Solving the problem of electrostatics with a dipole source by means of the duality method

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Abstract

Aim of this note is to give a simple and direct proof of the existence and uniqueness of the solution to the problem of electrostatics when the source (namely, the applied current density) is a dipole. The result is obtained by using the classical *duality method*.

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1. Introduction

The Maxwell equations read

$$
\begin{aligned}
&\operatorname{curl}\mathbf{H} - \boldsymbol{\epsilon}\tfrac{\partial \mathbf{E}}{\partial t} = \boldsymbol{\sigma}\mathbf{E} + \mathbf{J}_e \\
&\operatorname{curl}\mathbf{E} + \boldsymbol{\mu}\tfrac{\partial \mathbf{H}}{\partial t} = \mathbf{0} \\
&\operatorname{div}(\boldsymbol{\mu}\mathbf{H}) = 0\,,
\end{aligned}
$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively, $\mathbf{J}_e$ is the applied current density, $\boldsymbol{\epsilon}$ is the electric permittivity, $\boldsymbol{\mu}$ is the magnetic permeability and $\boldsymbol{\sigma}$ is the electric conductivity.

By disregarding the terms with the time derivative one obtains the static model

$$
\begin{aligned}
&\operatorname{curl}\mathbf{H} = \boldsymbol{\sigma}\mathbf{E} + \mathbf{J}_e \\
&\operatorname{curl}\mathbf{E} = \mathbf{0} \\
&\operatorname{div}(\boldsymbol{\mu}\mathbf{H}) = 0\,.
\end{aligned}
$$

Note that a consequence of the first equation is that $\operatorname{div}(\boldsymbol{\sigma}\mathbf{E} + \mathbf{J}_e) = 0$.

If the domain D where the problem is considered is simply-connected, one can introduce a scalar electric potential u such that $\mathbf{E} = -\operatorname{grad} u$ in D; it satisfies the electrostatics problem

$$\operatorname{div}(\boldsymbol{\sigma}\operatorname{grad} u) = \operatorname{div}\mathbf{J}_e \quad \text{in } D\,.$$

For many physical problems the conductivity $\boldsymbol{\sigma}$ vanishes outside a region Ω (a conductor), completely contained in D (one can think that $D \setminus \overline{\Omega}$ is a domain filled by air).

Moreover, it happens that also the applied current density vanishes outside Ω. Let us recall that, due to the properties of the div operator, the equation $\operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} u - \mathbf{J}_e) = 0$ in D can be always rewritten as $\operatorname{div}[(\boldsymbol{\sigma} \operatorname{grad} u - \mathbf{J}_e)_{|\Omega}] = 0$ in Ω, $\operatorname{div}[(\boldsymbol{\sigma} \operatorname{grad} u - \mathbf{J}_e)_{|D\setminus\overline{\Omega}}] = 0$ in $D \setminus \overline{\Omega}$ and $(\boldsymbol{\sigma} \operatorname{grad} u - \mathbf{J}_e)_{|\Omega} \cdot \mathbf{n} = (\boldsymbol{\sigma} \operatorname{grad} u - \mathbf{J}_e)_{|D\setminus\overline{\Omega}} \cdot \mathbf{n}$ on the interface $\partial\Omega$. Therefore, the assumption that $\boldsymbol{\sigma}$ and $\mathbf{J}_e$ are supported in Ω has the consequence that the electrostatics problem reads

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} u) = \operatorname{div} \mathbf{J}_e & \text{in } \Omega \\ (\boldsymbol{\sigma} \operatorname{grad} u) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{1}$$

Electroencephalography (EEG) is a non-invasive technique for detecting the brain activity from the measure of the electric field (or of its potential) on the head surface; in mathematical terms, an inverse problem in which one wants to determine the source, that has generated the electric field, by measuring the boundary value of the electric potential.

An interesting case is when the source is a dipole, namely, $\mathbf{J}_e = \mathbf{p}_0 \delta_{\mathbf{x}_0}$, where $\mathbf{p}_0 \neq \mathbf{0}$, $\mathbf{x}_0 \in \Omega$ and $\delta_{\mathbf{x}_0}$ denotes the Dirac delta distribution centered at $\mathbf{x}_0$; for instance, this is a mathematical model for an epileptic focus in human brain (see, e.g., Sarvas [1], He and Romanov [2], Ammari *et al.* [3], Albanese and Monk [4], Wolters *et al.* [5], Lew *et al.* [6]).

Let us also recall that in the brain the conductivity $\boldsymbol{\sigma}$ is non-constant, and even non-isotropic (see, e.g., Marin *et al.* [7]). Therefore, a reasonable model has to assume that $\boldsymbol{\sigma}$ is a symmetric and positive definite matrix, with non-constant entries σ_{lm}, $l, m = 1, 2, 3$.

In this case even the forward problem, that is the solution of (1) with the assigned $\mathbf{J}_e = \mathbf{p}_0 \delta_{\mathbf{x}_0}$, is a non-standard mathematical problem. Usually it is solved by means of the so-called *subtraction method* (see Awada *et al.* [8], Wolters *et al.* [5], Lew *et al.* [6]); it consists in writing the electric potential u as $u = K + w$, where K is the solution to

$$\operatorname{div}(\boldsymbol{\sigma}_0 \operatorname{grad} K) = \operatorname{div}(\mathbf{p}_0 \delta_{\mathbf{x}_0}) \,,$$

$\boldsymbol{\sigma}_0$ being the constant matrix $\boldsymbol{\sigma}(\mathbf{x}_0)$, and then looking for the solution w to

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} w) = -\operatorname{div}[(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) \operatorname{grad} K] & \text{in } \Omega \\ (\boldsymbol{\sigma} \operatorname{grad} w) \cdot \mathbf{n} = -(\boldsymbol{\sigma} \operatorname{grad} K) \cdot \mathbf{n} & \text{on } \partial\Omega \,. \end{cases} \tag{2}$$

It can be proved (see Wolters *et al.* [5], Lew *et al.* [6]) that K is a smooth function for $\mathbf{x} \neq \mathbf{x}_0$, and has a singularity like $|\mathbf{x} - \mathbf{x}_0|^{-2}$ for $\mathbf{x} \approx \mathbf{x}_0$. Therefore the Neumann datum $-\boldsymbol{\sigma} \operatorname{grad} K \cdot \mathbf{n}$ is smooth; on the contrary, the right hand side $-\operatorname{div}[(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) \operatorname{grad} K]$ has a singularity like $|\mathbf{x} - \mathbf{x}_0|^{-3}$ (provided that $\boldsymbol{\sigma}$ is Lipschitz continuous), hence it is not even locally summable. To overcome this difficulty, a suitable assumption on $\boldsymbol{\sigma}$ has to be made (see (3)).

Before going on let us make clear some notations. We denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with support strictly contained in Ω; by $C^k(\Omega)$ (respectively, $C^k(\overline{\Omega})$), $k = 0, 1, 2, \ldots$, the space of functions with k-order derivatives that are continuous in Ω (respectively, in $\overline{\Omega}$)); by $H^k(\Omega)$, $k = 0, 1, 2, \ldots$, the Sobolev space of (measurable) functions with k-order distributional derivatives that are square-summable in Ω (we also write $L^2(\Omega)$ instead of $H^0(\Omega)$); by $W^{k,p}(\Omega)$, $k = 0, 1, 2, \ldots$, $1 \leq p \leq \infty$,

$p \neq 2$, the space of (measurable) functions with k-order distributional derivatives that are either p-summable (when $p < \infty$) or essentially bounded (when $p = \infty$) in Ω (we also write $L^p(\Omega)$ instead of $W^{0,p}(\Omega)$); finally, by $H^1_0(\Omega)$ we indicate the space of functions belonging to $H^1(\Omega)$ and having the trace vanishing on $\partial\Omega$.

The assumptions in Wolters *et al.* [5], Lew *et al.* [6] are that $\boldsymbol{\sigma}$ is a symmetric and positive definite matrix, with entries σ_{lm} belonging to $L^\infty(\Omega)$ for $l, m = 1, 2, 3$, and moreover that the *homogeneity condition* is satisfied. This condition reads:

$$\text{there exists } r_0 > 0 \text{ such that } \boldsymbol{\sigma}(\mathbf{x}) \text{ is a constant matrix for each } \mathbf{x} \in B_{r_0}(\mathbf{x}_0), \tag{3}$$

where $B_{r_0}(\mathbf{x}_0) := \{\mathbf{x} \in \Omega \,|\, |\mathbf{x} - \mathbf{x}_0| < r_0\}$. In such a way the singularity of the right hand side $-\operatorname{div}[(\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) \operatorname{grad} K]$ disappears, and therefore these assumptions permit to write problem (2) in a variational form, where the right hand side turns out to be a linear and continuous functional in the Sobolev space $H^1(\Omega)$. Lax–Milgram lemma then gives that the solution w exists and is unique in $H^1(\Omega)$.

With our approach, that is based on the so-called *duality method* (see, e.g., Višik and Sobolev [9], Lions [10], Stampacchia [11]), we are able in particular to weaken the assumption on the conductivity, only requiring that $\boldsymbol{\sigma}$ is a symmetric and positive definite matrix, with entries σ_{lm} belonging to $L^\infty(\Omega)$ for $l, m = 1, 2, 3$, and moreover that

$$\text{there exists } r_0 > 0 \text{ such that } \sigma_{lm} \in W^{2,\infty}(B_{r_0}(\mathbf{x}_0)) \text{ for } l, m = 1, 2, 3\,. \tag{4}$$

An even weaker assumption is presented in (10).

2. The weak problem

The formal expression of the problem reads

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} u) = \operatorname{div}(\mathbf{p}_0 \delta_{\mathbf{x}_0}) & \text{in } \Omega \\ (\boldsymbol{\sigma} \operatorname{grad} u) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{5}$$

Clearly, the solution u is defined up to an additive constant.

We want to give a weak formulation of problem (5). Introduce the linear space

$$X := \{\varphi \in H^1(\Omega) \,|\, \varphi \in C^1(B_{r_*}(\mathbf{x}_0)), \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) \in H^1_0(\Omega), (\boldsymbol{\sigma} \operatorname{grad} \varphi) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,,$$

where $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$ and $0 < r_* < r_0$ is a fixed number. We proceed formally: multiplying the first equation in (5) by $\varphi \in X$, integrating in Ω and integrating by parts we readily find

$$\begin{aligned} \int_\Omega \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} u)\, \varphi &= -\int_\Omega (\boldsymbol{\sigma} \operatorname{grad} u) \cdot \operatorname{grad} \varphi + \int_{\partial\Omega} (\boldsymbol{\sigma} \operatorname{grad} u) \cdot \mathbf{n}\, \varphi \\ &= \int_\Omega u \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) - \int_{\partial\Omega} u\, (\boldsymbol{\sigma} \operatorname{grad} \phi) \cdot \mathbf{n} + \int_{\partial\Omega} (\boldsymbol{\sigma} \operatorname{grad} u) \cdot \mathbf{n}\, \varphi \\ &= \int_\Omega u \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) \end{aligned}$$

$$\int_\Omega \operatorname{div}(\mathbf{p}_0 \delta_{\mathbf{x}_0})\, \varphi = -\int_\Omega \mathbf{p}_0 \cdot \operatorname{grad} \varphi\, \delta_{\mathbf{x}_0} = -\,\mathbf{p}_0 \cdot \operatorname{grad} \varphi(\mathbf{x}_0)\,,$$

having taken into account the boundary conditions satisfied by u and φ. Since we know that $\operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) \in H^1_0(\Omega)$, the term $\int_\Omega u \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi)$ has a meaning also for $u \in H^{-1}(\Omega)$, the dual space of $H^1_0(\Omega)$, and has to be expressed as a duality pairing, say, $\langle u, \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi)\rangle$.

Let us denote by $\hat\eta$ a smooth function, vanishing on $\partial\Omega$ and strictly positive in Ω, and such that $\int_\Omega \hat\eta = 1$ (in particular, $\hat\eta \in H^1_0(\Omega)$). A condition which filters out additive constants, and therefore is suitable for assuring uniqueness of the solution u, is for instance $\langle u, \hat\eta\rangle = 0$.

We are now in a position to describe the weak formulation of (5) that we consider:

$$\text{find } u \in H^{-1}(\Omega) \; : \begin{cases} \langle u, \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi)\rangle = -\,\mathbf{p}_0 \cdot \operatorname{grad} \varphi(\mathbf{x}_0) & \forall\, \varphi \in X \\ \langle u, \hat\eta\rangle = 0\,. \end{cases} \tag{6}$$

3. Existence and uniqueness

This section is devoted to the proof of the existence and uniqueness of the solution u of problem (6). From now on $\Omega \subset \mathbb{R}^3$ will be an open connected bounded set with Lipschitz continuous boundary $\partial\Omega$.

Theorem 3.1. *There exists a solution u to (6). Moreover, $u \in L^q(\Omega)$ for each q with $1 \le q < \frac{3}{2}$.*

Proof. We use an approximation argument. Let us denote by δ_k a sequence of functions such that $\delta_k \in C^\infty_0(B_{r_*}(\mathbf{x}_0))$, $\delta_k \ge 0$, $\int_\Omega \delta_k = 1$ and $\int_\Omega \delta_k\, \xi \to \xi(\mathbf{x}_0)$ for each $\xi \in C^0(B_{r_*}(\mathbf{x}_0))$. We consider the solution $u_k \in H^1(\Omega)$ of the Neumann problem

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} u_k) = \operatorname{div}(\mathbf{p}_0 \delta_k) & \text{in } \Omega \\ (\boldsymbol{\sigma} \operatorname{grad} u_k) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \int_\Omega u_k\, \hat\eta = 0\,. \end{cases}$$

The existence and uniqueness of u_k is assured as $\int_\Omega \operatorname{div}(\mathbf{p}_0\delta_k) = \int_{\partial\Omega} \mathbf{p}_0 \cdot \mathbf{n}\, \delta_k = 0$, hence the compatibility condition is satisfied. In particular, by integrating by parts we see that u_k satisfies

$$\int_\Omega u_k \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) = -\int_\Omega \mathbf{p}_0 \cdot \operatorname{grad} \varphi\, \delta_k \quad \forall\, \varphi \in X\,.$$

Take now $\psi \in H^1_0(\Omega)$: we want to find an uniform estimate of $\langle u_k, \psi\rangle$. Consider the solution $\hat\varphi$ of the Neumann problem

$$\begin{cases} \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \hat\varphi) = \psi - \big(\int_\Omega \psi\big)\hat\eta & \text{in } \Omega \\ (\boldsymbol{\sigma} \operatorname{grad} \hat\varphi) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \int_\Omega \hat\varphi = 0\,. \end{cases} \tag{7}$$

Since $\int_\Omega \big[\psi - \big(\int_\Omega \psi\big)\hat\eta\big] = \int_\Omega \psi - \big(\int_\Omega \psi\big)\int_\Omega \hat\eta = 0$, we have a unique solution $\hat\varphi \in H^1(\Omega)$. On the other hand, we have $\big[\psi - \big(\int_\Omega \psi\big)\hat\eta\big] \in H^1_0(\Omega)$ and the regularity results for elliptic problems (see, e.g., Evans [12], Sect. 6.3.1) yield $\hat\varphi \in H^3(B_{r_*}(\mathbf{x}_0))$. The Sobolev immersion theorem (see, e.g., Evans [12], Sect. 5.6.3) also gives $\hat\varphi \in C^1(\overline{B_{r_*}(\mathbf{x}_0)})$, hence $\hat\varphi \in X$. Moreover, $\|\hat\varphi\|_{C^1(\overline{B_{r_*}(\mathbf{x}_0)})} \leq c_0 \|\psi\|_{H^1(\Omega)}$, where c_0 depends on $\boldsymbol{\sigma}$, $\hat\eta$, r_*, but not on ψ.

We are now in a position to obtain the needed estimate. We have

$$\begin{aligned}
|\langle u_k, \psi\rangle| &= \big| \textstyle\int_\Omega u_k\, \psi \big| = \big| \int_\Omega u_k \big[\psi - \big(\int_\Omega \psi\big)\hat\eta\big]\big| \\
&= \big| \textstyle\int_\Omega u_k \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \hat\varphi)\big| = \big| - \int_\Omega \mathbf{p}_0 \cdot \operatorname{grad} \hat\varphi\, \delta_k \big| \\
&\leq |\mathbf{p}_0|\, \| \operatorname{grad} \hat\varphi\|_{C^0(\overline{B_{r_*}(\mathbf{x}_0)})} \textstyle\int_\Omega \delta_k \leq c_0\, |\mathbf{p}_0| \|\psi\|_{H^1(\Omega)} \ .
\end{aligned}$$

In other words,

$$\|u_k\|_{H^{-1}(\Omega)} := \sup_{\psi \in H^1_0(\Omega)} \frac{|\langle u_k, \psi\rangle|}{\|\psi\|_{H^1(\Omega)}} \leq c_0\, |\mathbf{p}_0| \, .$$

We can thus select a subsequence (still denoted by u_k) which converges in $H^{-1}(\Omega)$ to $u \in H^{-1}(\Omega)$. In particular, for each $\varphi \in X$

$$\begin{aligned}
&\textstyle\int_\Omega u_k \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) = \langle u_k, \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi)\rangle \to \langle u, \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi)\rangle \\
&- \textstyle\int_\Omega \mathbf{p}_0 \cdot \operatorname{grad} \varphi\, \delta_k \to - \mathbf{p}_0 \cdot \operatorname{grad} \varphi(\mathbf{x}_0) \, .
\end{aligned}$$

Finally,

$$0 = \int_\Omega u_k\, \hat\eta = \langle u_k, \hat\eta\rangle \to \langle u, \hat\eta\rangle \, ,$$

and u is a solution to (6).

By the Sobolev immersion theorem we also know that $W^{2,p}(B_{r_*}(\mathbf{x}_0)) \subset C^1(\overline{B_{r_*}(\mathbf{x}_0)})$ for $p > 3$. Moreover, if we assume that $\psi \in L^p(\Omega)$, the regularity results for elliptic problems assure that the solution $\hat\varphi$ of (7) belongs to $W^{2,p}(B_{r_*}(\mathbf{x}_0))$ and that $\|\hat\varphi\|_{C^1(\overline{B_{r_*}(\mathbf{x}_0)})} \leq \hat c_0 \|\psi\|_{L^p(\Omega)}$ (here it is enough to assume that the conductivity belongs to $W^{1,\infty}(B_{r_0}(\mathbf{x}_0))$; see, e.g., Gilbarg and Trudinger [13], Sect. 9.5). Repeating the argument above we find

$$\big| \textstyle\int_\Omega u_k\, \psi \big| = \big| - \int_\Omega \mathbf{p}_0 \cdot \operatorname{grad} \hat\varphi\, \delta_k\big| \leq |\mathbf{p}_0|\, \| \operatorname{grad} \hat\varphi\|_{C^0(\overline{B_{r_*}(\mathbf{x}_0)})} \leq \hat c_0\, |\mathbf{p}_0| \|\psi\|_{L^p(\Omega)} \ .$$

Hence we have obtained that

$$\|u_k\|_{L^q(\Omega)} := \sup_{\psi \in L^p(\Omega)} \frac{|\int_\Omega u_k\, \psi|}{\|\psi\|_{L^p(\Omega)}} \leq \hat c_0\, |\mathbf{p}_0|$$

for q such that $\frac{1}{q} + \frac{1}{p} = 1$ (in particular, from $p > 3$ we have $q < \frac{3}{2}$). Passing to the limit with respect to k it is proved that $u \in L^q(\Omega)$. □

Theorem 3.2. *The solution u to (6) is unique.*

Proof. Let u be any solution to (6). For each $\psi \in H^1_0(\Omega)$, consider the solution $\hat\varphi$ of (7). Using it in (6) we find

$$\begin{aligned}
|\langle u, \psi\rangle| &= |\langle u, \psi - \big(\textstyle\int_\Omega \psi\big)\hat\eta\rangle| = |\langle u, \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \hat\varphi)\rangle| = | - \mathbf{p}_0 \cdot \operatorname{grad} \hat\varphi(\mathbf{x}_0)| \\
&\leq |\mathbf{p}_0|\, \| \operatorname{grad} \hat\varphi\|_{C^0(\overline{B_{r_*}(\mathbf{x}_0)})} \leq c_0\, |\mathbf{p}_0| \|\psi\|_{H^1(\Omega)} \ ,
\end{aligned}$$

hence $\|u\|_{H^{-1}(\Omega)} \leq c_0\, |\mathbf{p}_0|$, and uniqueness follows. □

Remark 3.3. *The approach we have presented is not based on the Hilbert structure of the Sobolev spaces $H^k(\Omega)$, but on duality. Therefore, renouncing to the choice of the summability exponent $p = 2$, one realizes that it is also possible to consider the problem*

$$\text{find } u_p \in L^{p^*}(\Omega) \, : \begin{cases} \int_\Omega u_p \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) = -\, \mathbf{p}_0 \cdot \operatorname{grad} \varphi(\mathbf{x}_0) & \forall\, \varphi \in X_p \\ \int_\Omega u_p \hat{\eta} = 0 \,, \end{cases} \tag{8}$$

where p is a fixed number satisfying $3 < p < +\infty$, p^* is the Hölder dual exponent defined by $\frac{1}{p^*} + \frac{1}{p} = 1$, and

$$X_p := \{\varphi \in W^{1,p}(\Omega) \,|\, \varphi \in C^1(B_{r_*}(\mathbf{x}_0)), \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) \in L^p(\Omega), (\boldsymbol{\sigma} \operatorname{grad} \varphi) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \,.$$

Proceeding as before, one proves the existence and uniqueness of a solution to (8). Since for $3 < s < p$ one has $L^{s^*}(\Omega) \subset L^{p^*}(\Omega)$ and $X_s \supset X_p$, it is readily verified that $u_s = u_p$ for all finite values $s, p > 3$, therefore we have solved the problem

$$\text{find } u \in \bigcap_{p>3} L^{p^*}(\Omega) \, : \begin{cases} \int_\Omega u \operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) = -\, \mathbf{p}_0 \cdot \operatorname{grad} \varphi(\mathbf{x}_0) & \forall\, \varphi \in \bigcup_{p>3} X_p \\ \int_\Omega u \, \hat{\eta} = 0 \,. \end{cases} \tag{9}$$

The Sobolev immersion $H^1_0(\Omega) \subset L^6(\Omega)$ yields $L^{6/5}(\Omega) \subset H^{-1}(\Omega)$, therefore $\bigcap_{p>3} L^{p^*}(\Omega) \subset H^{-1}(\Omega)$. On the other hand the theory of elliptic regularity applied to $\operatorname{div}(\boldsymbol{\sigma} \operatorname{grad} \varphi) \in H^1_0(\Omega)$ gives $\varphi \in W^{1,p_0}(\Omega)$ for a suitable $p_0 > 3$, therefore $X \subset \bigcup_{p>3} X_p$. In conclusion, the solution to (9) is the solution to (6).

It is worth noting that this L^p-approach, instead of (4), only requires that

$$\text{there exists } r_0 > 0 \text{ such that } \sigma_{lm} \in W^{1,\infty}(B_{r_0}(\mathbf{x}_0)) \text{ for } l, m = 1, 2, 3 \,, \tag{10}$$

namely, local Lipschitz regularity of the conductivity.

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