\, \psi_* = \mathbf{v}_D$ and

$$\begin{cases} \mathrm{div}\,(\boldsymbol{\varepsilon}\, \mathrm{grad}\, \psi_*) = 0 & \text{in } \Omega_D \backslash \Sigma , \\ \boldsymbol{\varepsilon}\, \mathrm{grad}\, \psi_* \cdot \mathbf{n}_D = 0 & \text{on } \Gamma_D \setminus \partial\Sigma , \\ \psi_* = \kappa^* & \text{on } \Gamma \setminus \partial\Sigma , \\ [\psi_*]_\Sigma = c^* , & \\ [\boldsymbol{\varepsilon}\, \mathrm{grad}\, \psi_* \cdot \mathbf{n}_D]_\Sigma = 0 , & \end{cases} \tag{25}$$

κ^* and c^* being constants. Since $\Gamma \cap \Sigma \neq \emptyset$ the constant c^* must be zero; therefore the unique solution of (25) is $\psi = \kappa^*$ and consequently $\mathbf{v}_D = \mathbf{0}$. The existence of the solution to (25) can be proved as in [1]. □

5 Finite element approximation

The variational formulations (18) and (19) are not suitable for finite element numerical approximation. In fact, a conforming finite element approximation based directly on them requires that $\boldsymbol{\varrho}_D$ is explicitly known. An alternative approach, that overcomes this difficulty, is based on a different decomposition of $\mathbf{H}_D$.

Let $\boldsymbol{\zeta}_D$ be the generalized gradient of a function $\eta \in H^1(\Omega_D \backslash \Sigma)$ such that $[\eta]_\Sigma = 1$. Then $\mathbf{curl}\, \boldsymbol{\zeta}_D = \mathbf{0}$ and $\int_{\partial\Gamma_J} \boldsymbol{\zeta}_D \cdot \mathbf{t} = 1$, but in general $\boldsymbol{\zeta}_D \notin H(\mathrm{div}\,; \Omega_D)$. Since Ω_D is simply connected, $\boldsymbol{\varrho}_D = \boldsymbol{\zeta}_D + \mathrm{grad}\, g^{\boldsymbol{\zeta}_D}$ for some $g^{\boldsymbol{\zeta}_D} \in H^1(\Omega_D)$. Hence $\mathbf{H}_D = \mathrm{grad}\, \psi_D + I\boldsymbol{\varrho}_D = \mathrm{grad}\, \psi_D + I(\boldsymbol{\zeta}_D +$

$\operatorname{grad} g^{\boldsymbol{\zeta}_D}) = \operatorname{grad} \widehat{\psi}_D + I\boldsymbol{\zeta}_D$, with $\widehat{\psi}_D \in H^1(\Omega_D)$ that depends on the choice of $\boldsymbol{\zeta}_D$. This alternative decomposition is not $L^2(\boldsymbol{\mu};\Omega_D)$-orthogonal and this has as a consequence that the corresponding weak formulation has some additional terms. In fact the voltage excitation problem now reads

$$\left\{\begin{array}{l} \text{Find } (\mathbf{E}_C, \widehat{\psi}_D, I) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \times \mathbb{C}\ : \\[2ex] \int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}}_C) \\ \qquad\qquad -i\omega \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \operatorname{grad} \widehat{\psi}_D - i\omega I \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\zeta}_D = -i\omega \int_{\Omega_C} \mathbf{J}_C \cdot \overline{\mathbf{w}}_C \\[2ex] -i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \operatorname{grad} \overline{\phi}_D + \omega^2 \int_{\Omega_D} \boldsymbol{\mu} \operatorname{grad} \widehat{\psi}_D \cdot \operatorname{grad} \overline{\phi}_D + \omega^2 I \int_{\Omega_D} \boldsymbol{\mu}\, \boldsymbol{\zeta}_D \cdot \operatorname{grad} \overline{\phi}_D = 0 \\[2ex] -i\omega\overline{Q} \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\zeta}_D + \omega^2 \overline{Q} \int_{\Omega_D} \boldsymbol{\mu} \operatorname{grad} \widehat{\psi}_D \cdot \boldsymbol{\zeta}_D + \omega^2 I\overline{Q} \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\zeta}_D \cdot \boldsymbol{\zeta}_D = -i\omega V\overline{Q} \\[2ex] \text{for all } (\mathbf{w}_C, \phi_D, Q) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \times \mathbb{C}\,, \end{array}\right. \tag{26}$$

while the current excitation problem reads

$$\left\{\begin{array}{l} \text{Find } (\mathbf{E}_C, \widehat{\psi}_D) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C}\ : \\[2ex] \int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{E}_C \cdot \mathbf{curl}\,\overline{\mathbf{w}}_C + i\omega\boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{w}}_C) - i\omega \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \operatorname{grad} \widehat{\psi}_D \\ \qquad\qquad\qquad\qquad = -i\omega \int_{\Omega_C} \mathbf{J}_C \cdot \overline{\mathbf{w}}_C + i\omega I \int_\Gamma \overline{\mathbf{w}}_C \times \mathbf{n}_C \cdot \boldsymbol{\zeta}_D \\[2ex] -i\omega \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \operatorname{grad} \overline{\phi}_D + \omega^2 \int_{\Omega_D} \boldsymbol{\mu} \operatorname{grad} \widehat{\psi}_D \cdot \operatorname{grad} \overline{\phi}_D = -\omega^2 I \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\zeta}_D \cdot \operatorname{grad} \overline{\phi}_D \\[2ex] \text{for all } (\mathbf{w}_C, \phi_D) \in \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C}\,. \end{array}\right. \tag{27}$$

Here below we present two different possible choices of $\boldsymbol{\zeta}_D$ in the framework of finite element approximation.

Let us now propose our finite element approximation schemes. We assume that Ω_C and Ω_D are Lipschitz polyhedral domains, and that $\{\mathcal{T}_h^C\}_h$ and $\{\mathcal{T}_h^D\}_h$ are two families of tetrahedral meshes of Ω_C and Ω_D, respectively. We employ the Nédélec curl-conforming edge elements of degree k, $\mathbf{N}_{C,h}^k$, to approximate the functions in $\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)$ and continuous nodal elements of degree k, $L_{D,h}^k$, to approximate the functions in $H^1(\Omega_D)$. Let us also set $\mathbf{W}_{C,h}^k := \mathbf{N}_{C,h}^k \cap \boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)$.

We consider two different approaches. The first one is a conforming method where the function $\boldsymbol{\zeta}_D$ is chosen independently of the mesh, while in the second approach we consider a function $\boldsymbol{\zeta}_D$ which is mesh dependent.

Let us start from the first approach. Let us assume that the family $\{\mathcal{T}_h^D\}_h$ is obtained by refining a coarse mesh $\mathcal{T}_{h^*}^D$. Then we can choose a set of faces of tetrahedrons in $\mathcal{T}_{h^*}^D$ such that the union is a 'cutting' surface $\Sigma \subset \Omega_D$. Let us denote by η_D^* the piecewise linear function taking value 1 at the nodes on one side of Σ, say Σ^+, and 0 at all the other nodes including those on Σ^-, the other side of Σ. Then we choose $\boldsymbol{\zeta}_D = \widetilde{\operatorname{grad} \eta_D^*} =: \boldsymbol{\lambda}_D$ (see [3]), that is independent of h.

The finite element approximation of the voltage excitation problem reads

$$\left\{\begin{array}{l} \text{Find } (\mathbf{E}_{C,h}, \widehat{\psi}_{D,h}, I_h) \in \mathbf{W}_{C,h}^k \times L_{D,h}^k/\mathbb{C} \times \mathbb{C}\ : \\[2ex] \mathcal{C}((\mathbf{E}_{C,h}, \widehat{\psi}_{D,h}, I_h), (\mathbf{w}_{C,h}, \phi_{D,h}, Q)) = -i\omega \int_{\Omega_C} \mathbf{J}_C \cdot \overline{\mathbf{w}}_{C,h} - i\omega V\overline{Q} \\[2ex] \text{for all } (\mathbf{w}_{C,h}, \phi_{D,h}, Q) \in \mathbf{W}_{C,h}^k \times L_{D,h}^k/\mathbb{C} \times \mathbb{C}\,, \end{array}\right. \tag{28}$$

where $\mathcal{C}(\cdot,\cdot)$ is the sesquilinear form, defined in $\boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C) \times H^1(\Omega_D)/\mathbb{C} \times \mathbb{C}$, associated to problem (26), namely,

$$
\begin{array}{l}
\mathcal{C}((\mathbf{v}_C,\varphi_D,K),(\mathbf{w}_C,\phi_D,Q):= \\
\qquad \int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C+i\omega\boldsymbol{\sigma}\mathbf{v}_C\cdot\overline{\mathbf{w}}_C) \\
\qquad\qquad +\omega^2\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\varphi_D\cdot\mathrm{grad}\,\overline{\phi}_D+\omega^2K\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\boldsymbol{\lambda}_D\cdot\boldsymbol{\lambda}_D \\
\qquad -i\omega\big[\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathrm{grad}\,\varphi_D+\int_\Gamma\mathbf{v}_C\times\mathbf{n}_C\cdot\mathrm{grad}\,\overline{\phi}_D\big] \\
\qquad -i\omega\big[K\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\lambda}_D+\overline{Q}\int_\Gamma\mathbf{v}_C\times\mathbf{n}_C\cdot\boldsymbol{\lambda}_D\big] \\
\qquad +\omega^2\big[K\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\overline{\phi}_D\cdot\boldsymbol{\lambda}_D+\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\varphi_D\cdot\boldsymbol{\lambda}_D\big]\,.
\end{array}
$$

Analogously, the finite element approximation of the current excitation problem reads

$$
\left\{
\begin{array}{l}
\text{Find } (\mathbf{E}_{C,h},\widehat{\psi}_{D,h})\in\mathbf{W}^k_{C,h}\times L^k_{D,h}/\mathbb{C}\ : \\[2ex]
\mathcal{A}((\mathbf{E}_{C,h},\mathrm{grad}\,\widehat{\psi}_{D,h}),(\mathbf{w}_{C,h},\mathrm{grad}\,\phi_{D,h})) \\
\qquad\qquad = -i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_C+i\omega I\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\lambda}_D-\omega^2I\int_{\Omega_D}\boldsymbol{\mu}\boldsymbol{\lambda}_D\cdot\mathrm{grad}\,\overline{\phi}_D \\[2ex]
\text{for all } (\mathbf{w}_{C,h},\phi_{D,h})\in\mathbf{W}^k_{C,h}\times L^k_{D,h}/\mathbb{C}\,.
\end{array}
\right.
\tag{29}
$$

From the coerciveness of $\mathcal{A}(\cdot,\cdot)$ it is easy to obtain the following result:

Proposition 5.1 *The sesquilinear form $\mathcal{C}(\cdot,\cdot)$ is coercive in $\boldsymbol{H}_{0,\Gamma_C}(\mathbf{curl}\,;\Omega_C)\times H^1(\Omega_D)/\mathbb{C}\times\mathbb{C}$.*

Proof. We notice that

$$
\begin{array}{rcl}
\mathcal{C}\big((\mathbf{w}_C,\phi_D,Q),(\mathbf{w}_C,\phi_D,Q)\big) & = & \mathcal{A}\big((\mathbf{w}_C,\mathrm{grad}\,\phi_D+Q\boldsymbol{\lambda}_D),(\mathbf{w}_C,\mathrm{grad}\,\phi_D+Q\boldsymbol{\lambda}_D)\big) \\
& \geq & \alpha(\|\mathbf{w}_C\|^2_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}+\|\mathrm{grad}\,\phi_D+Q\boldsymbol{\lambda}_D\|^2_{(L^2(\Omega_D))^3})\,,
\end{array}
$$

since $\mathcal{A}(\cdot,\cdot)$ is coercive on $\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)\times\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$. Moreover we know that $\boldsymbol{\varrho}_D=\boldsymbol{\lambda}_D+\mathrm{grad}\,g^{\boldsymbol{\lambda}_D}$, and we also have $\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\varphi_D\cdot\boldsymbol{\varrho}_D=0$ for each $\varphi_D\in H^1(\Omega_D)$. Since from the assumptions on $\boldsymbol{\mu}$ there exists two positive constants μ_* and μ^* such that $\mu_*\|\mathbf{v}_D\|^2_{(L^2(\Omega_D))^3}\leq\int_{\Omega_D}\boldsymbol{\mu}\mathbf{v}_D\cdot\overline{\mathbf{v}}_D\leq\mu^*\|\mathbf{v}_D\|^2_{(L^2(\Omega_D))^3}$ for all $\mathbf{v}_D\in(L^2(\Omega_D))^3$, it follows that

$$
\begin{array}{l}
\|\mathrm{grad}\,\phi_D+Q\boldsymbol{\lambda}_D\|^2_{(L^2(\Omega_D))^3}=\|\mathrm{grad}\,(\phi_D-Qg^{\boldsymbol{\lambda}_D})+Q\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3} \\
\qquad \geq\frac{1}{\mu^*}\int_{\Omega_D}\boldsymbol{\mu}\left[\mathrm{grad}\,(\phi_D-Qg^{\boldsymbol{\lambda}_D})+Q\boldsymbol{\varrho}_D\right]\cdot\left[\mathrm{grad}\,(\overline{\phi}_D-\overline{Q}g^{\boldsymbol{\lambda}_D})+\overline{Q}\boldsymbol{\varrho}_D\right] \\
\qquad =\frac{1}{\mu^*}\Big(\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,(\phi_D-Qg^{\boldsymbol{\lambda}_D})\cdot\mathrm{grad}\,(\overline{\phi}_D-\overline{Q}g^{\boldsymbol{\lambda}_D})+|Q|^2\int_{\Omega_D}\boldsymbol{\mu}\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D\Big) \\
\qquad \geq\frac{\mu_*}{\mu^*}\big(\|\mathrm{grad}\,(\phi_D-Qg^{\boldsymbol{\lambda}_D})\|^2_{(L^2(\Omega_D))^3}+|Q|^2\|\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3}\big)\,.
\end{array}
$$

Using that $\|f+g\|^2_{(L^2(\Omega_D))^3}\geq(1-\delta)\|f\|^2_{(L^2(\Omega_D))^3}+(1-\frac{1}{\delta})\|g\|^2_{(L^2(\Omega_D))^3}$ for each $\delta>0$, we obtain

$$
\begin{array}{l}
\|\mathrm{grad}\,\phi_D+Q\boldsymbol{\lambda}_D\|^2_{(L^2(\Omega_D))^3} \\
\quad \geq\frac{\mu_*}{\mu^*}(1-\delta)\|\mathrm{grad}\,\phi_D\|^2_{(L^2(\Omega_D))^3}+\frac{\mu_*}{\mu^*}(1-\frac{1}{\delta})|Q|^2\|\mathrm{grad}\,g^{\boldsymbol{\lambda}_D}\|^2_{(L^2(\Omega_D))^3}+\frac{\mu_*}{\mu^*}|Q|^2\|\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3} \\
\quad =\frac{\mu_*}{\mu^*}(1-\delta)\|\mathrm{grad}\,\phi_D\|^2_{(L^2(\Omega_D))^3}+\frac{\mu_*}{\mu^*}\big[(1-\frac{1}{\delta})\|\mathrm{grad}\,g^{\boldsymbol{\lambda}_D}\|^2_{(L^2(\Omega_D))^3}+\|\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3}\big]|Q|^2\,.
\end{array}
$$

Choosing δ such that

$$
\frac{\|\mathrm{grad}\,g^{\boldsymbol{\lambda}_D}\|^2_{(L^2(\Omega_D))^3}}{\|\mathrm{grad}\,g^{\boldsymbol{\lambda}_D}\|^2_{(L^2(\Omega_D))^3}+\|\boldsymbol{\varrho}_D\|^2_{(L^2(\Omega_D))^3}}<\delta<1\,,
$$

we have for some positive constant C

$$
\|\mathrm{grad}\,\phi_D+Q\boldsymbol{\lambda}_D\|^2_{(L^2(\Omega_D))^3}\geq C(\|\mathrm{grad}\,\phi_D\|^2_{(L^2(\Omega_D))^3}+|Q|^2)\,,
$$

so the coerciveness of $\mathcal{C}(\cdot,\cdot)$ follows. □

The optimality of the discrete solution of both problems is a consequence of Cea's Lemma: for the voltage excitation problem we have

$$\begin{aligned}&\|\mathbf{E}_C-\mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}+\|\mathrm{grad}\,\psi_D-\mathrm{grad}\,\psi_{D,h}\|_{(L^2(\Omega_D))^3}+|I-I_h|\\ &\quad\le C\inf_{(\mathbf{w}_{C,h},\phi_{D,h})\in\mathbf{W}^k_{C,h}\times L^k_{D,h}}\left(\|\mathbf{E}_C-\mathbf{w}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}+\|\mathrm{grad}\,\psi_D-\mathrm{grad}\,\phi_{D,h}\|_{(L^2(\Omega_D))^3}\right),\end{aligned}$$

and for the current excitation problem we find

$$\begin{aligned}&\|\mathbf{E}_C-\mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}+\|\mathrm{grad}\,\psi_D-\mathrm{grad}\,\psi_{D,h}\|_{(L^2(\Omega_D))^3}\\ &\quad\le C\inf_{(\mathbf{w}_{C,h},\phi_{D,h})\in\mathbf{W}^k_{C,h}\times L^k_{D,h}}\left(\|\mathbf{E}_C-\mathbf{w}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}+\|\mathrm{grad}\,\psi_D-\mathrm{grad}\,\phi_{D,h}\|_{(L^2(\Omega_D))^3}\right).\end{aligned}$$

Therefore, by standard density results, we obtain the convergence of the approximation for both problems. As usual, the precise order of convergence is related to the regularity of the solution $(\mathbf{E}_C,\psi_D)$.

In the second approach the function $\boldsymbol{\zeta}_D$ depends on h because it is the generalized gradient of a piecewise linear function $\eta_{D,h}$ with a jump one on a discrete 'cutting' surface Σ_h that depends on the mesh $\{\mathcal{T}_h^D\}_h$. This choice will be denoted by $\boldsymbol{\zeta}_D=\boldsymbol{\lambda}_D^h$. Notice that now we are not assuming that $\{\mathcal{T}_h^D\}_h$ is obtained by refining $\mathcal{T}_{h^*}^D$. This approach is similar to the one analyzed in [9] for the current excitation problem.

The sesquilinear form associated to Problem (26) now depends on h

$$\begin{aligned}&\mathcal{C}_h((\mathbf{v}_C,\varphi_D,K),(\mathbf{w}_C,\phi_D,Q)):=\\ &\qquad\textstyle\int_{\Omega_C}(\boldsymbol{\mu}^{-1}\mathbf{curl}\,\mathbf{v}_C\cdot\mathbf{curl}\,\overline{\mathbf{w}}_C+i\omega\boldsymbol{\sigma}\mathbf{v}_C\cdot\overline{\mathbf{w}}_C)\\ &\qquad\qquad\textstyle+\omega^2\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\varphi_D\cdot\mathrm{grad}\,\overline{\phi}_D+\omega^2K\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\boldsymbol{\lambda}_D^h\cdot\boldsymbol{\lambda}_D^h\\ &\qquad\textstyle-i\omega\big[\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\mathrm{grad}\,\varphi_D+\int_\Gamma\mathbf{v}_C\times\mathbf{n}_C\cdot\mathrm{grad}\,\overline{\phi}_D\big]\\ &\qquad\textstyle-i\omega\big[K\int_\Gamma\overline{\mathbf{w}}_C\times\mathbf{n}_C\cdot\boldsymbol{\lambda}_D^h+\overline{Q}\int_\Gamma\mathbf{v}_C\times\mathbf{n}_C\cdot\boldsymbol{\lambda}_D^h\big]\\ &\qquad\textstyle+\omega^2\big[K\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\overline{\phi}_D\cdot\boldsymbol{\lambda}_D^h+\overline{Q}\int_{\Omega_D}\boldsymbol{\mu}\,\mathrm{grad}\,\varphi_D\cdot\boldsymbol{\lambda}_D^h\big].\end{aligned}$$

However $\mathcal{C}_h((\mathbf{v}_C,\varphi_D,K),(\mathbf{w}_C,\phi_D,Q))=\mathcal{A}((\mathbf{v}_C,\mathrm{grad}\,\varphi_D+K\boldsymbol{\lambda}_D^h),(\mathbf{w}_C,\mathrm{grad}\,\phi_D+Q\boldsymbol{\lambda}_D^h))$. Hence the finite element approximation of the voltage excitation problem with this second approach reads

$$\left\{\begin{array}{l}\text{Find }(\mathbf{E}_{C,h},\widehat{\psi}_{D,h},I_h)\in\mathbf{W}^k_{C,h}\times L^k_{D,h}/\mathbb{C}\times\mathbb{C}\ :\\[2ex] \mathcal{A}((\mathbf{E}_{C,h},\mathrm{grad}\,\widehat{\psi}_{D,h}+I_h\boldsymbol{\lambda}_D^h),(\mathbf{w}_{C,h},\mathrm{grad}\,\phi_{D,h}+Q\boldsymbol{\lambda}_D^h))=-i\omega\int_{\Omega_C}\mathbf{J}_C\cdot\overline{\mathbf{w}}_{C,h}-i\omega V\overline{Q}\\[2ex] \text{for all }(\mathbf{w}_{C,h},\phi_{D,h},Q)\in\mathbf{W}^k_{C,h}\times L^k_{D,h}/\mathbb{C}\times\mathbb{C}\,.\end{array}\right.\tag{30}$$

Let us consider now the error estimate. Let us set $\mathbf{H}_{D,h}:=\mathrm{grad}\,\widehat{\psi}_{D,h}+I_h\boldsymbol{\lambda}_D^h\in\boldsymbol{H}^0(\mathbf{curl}\,;\Omega_D)$. From (26) and (30), we have the following equation for the error:

$$\mathcal{A}((\mathbf{E}_C-\mathbf{E}_{C,h},\mathbf{H}_D-\mathbf{H}_{D,h}),(\mathbf{w}_{C,h},\mathrm{grad}\,\phi_{D,h}+Q\boldsymbol{\lambda}_D^h))=0$$

for all $\mathbf{w}_{C,h}\in\mathbf{W}^k_{C,h}$, $\phi_{D,h}\in L^k_{D,h}$ and $Q\in\mathbb{C}$. Hence

$$\begin{aligned}&\|\mathbf{E}_C-\mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}+\|\mathbf{H}_D-\mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\\ &=\|\mathbf{E}_C-\mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}+\|\mathbf{H}_D-\mathrm{grad}\,\widehat{\psi}_{D,h}-I_h\boldsymbol{\lambda}_D^h\|_{(L^2(\Omega_D))^3}\\ &\qquad\le C\inf_{(\mathbf{w}_{C,h},\mathbf{z}_{D,h})\in\mathbf{W}^k_{C,h}\times\mathbf{Z}^k_{D,h}}\left(\|\mathbf{E}_C-\mathbf{w}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}+\|\mathbf{H}_D-\mathbf{z}_{D,h}\|_{(L^2(\Omega_D))^3}\right),\end{aligned}$$

where

$$\mathbf{Z}^k_{D,h}:=\mathrm{grad}\,L^k_{D,h}\,\oplus\,\mathrm{span}\,\{\boldsymbol{\lambda}_D^h\}\,.$$

An error estimate for the intensity is obtained by noticing that, from (13),

$$\int_{\Omega_D}\boldsymbol{\mu}(\mathbf{H}_D-\mathbf{H}_{D,h})\cdot\boldsymbol{\varrho}_D=(I-I_h)\int_{\Omega_D}\boldsymbol{\mu}\boldsymbol{\varrho}_D\cdot\boldsymbol{\varrho}_D\,.$$

Hence

$$|I-I_h|\le C\|\mathbf{H}_D-\mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\,,$$

where $C=\frac{\mu^*}{\mu_*}\|\boldsymbol{\varrho}_D\|^{-1}_{(L^2(\Omega_D))^3}$.

Remark 5.1 It is worth noting that a suitable choice of the discrete function $\mathbf{z}_{D,h}$ is easily performed. In fact, let us denote by $\mathbf{N}^k_{D,h}$ the space of Nédélec curl-conforming edge elements of degree k in $\mathcal{T}_h^D$, and $\mathbf{\Pi}_{D,h}$ the interpolation operator. If $\mathbf{H}_D$ is so regular that $\mathbf{\Pi}_{D,h}\mathbf{H}_D$ is well defined, then $\mathbf{\Pi}_{D,h}\mathbf{H}_D \in \mathbf{Z}^k_{D,h}$. In fact, $\mathbf{curl}\,(\mathbf{\Pi}_{D,h}\mathbf{H}_D - I\boldsymbol{\lambda}^h_D) = \mathbf{0}$ and $\int_{\partial\Gamma_J}(\mathbf{\Pi}_{D,h}\mathbf{H}_D - I\boldsymbol{\lambda}^h_D)\cdot\mathbf{t} = 0$. Consequently $\mathbf{\Pi}_{D,h}\mathbf{H}_D - I\boldsymbol{\lambda}^h_D = \operatorname{grad}\varphi_D$ for some $\varphi_D \in H^1(\Omega_D)$. Since $\mathbf{\Pi}_{D,h}\mathbf{H}_D - I\boldsymbol{\lambda}^h_D \in \mathbf{N}^k_{D,h}$, from Lemma 5.3, Chapter III in [17], $\varphi_{D|K}$ is a polynomial of degree k for each $K \in \mathcal{T}_{D,h}$, therefore $\varphi_D \in L^k_{D,h}$.

As a consequence, from standard interpolation estimates, for a regular solution $(\mathbf{E}_C, \mathbf{H}_D)$ it is straightforward to specify the order of convergence of the approximation method.

If one has no information about the regularity of the solution, by a density argument it is possible to prove the convergence of the finite element scheme provided that the permeability coefficient $\boldsymbol{\mu}$ is regular enough in Ω_D (say, a constant as in the usual physical case) or if the family of meshes $\{\mathcal{T}_h^D\}_h$ is obtained by refining a coarse mesh $\mathcal{T}_{h^*}^D$.

In fact, when $\boldsymbol{\mu}$ is constant we know that the harmonic field $\boldsymbol{\varrho}_D$ is regular enough to define the interpolation $\mathbf{\Pi}_{D,h}\boldsymbol{\varrho}_D$ (see [6], [2]). Since $\mathbf{H}_D = \operatorname{grad}\psi_D + I\boldsymbol{\varrho}_D$, a density argument applied to ψ_D permits to conclude the proof. In the other case, first we note that, as seen in Proposition 5.1, we can write $\boldsymbol{\varrho}_D = \operatorname{grad} g^{\boldsymbol{\lambda}_D} + \boldsymbol{\lambda}_D$. Then, knowing that $\{\mathcal{T}_h^D\}_h$ is a refinement of $\mathcal{T}_{h^*}^D$, it follows $\boldsymbol{\lambda}_D \in \mathbf{N}^k_{D,h}$. Therefore, as proved above, since $\mathbf{curl}\,\boldsymbol{\lambda}_D = \mathbf{0}$ in Ω_D we have $\boldsymbol{\lambda}_D = \mathbf{\Pi}_{D,h}\boldsymbol{\lambda}_D \in \mathbf{Z}^k_{D,h}$, and a density argument for $\psi_D + g^{\boldsymbol{\lambda}_D}$ gives the result. □

For the current excitation problem the finite element approach reads

$$\begin{cases} \text{Find } (\mathbf{E}_{C,h}, \widehat{\psi}_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}/\mathbb{C} : \\[2ex] \mathcal{A}((\mathbf{E}_{C,h}, \operatorname{grad}\widehat{\psi}_{D,h}), (\mathbf{w}_{C,h}, \operatorname{grad}\phi_{D,h})) \\ \qquad = -i\omega \int_{\Omega_C} \mathbf{J}_C \cdot \overline{\mathbf{w}}_{C,h} + i\omega I \int_\Gamma \overline{\mathbf{w}}_{C,h} \times \mathbf{n}_C \cdot \boldsymbol{\lambda}^h_D - \omega^2 I \int_{\Omega_D} \boldsymbol{\mu}\boldsymbol{\lambda}^h_D \cdot \operatorname{grad}\overline{\phi}_{D,h} \\[2ex] \text{for all } (\mathbf{w}_{C,h}, \phi_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}/\mathbb{C}. \end{cases} \tag{31}$$

Recall that $\mathbf{H}_D = \operatorname{grad}\widehat{\psi}_D + I\boldsymbol{\lambda}^h_D$ for some $\widehat{\psi}_D \in H^1(\Omega_D)$ (that in fact depends on h). Setting $\mathbf{H}_{D,h} = \operatorname{grad}\widehat{\psi}_{D,h} + I\boldsymbol{\lambda}^h_D$ from (27) and (31) we have the following equation for the error

$$\begin{aligned} &\mathcal{A}((\mathbf{E}_C - \mathbf{E}_{C,h}, \mathbf{H}_D - \mathbf{H}_{D,h}), (\mathbf{w}_{C,h}, \operatorname{grad}\phi_{D,h})) \\ &\quad = \mathcal{A}((\mathbf{E}_C - \mathbf{E}_{C,h}, \operatorname{grad}\widehat{\psi}_D - \operatorname{grad}\widehat{\psi}_{D,h}), (\mathbf{w}_{C,h}, \operatorname{grad}\phi_{D,h})) = 0 \end{aligned}$$

for each $(\mathbf{w}_{C,h}, \phi_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}/\mathbb{C}$. Therefore, the coerciveness of $\mathcal{A}(\cdot,\cdot)$ gives

$$\begin{aligned} &\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3} \\ &\quad = \|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_C)} + \|\operatorname{grad}\widehat{\psi}_D - \operatorname{grad}\widehat{\psi}_{D,h}\|_{(L^2(\Omega_D))^3} \\ &\quad \le C\big(\|\mathbf{E}_C - \mathbf{w}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_C)} + \|\operatorname{grad}\widehat{\psi}_D - \operatorname{grad}\phi_{D,h}\|_{(L^2(\Omega_D))^3}\big) \end{aligned}$$

for each $(\mathbf{w}_{C,h}, \phi_{D,h}) \in \mathbf{W}^k_{C,h} \times L^k_{D,h}$. Therefore

$$\begin{aligned} &\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3} \\ &\quad \le C \inf_{(\mathbf{w}_{C,h}, \mathbf{z}_{D,h}) \in \mathbf{W}^k_{C,h} \times \mathbf{Z}^k_{D,h}(I)} \big(\|\mathbf{E}_C - \mathbf{w}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_C)} + \|\mathbf{H}_D - \mathbf{z}_{D,h}\|_{(L^2(\Omega_D))^3}\big), \end{aligned}$$

where

$$\mathbf{Z}^k_{D,h}(I) := \operatorname{grad} L^k_{D,h} + I\boldsymbol{\lambda}^h_D .$$

The convergence of the approximation scheme can be proved following the arguments presented in Remark 5.1 (the only difference is that now we work with the space $\mathbf{Z}^k_{D,h}(I)$ instead of $\mathbf{Z}^k_{D,h}$, and this fact gives no problem to the procedure).

Once we have obtained $\mathbf{E}_{C,h}$ and $\widehat{\psi}_{D,h}$ we can compute

$$V_h := \int_\Gamma \mathbf{E}_{C,h} \times \mathbf{n}_C \cdot \boldsymbol{\lambda}^h_D + i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_{D,h} \cdot \boldsymbol{\lambda}^h_D .$$

This quantity is an approximation of the voltage, that, from (16), can be written as

$$V = \int_\Gamma \mathbf{E}_C \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_D + i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_D \cdot \boldsymbol{\varrho}_D \,.$$

In fact, let us introduce the auxiliary quantity

$$\widehat{V}_h := \int_\Gamma \mathbf{E}_{C,h} \times \mathbf{n}_C \cdot \boldsymbol{\varrho}_D + i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_{D,h} \cdot \boldsymbol{\varrho}_D \,.$$

We easily have

$$|V - \widehat{V}_h| \le C_1(\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}) \,.$$

On the other hand, taking $\mathbf{w}_{C,h} = \mathbf{0}$ in (31), it is easy to see that

$$V_h = \int_\Gamma \mathbf{E}_{C,h} \times \mathbf{n}_C \, (\operatorname{grad} \phi_{D,h} + \boldsymbol{\lambda}_D^h) + i\omega \int_{\Omega_D} \boldsymbol{\mu}\mathbf{H}_{D,h} \cdot (\operatorname{grad} \phi_{D,h} + \boldsymbol{\lambda}_D^h)$$

for all $\phi_{D,h} \in L^k_{D,h}$. Thus

$$|\widehat{V}_h - V_h| \le C_2 \left(\|\mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\right) \|\boldsymbol{\varrho}_D - (\operatorname{grad} \phi_{D,h} + \boldsymbol{\lambda}_D^h)\|_{(L^2(\Omega_D))^3} \,,$$

for all $\phi_{D,h} \in L^k_{D,h}$. Therefore

$$\begin{aligned} |V - V_h| \;& \le C_1 \left(\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\right) \\ & \qquad + C_2 \left(\|\mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\right) \inf_{z_{D,h} \in \mathbf{Z}^k_{D,h}(1)} \|\boldsymbol{\varrho}_D - z_{D,h}\|_{(L^2(\Omega_D))^3} \\ & \le \left(C_1 + C_2 \inf_{z_{D,h} \in \mathbf{Z}^k_{D,h}(1)} \|\boldsymbol{\varrho}_D - z_{D,h}\|_{(L^2(\Omega_D))^3}\right) \\ & \qquad\qquad \times \left(\|\mathbf{E}_C - \mathbf{E}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D - \mathbf{H}_{D,h}\|_{(L^2(\Omega_D))^3}\right) \\ & \qquad + C_2 \left(\|\mathbf{E}_C\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)} + \|\mathbf{H}_D\|_{(L^2(\Omega_D))^3}\right) \inf_{z_{D,h} \in \mathbf{Z}^k_{D,h}(1)} \|\boldsymbol{\varrho}_D - z_{D,h}\|_{(L^2(\Omega_D))^3} \,. \end{aligned}$$

If the permeability coefficient $\boldsymbol{\mu}$ is a constant in Ω_D or if the family of meshes $\{\mathcal{T}_h^D\}_h$ is a refinement of a coarse mesh $\mathcal{T}_{h^*}^D$, the convergence can be proved as in Remark 5.1,

6 Numerical results

The finite element method presented above has been implemented in Matlab, using Nédélec edge elements of first order for the electric field in the conductor, and scalar Lagrangian P_1 elements for the magnetic potential in the insulator.

The method has been tested by solving a problem with a known analytical solution. Since this problem has been already presented in [7], we just give a brief description of it, and refer the reader to the quoted paper for details.

The conducting domain Ω_C and the whole domain Ω are two coaxial cylinders of radii R_C and R_D, respectively, with height L. An alternating current of intensity $\mathbb{I}(t) = I\cos(\omega t)$ is traversing the conductor in the axial direction. Supposing that the physical parameters $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are constant scalars, the solution of the problem in cylindrical coordinates is given by

$$\begin{aligned} \mathbf{E}_C(r,\theta,z) &= \frac{\gamma}{2\pi R_C \boldsymbol{\sigma}} \frac{\mathcal{I}_0(\gamma r)}{\mathcal{I}_1(\gamma R_C)} \, \mathbf{e}_z \quad \text{in } \Omega_C, \\ \mathbf{H}_C(r,\theta,z) &= \frac{I}{2\pi R_C} \frac{\mathcal{I}_1(\gamma r)}{\mathcal{I}_1(\gamma R_C)} \, \mathbf{e}_\theta \quad \text{in } \Omega_C, \\ \mathbf{H}_D(r,\theta,z) &= \frac{I}{2\pi r} \, \mathbf{e}_\theta \quad \text{in } \Omega_D, \end{aligned}$$

where $\mathcal{I}_0$ and $\mathcal{I}_1$ denote the modified Bessel functions of the first kind and order 0 and 1, respectively, and $\gamma = \sqrt{i\omega\boldsymbol{\mu}\boldsymbol{\sigma}}$. Moreover, for this particular geometry it holds $\boldsymbol{\varrho}_D = \frac{1}{2\pi r}\,\mathbf{e}_\theta$, so $\mathbf{H}_D = I\boldsymbol{\varrho}_D$.

Once the fields and the function $\boldsymbol{\varrho}_D$ are known, the value of V is computed from the expression (20) to obtain

$$V = \frac{\gamma LI}{2\pi\boldsymbol{\sigma}R_C}\frac{\mathcal{I}_0(\gamma R_C)}{\mathcal{I}_1(\gamma R_C)} + i\omega\boldsymbol{\mu}\frac{LI}{2\pi}\ln\frac{R_D}{R_C}.$$

For our particular case we have used the following geometry and data

$$\begin{aligned} R_C &= 0.25 \text{ m}, \\ R_D &= 0.5 \text{ m}, \\ L &= 0.25 \text{ m}, \\ \boldsymbol{\sigma} &= 151565.8\ (\Omega\text{m})^{-1}, \\ \boldsymbol{\mu} &= 4\pi 10^{-7}\ \text{Hm}^{-1}, \\ \omega &= 50 \times 2\pi \text{ rad/s}, \end{aligned}$$

and either assigned current intensity or voltage,

$$I = 10^4 \text{ A}, \quad \text{or} \quad V = 0.08979 + 0.14680i,$$

where the value of V has been computed for an intensity of 10^4 A.

To test the order of convergence, the problem has been solved in four successively refined meshes, for either assigned intensity or voltage. We notice that the only approach implemented in our program is that in which the function λ_D^h depends on the mesh, namely, problems (30) and (31). We present in Tables 1 and 2 the relative errors of our numerical solutions against the analytical solution, that have been set as follows:

$$e_E = \frac{\|E_C - E_{C,h}\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}}{\|E_C\|_{\boldsymbol{H}(\mathbf{curl}\,;\Omega_C)}}, \qquad e_V = \frac{|V - V_h|}{|V|}$$
$$e_H = \frac{\|H_D - H_{D,h}\|_{(L^2(\Omega_D))^3}}{\|H_D\|_{(L^2(\Omega_D))^3}}, \qquad e_I = \frac{|I - I_h|}{|I|}.$$

Finally, Figures 4 and 5 show the plots in a log-log scale of the relative errors versus the degrees of freedom. A linear dependence on the mesh size is obtained for the errors of electric and magnetic fields, either for assigned intensity or voltage. This dependence turns out to be more than linear for the errors of voltage and intensity.

Elements	DoF	e_E	e_H	e_V
2304	1684	0.2341	0.1693	0.0312
18432	11240	0.1132	0.0847	0.0089
62208	35580	0.0750	0.0567	0.0048
147456	81616	0.0561	0.0425	0.0018

Table 1: Relative errors for assigned intensity.

Elements	DoF	e_E	e_H	e_I
2304	1685	0.2336	0.1685	0.0274
18432	11241	0.1132	0.0847	0.0085
62208	35581	0.0750	0.0566	0.0041
147456	81617	0.0561	0.0425	0.0024

Table 2: Relative errors for assigned voltage.

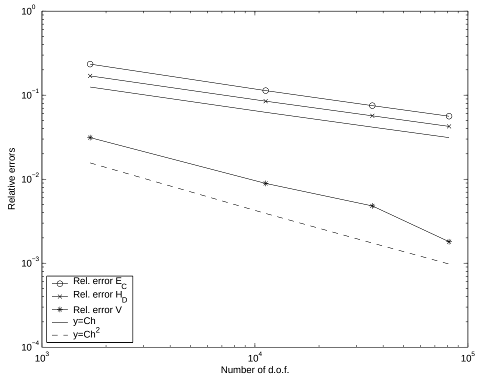

Figure 4: Relative error versus number of d.o.f. (assigned intensity).

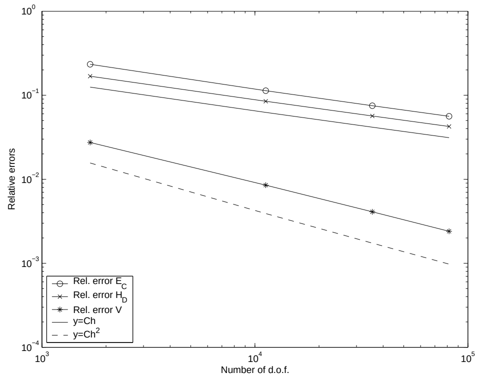

Figure 5: Relative error versus number of d.o.f. (assigned voltage).

The method has been also applied to a more realistic problem which was presented in [9]. In this case the domain is a cylindrical electric furnace with three ELSA electrodes equally distanced. The dimensions of the furnace are the following: furnace height: 2 m.; furnace diameter: 8.88 m.; electrodes height: 1.25 m.; electrodes diameter: 1 m.; distance from the center of the electrodes to the wall: 3 m.

The three ELSA electrodes inside the furnace are formed by a graphite core of 0.4 m. of diameter, and an outer part of Söderberg paste. The electric current enters the electrodes through horizontal copper bars of rectangular section (0.07 m. $\times$ 0.25 m.), connecting the top of the electrode with the external boundary.

For the simulation we have considered the angular frequency $\omega = 50 \times 2\pi$ rad/s, and the electric conductivities $\sigma = 10^6(\Omega\text{m})^{-1}$ for graphite, $\sigma = 10^4(\Omega\text{m})^{-1}$ for Söderberg paste, and $\sigma = 5 \times 10^6(\Omega\text{m})^{-1}$ for copper. We have imposed an intensity of $I_j = 7 \times 10^4 A$ for each electrode, using the approach that has been explained at the end of Section 3 for the case of a non-connected conductor. With the same notation used there, the boundaries $\Gamma_{E,j}$ correspond to the contacts of the copper bars on the boundary of the furnace, and $\Gamma_{J,j}$ to the bottom of the electrodes.

In Figure 6 we present the modulus of the magnetic potential, i.e., $|\widehat{\psi}_{D,h} + \sum_{j=1}^{3} I_j \eta_{D,j,h}|$, where $\eta_{D,j,h}$ are the piecewise linear functions with a jump of height 1 on the 'cutting' surfaces $\Sigma_{j,h}$. In Figures 7 and 8 the modulus of the current density $\mathbf{J}_h = \sigma \mathbf{E}_{C,h}$ on a horizontal and a vertical section of one electrode is shown.

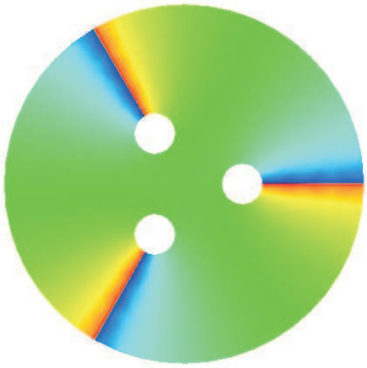

Figure 6: Magnetic potential in the dielectric.

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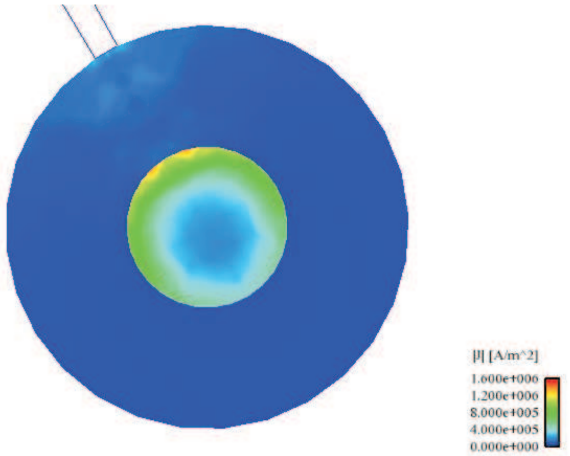

Figure 7: $|\mathbf{J}_h|$ on a horizontal section of one electrode.

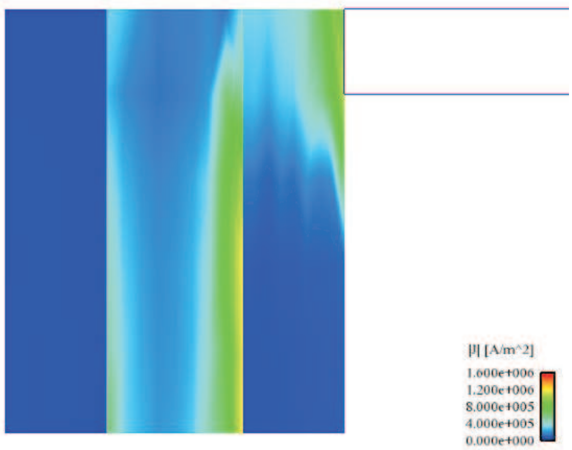

Figure 8: $|\mathbf{J}_h|$ on a vertical section of one electrode.

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