

# Solving the curl-div system using divergence-free or curl-free finite elements

Alberto Valli

Dipartimento di Matematica, Università di Trento, Italy

or:

Why I say to my students that divergence-free finite elements are not a suitable option?

Joint papers with:

**Ana Alonso Rodríguez**

Dipartimento di Matematica, Università di Trento, Italy

and

**Enrico Bertolazzi**

Dipartimento di Ingegneria Industriale, Università di Trento, Italy

**Jessika Camaño**

Departamento de Matematica y Fisica Aplicadas, Universidad Catolica de la Santisima Concepcion, Chile

**Riccardo Ghiloni**

Dipartimento di Matematica, Università di Trento, Italy

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# Introduction

## The aim

Aim of this talk is the analysis of the following **three problems** and of their mutual relations:

- (a) finding **finite element potentials**, namely, solving by means of finite elements the problems  $\mathbf{grad} \psi = \mathbf{H}$ ,  $\mathbf{curl} \mathbf{A} = \mathbf{B}$ ,  $\mathbf{div} \mathbf{v} = G$ ;
- (b) finding suitable basis functions for the spaces of **curl-free** or **divergence-free finite elements**;
- (c) based on (a) and (b), devising **simple finite element schemes** for the solution of the **curl-div system**, which reads

$$\begin{cases} \mathbf{curl} \mathbf{u} = \mathbf{B} & \text{in } \Omega \\ \mathbf{div} \mathbf{u} = G & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = \mathbf{a} \text{ (or } \mathbf{u} \cdot \mathbf{n} = b) & \text{on } \partial\Omega. \end{cases} \quad (1)$$

# Finite element potentials

## First results

Determining the **necessary and sufficient** conditions for assuring that a function defined in a bounded domain  $\Omega \subset \mathbb{R}^3$  is the gradient of a scalar potential, or the curl of a vector potential, or the divergence of a vector field is one of the **most classical problem** of vector analysis.

The answer is well-known, and shows an interesting interplay of **differential calculus** and **topology** (see, e.g., Cantarella et al. (2002)).



## First results (cont'd)

- a vector field is the **gradient** of a scalar potential if and only if it is **curl free** and its **line integral** is vanishing on all the closed curves that furnish a basis of the **first homology group** of  $\overline{\Omega}$ ;
- a vector field is the **curl** of a vector potential if and only if it is **divergence free** and its **flux** is vanishing across all the closed surfaces that furnish a basis of the **second homology group** of  $\overline{\Omega}$ , or, equivalently, across (all but one) the connected components of  $\partial\Omega$ ;
- **each** scalar function is the **divergence** of a vector field.

## First results (cont'd)

However, this theoretical result only clarifies when the answer is positive, and does not say how to determine an explicit and efficient procedure for constructing **finite element** potentials.

Our approach is based on (simple) tools from **algebraic topology** and **graph theory**. We suppose to have:

- a **basis**  $\sigma_n$ ,  $n = 1, \dots, g$ , of the first homology group of  $\overline{\Omega}$ ;
- a **basis**  $\hat{\sigma}_n$ ,  $n = 1, \dots, g$ , of the first homology group of  $\mathbb{R}^3 \setminus \Omega$ ;
- a **spanning tree**  $\mathcal{S}_h$  of the graph given by the nodes and the edges of the mesh  $\mathcal{T}_h$ .

[**Note**: an easy way for constructing  $\sigma_n$  and  $\hat{\sigma}_n$  is presented in Hiptmair and Ostrowski (2002); the determination of a spanning tree is a standard procedure in graph theory.]

## First results (cont'd)

Let us also introduce the finite element spaces we will use:

- the space  $L_h$  of **continuous piecewise-linear elements**, with dimension  $n_v$ , the number of vertices in  $\mathcal{T}_h$ ;
- the space  $N_h$  of **Nédélec edge elements of degree 1**, with dimension  $n_e$ , the number of edges in  $\mathcal{T}_h$ ;
- the space  $RT_h$  of **Raviart-Thomas elements of degree 1**, with dimension  $n_f$ , the number of faces in  $\mathcal{T}_h$ ;
- the space  $PC_h$  of (discontinuous) **piecewise-constant elements**, with dimension  $n_t$ , the number of tetrahedra in  $\mathcal{T}_h$ .

## The grad problem

We want to solve  $\mathbf{grad} \psi_h = \mathbf{H}_h$  in the finite element context. This is an easy problem, and the only reason for considering it is that it is useful for understanding better the procedures needed for the other two problems.

The “right” finite elements are:  $\psi_h \in L_h$  a **piecewise-linear nodal** element,  $\mathbf{H}_h \in N_h$  a lowest order Nédélec **edge** element, and we only have to impose that the **line integral** of  $\mathbf{grad} \psi_h$  and  $\mathbf{H}_h$  on each **edge** of the mesh  $\mathcal{T}_h$  is the same.

The fundamental theorem of calculus says that

$$\psi_h(v_b) - \psi_h(v_a) = \int_e \mathbf{grad} \psi_h \cdot \boldsymbol{\tau} = \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} \quad (2)$$

for an edge  $e = [v_a, v_b]$ . Hence the linear system associated to  $\mathbf{grad} \psi_h = \mathbf{H}_h$  has exactly **two non-zero values** per row.

## The grad problem (cont'd)

Starting from a root  $v_*$  of the **spanning tree**  $\mathcal{S}_h$ , where, for the sake of uniqueness, we impose  $\psi_h(v_*) = 0$ , for an edge  $e' = [v_*, \hat{v}] \in \mathcal{S}_h$  we compute

$$\psi_h(\hat{v}) = \psi_h(v_*) + \int_{e'} \mathbf{H}_h \cdot \boldsymbol{\tau};$$

since  $\mathcal{S}_h$  is a spanning tree, going on in this way we can visit **all** the nodes of  $\mathcal{T}_h$ .

The spanning tree is therefore a tool for selecting the rows for which, using the additional equation  $\psi_h(v_*) = 0$ , one can **eliminate** the unknowns **one after the other**.

We have thus found a nodal element  $\psi_h$  such that its gradient has line integral on all the edges of the **spanning tree** equal to that of  $\mathbf{H}_h$ . Then is easy to show that the same is true for **all** the other edges.

## The grad problem (cont'd)

In other words, we have given a **constructive** way for solving the problem

$$\begin{cases} \mathbf{grad} \psi_h = \mathbf{H}_h & \text{in } \Omega \\ \psi_h(v_*) = 0. \end{cases} \quad (3)$$

Since it can be easily proved that  $n_e > n_v - 1 + g$  (the  $n_e$  edges of the graph are more than the  $n_v - 1$  edges in the spanning tree plus  $g$  edges, one for each homological cycle), this is a **full rank overdetermined** system with  $n_e + 1$  equations and  $n_v$  unknowns.

Problems with a similar structure will appear in the sequel.

## The curl problem

We want to solve  $\mathbf{curl} \mathbf{A}_h = \mathbf{B}_h$  in the finite element context.

The “right” finite elements are:  $\mathbf{A}_h \in N_h$  a lowest order Nédélec **edge** element,  $\mathbf{B}_h \in RT_h$  a lowest order Raviart–Thomas **face** element, and we only have to impose that the **flux** of  $\mathbf{curl} \mathbf{A}_h$  and  $\mathbf{B}_h$  on each **face** of the mesh  $\mathcal{T}_h$  is the same.

The **Stokes theorem** assures that

$$\int_{e_1} \mathbf{A}_h \cdot \boldsymbol{\tau} + \int_{e_2} \mathbf{A}_h \cdot \boldsymbol{\tau} + \int_{e_3} \mathbf{A}_h \cdot \boldsymbol{\tau} = \int_f \mathbf{curl} \mathbf{A}_h \cdot \boldsymbol{\nu}_f = \int_f \mathbf{B}_h \cdot \boldsymbol{\nu}_f, \quad (4)$$

where  $\partial f = e_1 \cup e_2 \cup e_3$ , hence the linear system associated to  $\mathbf{curl} \mathbf{A}_h = \mathbf{B}_h$  has exactly **three non-zero values** for each row.

## The curl problem (cont'd)

With respect to the preceding case:

- **three** unknowns per row instead of two.

Therefore, in order to devise an **efficient elimination algorithm**, it is useful to **fix** the value of other unknowns.

The best situation should occur when the number of the new equations agrees with the dimension of the **kernel** of the curl operator.

Since this kernel is given by the **gradients of nodal elements** plus the space generated by the basis of the **first de Rham cohomology group** of  $\Omega$ , we see that its dimension is equal to  $n_v - 1 + g$ .



## The curl problem (cont'd)

Having this in mind, we are led to the problem

$$\begin{cases} \mathbf{curl} \mathbf{A}_h = \mathbf{B}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{A}_h \cdot d\mathbf{s} = \rho_n & \forall n = 1, \dots, g \\ \int_{e'} \mathbf{A}_h \cdot \boldsymbol{\tau} = 0 & \forall e' \in \mathcal{S}_h, \end{cases} \quad (5)$$

for arbitrarily given constants  $\rho_n$ .

Since the number of edges  $e'$  in  $\mathcal{S}_h$  is  $n_v - 1$ , (5)<sub>3</sub> can be seen as a “**filter**” for **gradients**. On the other hand, homology and cohomology are in duality, hence (5)<sub>2</sub> can be seen as a “**filter**” for **cohomology fields**.

This is a **full rank overdetermined** system, with  $n_f + g + n_v - 1$  equations and  $n_e$  unknowns [recall that the **Euler–Poincaré formula** says that  $n_f + g + n_v - 1 = n_e + n_t + p$ ]. It is not difficult to prove that **it has a unique solution**.

## Webb–Forghani algorithm

Webb and Forghani (1989) proposed this **solution algorithm**:

### Algorithm

- 1 *take a face  $f$  for which at least one edge unknown has not yet been assigned*
  - 1 *if exactly one edge unknown is not determined, compute its value from the Stokes relation (4)*
  - 2 *if two or three edge unknowns are not determined, pass to another face.*

This is a simple **elimination procedure** for solving the linear system at hand, and it is quite efficient, as the computational cost is **linearly dependent** on the number of unknowns.

The **weak point** is that:

- it **can stop** without having determined all the edge unknowns (even in simple topological situations!)

## An explicit formula for the vector potential

- **Cure:** devise an **explicit formula** for the solution to (5).

(We are able to do that if  $\mathbf{B}_h \cdot \mathbf{n} = 0$  on  $\partial\Omega$ , a quite natural condition in the most interesting physical situations, and for a suitable choice of the constants  $\rho_n$ .)

The explicit formula permits to **restart** the algorithm in case it stops (but it is better not using it for all the degrees of freedom, as it would be more expensive than the Webb–Forghani algorithm).

## The div problem

We want to solve  $\operatorname{div} \mathbf{v}_h = G_h$  in the finite element context.

The “right” finite elements are:  $\mathbf{v}_h \in RT_h$  a lowest order Raviart–Thomas **face** element,  $G_h \in PC_h$  a **piecewise-constant** element, and we have only to impose that the **integral** of  $\operatorname{div} \mathbf{v}_h$  and of  $G_h$  on each **element** of the mesh  $\mathcal{T}_h$  is the same.

The **Gauss theorem** says that

$$\begin{aligned} \int_{f_1} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_2} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_3} \mathbf{v}_h \cdot \boldsymbol{\nu}_f + \int_{f_4} \mathbf{v}_h \cdot \boldsymbol{\nu}_f \\ = \int_K \operatorname{div} \mathbf{v}_h = \int_K G_h, \end{aligned} \quad (6)$$

where  $\partial K = f_1 \cup f_2 \cup f_3 \cup f_4$ , hence the linear system associated to  $\operatorname{div} \mathbf{v}_h = G_h$  has exactly **four unknowns** per row.

## The div problem (cont'd)

For having **well-posedness** of the system, we want to **add** equations by fixing the value of some unknowns. Similarly to what done before we start by analyzing the **dimension** of the **kernel** of the divergence operator.

This kernel is given by the **curl of the Nédélec elements** plus the space generated by the basis of the **second** de Rham cohomology group of  $\Omega$ .

If we denote by  $(\partial\Omega)_0, \dots, (\partial\Omega)_p$  the **connected components** of  $\partial\Omega$ , we know that the dimension of the second de Rham cohomology group of  $\Omega$  is equal to  $p$ .

## The div problem (cont'd)

On the other hand, it is easy to check that the **dimension** of the space of the curl of the Nédélec elements is equal to the number of the edges **minus** the dimension of the kernel of the curl operator: hence, it is equal to  $n_e - n_v + 1 - g$ .

By the **Euler–Poincaré formula** we have

$$n_v - n_e + n_f - n_t = 1 - g + p,$$

hence the dimension of the space of the curl of the Nédélec elements can be rewritten as  $n_f - n_t - p$ .

In conclusion, besides the **topological** conditions

$$\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = c_r, \quad r = 1, \dots, p,$$

that are a **filter for the cohomology fields**, we could add  $n_f - n_t - p$  equations.

## A dual graph

To do that, let us note that an internal face **connects** two tetrahedra, while a boundary face **connects** a tetrahedron and a connected component of  $\partial\Omega$ .

We can therefore consider the following (connected) **dual graph**  $\mathcal{G}_h$ : the dual **vertices** are  $W = T \cup \Sigma$ , where the elements of  $T$  are the tetrahedra of the mesh and the elements of  $\Sigma$  are the  $p + 1$  connected components of  $\partial\Omega$ ; the set of dual **arcs** is  $F$ , the set of the faces of the mesh.

## A dual graph (cont'd)

The **number** of dual vertices is equal to  $n_t + p + 1$ , hence a **spanning tree**  $\mathcal{M}_h$  of  $\mathcal{G}_h$  has  $n_t + p$  dual arcs (and consequently its **cotree** has  $n_f - n_t - p$  dual arcs).

Therefore the linear system

$$\begin{cases} \operatorname{div} \mathbf{v}_h = G_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = c_r & \forall r = 1, \dots, p \\ \int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0 & \forall f \notin \mathcal{M}_h \end{cases} \quad (7)$$

is a **square linear system** of  $n_f$  equations and unknowns.

It can be shown that this system **has a unique solution**.



## Well-posedness of (7)

The procedure is **constructive**, similar to the elimination procedure used for the grad problem but now going along the **dual spanning tree**, starting from the **leaves**. (Let us recall that a leaf of a spanning tree  $\mathcal{M}_h$  is a vertex of  $W$  that has **only one arc** of  $\mathcal{M}_h$  incident to it.)

We can reduce the problem to the faces  $f \in \mathcal{M}_h$ . If  $w$  (a tetrahedron or a connected component) is a **leaf** of  $\mathcal{M}_h$ , then on it there is **only one face**  $f(w)$  belonging to the **spanning tree**  $\mathcal{M}_h$ , therefore the value of the flux of  $\mathbf{v}_h$  on  $f(w)$  can be computed by the Gauss theorem, if  $w$  is a tetrahedron or the connected component  $(\partial\Omega)_0$ , or by the equation  $\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = c_r$ , if  $w$  is the connected component  $(\partial\Omega)_r$ ,  $r = 1, \dots, p$  (recall that we know that  $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0$  for all  $f \notin \mathcal{M}_h$ ).

## Well-posedness of (7) (cont'd)

We can **iterate** this argument: if we remove from the spanning tree  $\mathcal{M}_h$  a leaf and its corresponding incident arc, the remaining graph is **still** a tree. After a finite number of steps the remaining tree reduces to just one vertex, and the result is that  $\int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f$  is known for all  $f \in F$ .

## A summary

	from	to	unknowns	equations
<b>grad</b>	$L_h$	$N_h$	$n_v$	$n_e + 1 (> n_v)$
<b>curl</b>	$N_h$	$RT_h$	$n_e$	$n_f + g + n_v - 1 = n_e + n_t + p$
<b>div</b>	$RT_h$	$PC_h$	$n_f$	$n_t + p + n_e - n_v + 1 - g = n_f$

Table: Finite element potentials.

## Curl-free or divergence-free finite elements

## Curl-free finite elements

The problem of describing in a suitable way **curl-free finite elements** is quite easy. In fact, it is straightforward to find a basis of the finite element space

$$\mathcal{V}_{0,h} = \{ \mathbf{v}_h \in N_h \mid \mathbf{curl} \mathbf{v}_h = \mathbf{0} \text{ in } \Omega, \int_{\sigma_n} \mathbf{v}_h \cdot d\mathbf{s} = 0 \forall n = 1, \dots, g \}, \quad (8)$$

as this space is **coincident** with **grad**  $L_h$  (indeed, the conditions  $\int_{\sigma_n} \mathbf{v}_h \cdot d\mathbf{s} = 0$  are filtering all the **curl-free vector fields that are not gradients**, namely, the fields belonging to the first de Rham cohomology group).

Thus we have only to identify and eliminate the **kernel** of the gradient operator: the **constants**. In conclusion, a basis for  $\mathcal{V}_{0,h}$  is simply given by **grad**  $\Phi_h^i$ ,  $i = 1, \dots, n_V - 1$ , where  $\Phi_h^i$ ,  $i = 1, \dots, n_V$ , are the standard nodal basis functions of  $L_h$ .

## Divergence-free finite elements

A more complicated situation arises for **divergence-free finite elements**. In fact, we start considering the space

$$\mathcal{W}_{0,h} = \{ \mathbf{v}_h \in RT_h \mid \operatorname{div} \mathbf{v}_h = 0 \text{ in } \Omega, \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = 0 \forall r = 1, \dots, p \}, \quad (9)$$

and it is **easy** to check that  $\mathcal{W}_{0,h} = \mathbf{curl} N_h$  (the conditions  $\int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = 0$  are filtering all the **divergence-free vector fields that are not curls**, namely, the fields belonging to the second de Rham cohomology group).

However, the problem is that

- the **kernel** of the curl operator is **large**: it contains the **gradients** of elements in  $L_h$  and the fields belonging to the **first de Rham cohomology group**, and has dimension equal to  $n_V - 1 + g$ .

## Divergence-free finite elements (cont'd)

Thus we need:

- to devise a strategy for **selecting**  $n_e - n_v + 1 - g$  edges in order that the associated edge element basis functions have **linearly independent curls**.

Results in this direction were obtained by Hecht (1981), Dubois (1990) and Scheichl (2002) for a simply-connected domain, and by Rapetti et al. (2003) for a  $\kappa$ -fold torus. A different approach, based on an algebraic point of view and the use of the dual graph, is due to Alotto and Perugia (1999).

Here we present a **general procedure** for the determination of a set of locally-supported basis functions of  $\mathcal{W}_{0,h}$ , together with an **easy proof** of its effectiveness.

## Divergence-free finite elements (cont'd)

Let us assume for a while that  $\Omega$  is simply-connected (therefore we have  $g = 0$ ). Consider all the edges not belonging to the spanning tree  $\mathcal{S}_h$ , namely, belonging to the **cotree**  $\mathcal{C}_h$ ; their number is  $n_e - n_v + 1$ . The result is:

- A **basis** of  $\mathcal{W}_{0,h}$  is given by **curl**  $\mathbf{w}_h^j$ , for the indices  $j$  such that the corresponding edges  $e_j$  belong to the **cotree**  $\mathcal{C}_h$  (say,  $j = 1, \dots, n_e - n_v + 1$ ).



## Divergence-free finite elements (cont'd)

The **proof** is quite simple and reads as follows: from

$$0 = \sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{curl} \mathbf{w}_h^j = \mathbf{curl} \left( \sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j \right)$$

we can conclude that  $\sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j$  is a **gradient**, say,  $\mathbf{grad} \varphi_h$ . This is an element of  $N_h$  for which all the degrees of freedom associated to the edges **belonging to the spanning tree** are vanishing. Hence  $\varphi_h$  is constant, and  $\sum_{j=1}^{n_e - n_v + 1} \alpha_j \mathbf{w}_h^j = \mathbf{0}$ . We can thus conclude that  $\alpha_j = 0$  for all  $j = 1, \dots, n_e - n_v + 1$ , since  $\{\mathbf{w}_h^j\}_{j=1}^{n_e - n_v + 1}$  are linearly independent.

## Divergence-free finite elements (cont'd)

The **general** topological case needs the identification of  $g$  **additional** edges to discard: a possible option is to select one edge for each **basis element**  $\sigma_n$  of the **first homology group** of  $\bar{\Omega}$ , having constructed the spanning tree in such a way that all the other edges of  $\sigma_n$  **belong to it**. (For definiteness, suppose these edges are associated to the indices  $j = 1, \dots, g$ : the union of the spanning tree  $\mathcal{S}_h$  and these additional  $g$  edges was called **belted tree** in Bossavit (1998), Rapetti et al. (2003).)

With this choice we have that the line integral of  $\sum_{j=g+1}^{n_e-n_v+1} \alpha_j \mathbf{w}_h^j$  over  $\sigma_n$  **vanishes** for each  $n = 1, \dots, g$  (all the edges contained in  $\sigma_n$  belong to the belted tree, namely, they correspond to indices **smaller** than  $g + 1$  or **larger** than  $n_e - n_v + 1$ ). Therefore  $\sum_{j=g+1}^{n_e-n_v+1} \alpha_j \mathbf{w}_h^j$  is a gradient, and the argument develops as before.

# Solving the curl-div system

## First case: $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$

The problem at hand (slightly **more general** than the one previously presented) reads

$$\left\{ \begin{array}{ll} \mathbf{curl}(\eta\mathbf{u}) = \mathbf{B} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = G & \text{in } \Omega \\ (\eta\mathbf{u}) \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega \\ \int_{(\partial\Omega)_r} \mathbf{u} \cdot \mathbf{n} = \alpha_r & \forall r = 1, \dots, p, \end{array} \right. \quad (10)$$

where  $\eta$  is a **symmetric matrix**, uniformly positive definite in  $\Omega$ , with entries belonging to  $L^\infty(\Omega)$ ,  $\mathbf{B} \in (L^2(\Omega))^3$ ,  $G \in L^2(\Omega)$ ,  $\mathbf{a} \in H^{-1/2}(\operatorname{div}_\tau; \partial\Omega)$  [the space of **tangential traces** of vector fields belonging to  $H(\mathbf{curl}; \Omega)$ ],  $\alpha \in \mathbb{R}^p$ .

## First case: $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The data satisfy the **necessary** conditions  $\operatorname{div} \mathbf{B} = 0$  in  $\Omega$ ,  $\int_{\Omega} \mathbf{B} \cdot \boldsymbol{\rho} + \int_{\partial\Omega} \mathbf{a} \cdot \boldsymbol{\rho} = 0$  for each  $\boldsymbol{\rho} \in \mathcal{H}(m)$ , and  $\mathbf{B} \cdot \mathbf{n} = \operatorname{div}_{\tau} \mathbf{a}$  on  $\partial\Omega$ . Here  $\mathcal{H}(m)$  is the space of **Neumann harmonic fields**, namely,

$$\mathcal{H}(m) = \{ \boldsymbol{\rho} \in (L^2(\Omega))^3 \mid \mathbf{curl} \boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \operatorname{div} \boldsymbol{\rho} = 0 \text{ in } \Omega, \boldsymbol{\rho} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \}.$$

The **first step** of the procedure is to find a vector field  $\mathbf{u}^* \in (L^2(\Omega))^3$  satisfying

$$\begin{cases} \operatorname{div} \mathbf{u}^* = G & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{u}^* \cdot \mathbf{n} = \alpha_r & \forall r = 1, \dots, p. \end{cases} \quad (11)$$

## First case: $\mathbf{u} \times \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

Then the vector field  $\mathbf{W} = \mathbf{u} - \mathbf{u}^*$  has to satisfy

$$\left\{ \begin{array}{ll} \mathbf{curl}(\eta\mathbf{W}) = \mathbf{B} - \mathbf{curl}(\eta\mathbf{u}^*) & \text{in } \Omega \\ \operatorname{div} \mathbf{W} = 0 & \text{in } \Omega \\ (\eta\mathbf{W}) \times \mathbf{n} = \mathbf{a} - (\eta\mathbf{u}^*) \times \mathbf{n} & \text{on } \partial\Omega \\ \int_{(\partial\Omega)_r} \mathbf{W} \cdot \mathbf{n} = 0 & \forall r = 1, \dots, p. \end{array} \right. \quad (12)$$

The **second step** is to devise a **variational formulation** of (12).

## A variational formulation for the first case

Multiplying the first equation by a **test function**  $\mathbf{v} \in H(\mathbf{curl}; \Omega)$ , integrating in  $\Omega$  and integrating by parts we find:

$$\begin{aligned} \int_{\Omega} \mathbf{B} \cdot \mathbf{v} &= \int_{\Omega} \mathbf{curl} [\eta(\mathbf{W} + \mathbf{u}^*)] \cdot \mathbf{v} \\ &= \int_{\Omega} \eta(\mathbf{W} + \mathbf{u}^*) \cdot \mathbf{curl} \mathbf{v} - \int_{\partial\Omega} [\eta(\mathbf{W} + \mathbf{u}^*) \times \mathbf{n}] \cdot \mathbf{v} \\ &= \int_{\Omega} \eta \mathbf{W} \cdot \mathbf{curl} \mathbf{v} + \int_{\Omega} \eta \mathbf{u}^* \cdot \mathbf{curl} \mathbf{v} - \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v}. \end{aligned}$$

Let us introduce **the space**

$$\mathcal{W}_0 = \left\{ \mathbf{v} \in H(\mathbf{div}; \Omega) \mid \begin{aligned} &\mathbf{div} \mathbf{v} = 0 \text{ in } \Omega, \\ &\int_{(\partial\Omega)_r} \mathbf{v} \cdot \mathbf{n} = 0 \quad \forall r = 1, \dots, p \end{aligned} \right\}. \quad (13)$$

It is readily seen that  $\mathcal{W}_0 = \mathbf{curl} [H(\mathbf{curl}; \Omega)]$ .

## A variational formulation for the first case (cont'd)

The vector field  $\mathbf{W}$  is thus a solution to

$$\mathbf{W} \in \mathcal{W}_0 :$$

$$\int_{\Omega} \eta \mathbf{W} \cdot \mathbf{curl} \mathbf{v} = \int_{\Omega} \mathbf{B} \cdot \mathbf{v} - \int_{\Omega} \eta \mathbf{u}^* \cdot \mathbf{curl} \mathbf{v} + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v} \quad (14)$$

$$\forall \mathbf{v} \in H(\mathbf{curl}; \Omega).$$

More precisely,  $\mathbf{W}$  is the unique solution of that problem: in fact, assuming  $\mathbf{B} = \mathbf{u}^* = \mathbf{a} = \mathbf{0}$ , and taking  $\mathbf{v}$  such that  $\mathbf{curl} \mathbf{v} = \mathbf{W}$ , it follows at once  $\int_{\Omega} \eta \mathbf{W} \cdot \mathbf{W} = 0$ , hence  $\mathbf{W} = \mathbf{0}$ .



## Finite element approximation of the first case

The finite element approximation follows the **same steps**.

The **first one** is finding a finite element **potential**  $\mathbf{u}_h^* \in RT_h$  such that

$$\begin{cases} \operatorname{div} \mathbf{u}_h^* = G_h & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{u}_h^* \cdot \mathbf{n} = \alpha_r & \forall r = 1, \dots, p, \end{cases} \quad (15)$$

where  $G_h \in PC_h$  is the **piecewise-constant interpolant**  $I_h^{PC} G$  of  $G$ . This can be done as in (7).

## Finite element approximation of the first case (cont'd)

The **second step** concerns the numerical approximation of problem (14). The **natural choice** for the finite element space is clearly the space  $\mathcal{W}_{0,h}$  introduced in (9). The **finite element approximation** of (14) reads as follows:

$$\begin{aligned} \mathbf{W}_h \in \mathcal{W}_{0,h} : \\ \int_{\Omega} \eta \mathbf{W}_h \cdot \mathbf{curl} \mathbf{v}_h \\ = \int_{\Omega} \mathbf{B} \cdot \mathbf{v}_h - \int_{\Omega} \eta \mathbf{u}_h^* \cdot \mathbf{curl} \mathbf{v}_h + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v}_h \end{aligned} \quad (16)$$

$$\forall \mathbf{v}_h \in \mathbf{N}_h^*,$$

where

$$\mathbf{N}_h^* = \text{span}\{\mathbf{w}_h^j\}_{j=g+1}^{n_e-n_v+1}. \quad (17)$$

## Finite element approximation of the first case (cont'd)

The corresponding **algebraic problem** is a **square** linear system of dimension  $n_e - n_v + 1 - g$ , and it is **uniquely solvable**. In fact, we note that  $\mathcal{W}_{0,h} = \mathbf{curl} N_h^*$ , hence we can choose  $\mathbf{v}_h^* \in N_h^*$  such that  $\mathbf{curl} \mathbf{v}_h^* = \mathbf{W}_h$ ; from (16) we find at once  $\mathbf{W}_h = \mathbf{0}$ , provided that  $\mathbf{G} = \mathbf{u}_h^* = \mathbf{a} = \mathbf{0}$ .

The **convergence** of this finite element scheme is easily shown by **standard arguments**. For the sake of completeness, let us present the proof.

## Convergence of the approximation for the first case

**Theorem A.** Let  $\mathbf{W} \in \mathcal{W}_0$  and  $\mathbf{W}_h \in \mathcal{W}_{0,h}$  be the solutions of problem (14) and (16), respectively. Set  $\mathbf{u} = \mathbf{W} + \mathbf{u}^*$  and  $\mathbf{u}_h = \mathbf{W}_h + \mathbf{u}_h^*$ , where  $\mathbf{u}^* \in H(\operatorname{div}; \Omega)$  and  $\mathbf{u}_h^* \in RT_h$  are solutions to problem (11) and (15), respectively. Assume that  $\mathbf{u}$  is regular enough, so that the interpolant  $I_h^{RT} \mathbf{u}$  is defined. Then the following **error estimate** holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{div}; \Omega)} \leq c_0 (\|\mathbf{u} - I_h^{RT} \mathbf{u}\|_{L^2(\Omega)} + \|G - I_h^{PC} G\|_{L^2(\Omega)}). \quad (18)$$

## Convergence of the approximation for the first case (cont'd)

**Proof.** Since  $N_h^* \subset H(\mathbf{curl}; \Omega)$ , we can choose  $\mathbf{v} = \mathbf{v}_h \in N_h^*$  in (14). By subtracting (16) from (14) we end up with

$$\int_{\Omega} \boldsymbol{\eta}[(\mathbf{W} + \mathbf{u}^*) - (\mathbf{W}_h + \mathbf{u}_h^*)] \cdot \mathbf{curl} \mathbf{v}_h = 0 \quad \forall \mathbf{v}_h \in N_h^*,$$

namely, the **consistency** property

$$\int_{\Omega} \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{curl} \mathbf{v}_h = 0 \quad \forall \mathbf{v}_h \in N_h^*. \quad (19)$$

Then from  $\mathcal{W}_{0,h} = \mathbf{curl} N_h^*$  we can write  $\mathbf{W}_h = \mathbf{curl} \mathbf{v}_h^*$  for a suitable  $\mathbf{v}_h^* \in N_h^*$ , and using (19) we find

## Convergence of the approximation for the first case (cont'd)

$$\begin{aligned}
 c_1 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}^2 &\leq \int_{\Omega} \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{u}_h) \\
 &= \int_{\Omega} \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{W}_h - \mathbf{u}_h^*) \\
 &= \int_{\Omega} \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{curl} \mathbf{v}_h^* - \mathbf{u}_h^*) \\
 &= \int_{\Omega} \boldsymbol{\eta}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{curl} \mathbf{v}_h - \mathbf{u}_h^*) \\
 &\leq c_2 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \|\mathbf{u} - \boldsymbol{\Phi}_h - \mathbf{u}_h^*\|_{L^2(\Omega)} \quad \forall \boldsymbol{\Phi}_h \in \mathcal{W}_{0,h}.
 \end{aligned}$$

We can choose  $\boldsymbol{\Phi}_h = (I_h^{RT} \mathbf{u} - \mathbf{u}_h^*) \in \mathcal{W}_{0,h}$ ; in fact,

$\operatorname{div} (I_h^{RT} \mathbf{u}) = I_h^{PC} (\operatorname{div} \mathbf{u}) = I_h^{PC} \mathbf{G} = \mathbf{G}_h$  and

$\int_{(\partial\Omega)_r} I_h^{RT} \mathbf{u} \cdot \mathbf{n} = \int_{(\partial\Omega)_r} \mathbf{u} \cdot \mathbf{n} = \alpha_r$  for each  $r = 1, \dots, p$ . Then it

follows at once  $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq \frac{c_2}{c_1} \|\mathbf{u} - I_h^{RT} \mathbf{u}\|_{L^2(\Omega)}$ .

Finally,  $\operatorname{div} (\mathbf{u} - \mathbf{u}_h) = \mathbf{G} - \mathbf{G}_h = \mathbf{G} - I_h^{PC} \mathbf{G}$ . □

## Convergence of the approximation for the first case (cont'd)

Note that a **sufficient** condition for **defining** the interpolant of  $\mathbf{u}$  is that  $\mathbf{u} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$ ,  $\delta > 0$ . This is **satisfied** if, e.g.,  $\boldsymbol{\eta}$  is a **scalar Lipschitz function** in  $\overline{\Omega}$  and  $\mathbf{a} \in (H^\gamma(\partial\Omega))^3$ ,  $\gamma > 0$ .

## The algebraic problem for the first case

The solution  $\mathbf{W}_h \in \mathcal{W}_{0,h}$  can be written in terms of the basis as  $\mathbf{W}_h = \sum_{j=g+1}^{n_e-n_v+1} W_j \mathbf{curl} \mathbf{w}_{h,j}$ . Hence the finite dimensional problem (16) can be rewritten as

$$\begin{aligned} \sum_{j=g+1}^{n_e-n_v+1} W_j \int_{\Omega} \eta \mathbf{curl} \mathbf{w}_{h,j} \cdot \mathbf{curl} \mathbf{w}_{h,m} \\ = \int_{\Omega} \mathbf{B} \cdot \mathbf{w}_{h,m} - \int_{\Omega} \eta \mathbf{u}_h^* \cdot \mathbf{curl} \mathbf{w}_{h,m} + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{w}_{h,m}, \end{aligned} \quad (20)$$

for each  $m = g + 1, \dots, n_e - n_v + 1$ .

The **matrix**  $\mathbf{K}^*$  with entries

$$K_{mj}^* = \int_{\Omega} \eta \mathbf{curl} \mathbf{w}_{h,j} \cdot \mathbf{curl} \mathbf{w}_{h,m}$$

is clearly **symmetric** and **positive definite**, as the vector fields  $\mathbf{curl} \mathbf{w}_{h,j}$  are **linearly independent**.



## Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$

The problem at hand reads

$$\left\{ \begin{array}{l} \mathbf{curl} \mathbf{u} = \mathbf{B} \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{u}) = G \\ \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n} = b \\ \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n \quad \forall n = 1, \dots, g, \end{array} \right. \quad (21)$$

where  $\boldsymbol{\mu}$  is a **symmetric matrix**, uniformly positive definite in  $\Omega$ , with entries belonging to  $L^\infty(\Omega)$ ,  $\mathbf{B} \in (L^2(\Omega))^3$ ,  $G \in L^2(\Omega)$ ,  $b \in H^{-1/2}(\partial\Omega)$ ,  $\boldsymbol{\beta} \in \mathbb{R}^g$ .

## Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

The data satisfy the **necessary** conditions  $\operatorname{div} \mathbf{B} = 0$  in  $\Omega$ ,  $\int_{\Omega} G = \int_{\partial\Omega} b$ ; moreover, in order that the line integral of  $\mathbf{u}$  on  $\sigma_n$  has a meaning, we also assume that  $\mathbf{B} \cdot \mathbf{n} = 0$  on  $\partial\Omega$  (which is more restrictive than the necessary condition  $\int_{(\partial\Omega)_r} \mathbf{B} \cdot \mathbf{n} = 0$  for each  $r = 1, \dots, p$ ).

The **first step** of the procedure is to find a vector field  $\mathbf{u}^* \in (L^2(\Omega))^3$  satisfying

$$\begin{cases} \operatorname{curl} \mathbf{u}^* = \mathbf{B} & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}^* \cdot d\mathbf{s} = \beta_n & \forall n = 1, \dots, g. \end{cases} \quad (22)$$

## Second case: $\mathbf{u} \cdot \mathbf{n}$ assigned on $\partial\Omega$ (cont'd)

Then the vector field  $\mathbf{V} = \mathbf{u} - \mathbf{u}^*$  has to satisfy

$$\left\{ \begin{array}{ll} \mathbf{curl} \mathbf{V} = \mathbf{0} & \text{in } \Omega \\ \operatorname{div}(\mu \mathbf{V}) = G - \operatorname{div}(\mu \mathbf{u}^*) & \text{in } \Omega \\ (\mu \mathbf{V}) \cdot \mathbf{n} = b - (\mu \mathbf{u}^*) \cdot \mathbf{n} & \text{on } \partial\Omega \\ \oint_{\sigma_n} \mathbf{V} \cdot d\mathbf{s} = 0 & \forall n = 1, \dots, g, \end{array} \right. \quad (23)$$

The **second step** is to devise a **variational formulation** of (23).

## A variational formulation for the second case

Multiplying the second equation by a **test** function  $\varphi \in H^1(\Omega)$ , integrating in  $\Omega$  and integrating by parts we find:

$$\begin{aligned} \int_{\Omega} G \varphi &= \int_{\Omega} \operatorname{div} [\boldsymbol{\mu}(\mathbf{V} + \mathbf{u}^*)] \varphi \\ &= - \int_{\Omega} \boldsymbol{\mu}(\mathbf{V} + \mathbf{u}^*) \cdot \mathbf{grad} \varphi + \int_{\partial\Omega} [\boldsymbol{\mu}(\mathbf{V} + \mathbf{u}^*) \cdot \mathbf{n}] \varphi \\ &= - \int_{\Omega} \boldsymbol{\mu} \mathbf{V} \cdot \mathbf{grad} \varphi - \int_{\Omega} \boldsymbol{\mu} \mathbf{u}^* \cdot \mathbf{grad} \varphi + \int_{\partial\Omega} b \varphi. \end{aligned}$$

Let us introduce **the space**

$$\mathcal{V}_0 = \{ \mathbf{v} \in H(\mathbf{curl}; \Omega) \mid \mathbf{curl} \mathbf{v} = \mathbf{0} \text{ in } \Omega, \quad \oint_{\sigma_n} \mathbf{v} \cdot d\mathbf{s} = 0 \quad \forall n = 1, \dots, g \}. \quad (24)$$

Note that  $\mathcal{V}_0 = \mathbf{grad} [H^1(\Omega)]$ .

## A variational formulation for the second case (cont'd)

The vector field  $\mathbf{V}$  is thus **a solution** to

$$\mathbf{V} \in \mathcal{V}_0 :$$

$$\int_{\Omega} \mu \mathbf{V} \cdot \mathbf{grad} \varphi = - \int_{\Omega} G \varphi - \int_{\Omega} \mu \mathbf{u}^* \cdot \mathbf{grad} \varphi + \int_{\partial\Omega} b \varphi \quad (25)$$

$$\forall \varphi \in H^1(\Omega).$$

It is easily seen that  $\mathbf{V}$  is indeed the **unique** solution of that problem: in fact, assuming  $G = b = 0$ ,  $\mathbf{u}^* = \mathbf{0}$ , and taking  $\varphi$  such that  $\mathbf{grad} \varphi = \mathbf{V}$ , it follows at once  $\int_{\Omega} \mu \mathbf{V} \cdot \mathbf{V} = 0$ , hence  $\mathbf{V} = \mathbf{0}$ .

## Finite element approximation of the second case

The finite element approximation follows the **same steps**.

The **first one** is finding a finite element **potential**  $\mathbf{u}_h^* \in N_h$  such that

$$\begin{cases} \mathbf{curl} \mathbf{u}_h^* = \mathbf{B}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}_h^* \cdot d\mathbf{s} = \beta_n & \forall n = 1, \dots, g, \end{cases} \quad (26)$$

where  $\mathbf{B}_h \in RT_h$  is the **Raviart–Thomas interpolant**  $I_h^{RT} \mathbf{B}$  of  $\mathbf{B}$  (we therefore assume that  $\mathbf{B}$  is so regular that its interpolant  $I_h^{RT} \mathbf{B}$  is defined; for instance, as already recalled, it is enough to assume  $\mathbf{B} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$ ,  $\delta > 0$ ). The construction of  $\mathbf{u}_h^*$  can be done as in (5).

## Finite element approximation of the second case (cont'd)

The **second step** is related to the numerical approximation of problem (25). The **natural choice** for the finite element space is clearly the space  $\mathcal{V}_{0,h}$  introduced in (8). The **finite element approximation** of (25) reads as follows:

$$\begin{aligned} \mathbf{V}_h \in \mathcal{V}_{0,h} : \\ \int_{\Omega} \mu \mathbf{V}_h \cdot \mathbf{grad} \varphi_h = \\ - \int_{\Omega} G \varphi_h - \int_{\Omega} \mu \mathbf{u}_h^* \cdot \mathbf{grad} \varphi_h + \int_{\partial\Omega} b \varphi_h \end{aligned} \quad (27)$$

$$\forall \varphi_h \in L_h^*,$$

where

$$L_h^* = \text{span}\{\psi_{h,i}\}_{i=1}^{n_v-1} = \{\varphi_h \in L_h \mid \varphi_h(\mathbf{v}_{n_v}) = 0\}. \quad (28)$$

## Finite element approximation of the second case (cont'd)

The corresponding **algebraic problem** is a **square** linear system of dimension  $n_V - 1$ , and it is **uniquely solvable**. In fact, since  $\mathcal{V}_{0,h} = \mathbf{grad} L_h^*$ , we can choose  $\varphi_h^* \in L_h^*$  such that  $\mathbf{grad} \varphi_h^* = \mathbf{V}_h$ ; from (27) we find at once  $\mathbf{V}_h = \mathbf{0}$ , provided that  $G = b = 0$ ,  $\mathbf{u}_h^* = \mathbf{0}$ .

The **convergence** of this finite element scheme is easily shown by **standard arguments**.



## Convergence of the approximation for the second case

**Theorem B.** Let  $\mathbf{V} \in \mathcal{V}_0$  and  $\mathbf{V}_h \in \mathcal{V}_{0,h}$  be the solutions of problem (25) and (27), respectively. Set  $\mathbf{u} = \mathbf{V} + \mathbf{u}^*$  and  $\mathbf{u}_h = \mathbf{V}_h + \mathbf{u}_h^*$ , where  $\mathbf{u}^* \in H(\mathbf{curl}; \Omega)$  and  $\mathbf{u}_h^* \in N_h$  are solutions to problem (22) and (26), respectively. Assume that  $\mathbf{u}$  and  $\mathbf{B}$  are regular enough, so that the interpolants  $I_h^N \mathbf{u}$  and  $I_h^{RT} \mathbf{B}$  are defined. Then the following **error estimate** holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathbf{curl}; \Omega)} \leq c_0 (\|\mathbf{u} - I_h^N \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{B} - I_h^{RT} \mathbf{B}\|_{L^2(\Omega)}). \quad (29)$$

## Convergence of the approximation for the second case (cont'd)

**Proof.** Since  $L_h^* \subset H^1(\Omega)$ , we can choose  $\varphi = \varphi_h \in L_h^*$  in (25). By subtracting (27) from (25) we end up with

$$\int_{\Omega} \mu[(\mathbf{V} + \mathbf{u}^*) - (\mathbf{V}_h + \mathbf{u}_h^*)] \cdot \mathbf{grad} \varphi_h = 0 \quad \forall \varphi_h \in L_h^*,$$

namely, the **consistency** property

$$\int_{\Omega} \mu(\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{grad} \varphi_h = 0 \quad \forall \varphi_h \in L_h^*. \quad (30)$$

Then, since  $\mathcal{V}_{0,h} = \mathbf{grad} L_h^*$  and thus  $\mathbf{V}_h = \mathbf{grad} \varphi_h^*$  for a suitable  $\varphi_h^* \in L_h^*$ , from (30) we find

## Convergence of the approximation for the second case (cont'd)

$$\begin{aligned}
 c_1 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}^2 &\leq \int_{\Omega} \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{u}_h) \\
 &= \int_{\Omega} \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{V}_h - \mathbf{u}_h^*) \\
 &= \int_{\Omega} \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{grad} \varphi_h^* - \mathbf{u}_h^*) \\
 &= \int_{\Omega} \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{grad} \varphi_h - \mathbf{u}_h^*) \\
 &\leq c_2 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \|\mathbf{u} - \boldsymbol{\Psi}_h - \mathbf{u}_h^*\|_{L^2(\Omega)} \quad \forall \boldsymbol{\Psi}_h \in \mathcal{V}_{0,h}.
 \end{aligned}$$

We can choose  $\boldsymbol{\Psi}_h = (I_h^N \mathbf{u} - \mathbf{u}_h^*) \in \mathcal{V}_{0,h}$ ; in fact,

$\mathbf{curl} (I_h^N \mathbf{u}) = I_h^{RT} (\mathbf{curl} \mathbf{u}) = I_h^{RT} \mathbf{B} = \mathbf{B}_h$  and

$\oint_{\sigma_n} I_h^N \mathbf{u} \cdot d\mathbf{s} = \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n$  for each  $n = 1, \dots, g$ . Then we find

at once  $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq \frac{c_2}{c_1} \|\mathbf{u} - I_h^N \mathbf{u}\|_{L^2(\Omega)}$ .

Moreover,  $\mathbf{curl} (\mathbf{u} - \mathbf{u}_h) = \mathbf{B} - \mathbf{B}_h = \mathbf{B} - I_h^{RT} \mathbf{B}$ . □

## Convergence of the approximation for the second case (cont'd)

Note that a **sufficient** condition for defining the interpolants of  $\mathbf{u}$  and  $\mathbf{B} = \mathbf{curl} \mathbf{u}$  is that they both belong to  $(H^{\frac{1}{2}+\delta}(\Omega))^3$ ,  $\delta > 0$ . Thus one has to assume that  $\mathbf{B} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$ ; moreover,  $\mathbf{u}$  belongs to  $(H^{\frac{1}{2}+\delta}(\Omega))^3$  if, for instance,  $\mu$  is a **scalar Lipschitz function** in  $\overline{\Omega}$  and  $b \in H^\gamma(\Omega)$ ,  $\gamma > 0$ .

## The algebraic problem for the second case

The solution  $\mathbf{V}_h \in \mathcal{V}_{0,h}$  is given by  $\mathbf{V}_h = \sum_{i=1}^{n_v-1} V_i \mathbf{grad} \psi_{h,i}$ .  
 Hence the finite dimensional problem (27) can be rewritten as

$$\begin{aligned} \sum_{i=1}^{n_v-1} V_i \int_{\Omega} \mu \mathbf{grad} \psi_{h,i} \cdot \mathbf{grad} \psi_{h,l} \\ = - \int_{\Omega} G \psi_{h,l} - \int_{\Omega} \mu \mathbf{u}_h^* \cdot \mathbf{grad} \psi_{h,l} + \int_{\partial\Omega} b \psi_{h,l}, \end{aligned} \quad (31)$$

for each  $l = 1, \dots, n_v - 1$ .

The **matrix**  $\mathbf{K}^*$  with entries

$$K_{li}^* = \int_{\Omega} \mu \mathbf{grad} \varphi_{h,i} \cdot \mathbf{grad} \varphi_{h,l}$$

is clearly **symmetric** and **positive definite**.

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## Other problems

- **Darcy problem** (for simplicity, assume  $\partial\Omega$  connected)

$$\begin{cases} \kappa \mathbf{u} - \mathbf{grad} p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ p|_{\partial\Omega} = \varphi & \text{on } \partial\Omega. \end{cases}$$

In weak form:

$$\mathbf{u} \in \mathcal{W}_0 :$$

$$\int_{\Omega} \kappa \mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\partial\Omega} \varphi \mathbf{v} \cdot \mathbf{n}$$

$$\forall \mathbf{v} \in \mathcal{W}_0.$$

- **elliptic problems in mixed formulation** (for simplicity, assume  $\partial\Omega$  connected)

$$\begin{cases} \kappa \mathbf{w} - \mathbf{grad} q = \mathbf{0} & \text{in } \Omega \\ \operatorname{div} \mathbf{w} = g & \text{in } \Omega \\ q|_{\partial\Omega} = \eta & \text{on } \partial\Omega. \end{cases}$$

In weak form:

$$\mathbf{W} \in \mathcal{W}_0 :$$

$$\int_{\Omega} \kappa \mathbf{W} \cdot \mathbf{v} = - \int_{\Omega} \kappa \mathbf{u}^* \cdot \mathbf{v} + \int_{\partial\Omega} \eta \mathbf{v} \cdot \mathbf{n}$$

$$\forall \mathbf{v} \in \mathcal{W}_0,$$

where  $\operatorname{div} \mathbf{u}^* = g$  in  $\Omega$  and  $\mathbf{w} = \mathbf{W} + \mathbf{u}^*$ .

## The grad problem (back to it)

Having found a nodal element  $\psi_h$  such that its gradient has line integral on all the edges of the **spanning tree** equal to that of  $\mathbf{H}_h$ , what about the edges **not** belonging to the spanning tree?

For each node  $v_i$ ,  $v_i \neq v_*$ , let us denote by  $C_{v_i}$  the set of edges in  $\mathcal{S}_h$  joining  $v_*$  to  $v_i$ . Given an edge  $e = [v_a, v_b]$  not belonging to  $\mathcal{S}_h$ , we define the **cycle**  $D_e = C_{v_a} + e - C_{v_b}$ .

Since  $\mathbf{H}_h$  is a gradient (it is **curl-free** and its line integral on all the cycles  $\sigma_n$  **vanishes**), its line integral on  $D_e$  vanishes. Therefore we have

$$\begin{aligned} 0 &= \oint_{D_e} \mathbf{H}_h \cdot d\mathbf{s} = \psi_h(v_a) + \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} - \psi_h(v_b) \\ &= \int_e \mathbf{H}_h \cdot \boldsymbol{\tau} - \int_e \mathbf{grad} \psi_h \cdot \boldsymbol{\tau}. \end{aligned}$$

## An explicit formula for the vector potential

- **Devise** an **explicit formula** for the solution to (5).

(We are able to do that if  $\mathbf{B}_h \cdot \mathbf{n} = 0$  on  $\partial\Omega$ , a quite natural condition in the most interesting physical situations, and for a suitable choice of the constants  $\rho_n$ .)

The idea is the following. Define the **Biot–Savart** field

$$\mathbf{H}^{BS}(\mathbf{x}) = \frac{1}{4\pi} \int_{\Omega} \mathbf{B}_h(\mathbf{y}) \times \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} d\mathbf{y},$$

and set  $\rho_n = \oint_{\sigma_n} \mathbf{H}^{BS} \cdot d\mathbf{s}$  in (5).

One has  $\mathbf{curl} \mathbf{H}^{BS} = \mathbf{B}_h$  in  $\Omega$  (here the condition  $\mathbf{B}_h \cdot \mathbf{n} = 0$  on  $\partial\Omega$  has played a role). Hence the Nédélec interpolant  $\Pi^{N_h} \mathbf{H}^{BS}$  satisfies (5)<sub>1</sub> and (5)<sub>2</sub>.

## An explicit formula for the vector potential (cont'd)

To find the solution to (5), we can correct  $\Pi^{N_h} \mathbf{H}^{BS}$  by a gradient, namely, construct the nodal element  $\phi_h$  whose gradient has the **same line integral** of  $\mathbf{H}^{BS}$  on the edges of the spanning tree  $\mathcal{S}_h$ .

The Nédélec finite element  $\mathbf{A}_h = \Pi^{N_h} \mathbf{H}^{BS} - \mathbf{grad} \phi_h$  is the **solution** to (5).

To express its **degrees of freedom**, we proceed as follows. For each edge  $e \notin \mathcal{S}_h$ , we define the **cycle**  $D_e$  as before (the edges from the root of the spanning tree to the first vertex of  $e$ , the edge  $e$ , the edges from the second vertex of  $e$  to the root of the spanning tree).

## An explicit formula for the vector potential (cont'd)

The cycle  $D_e$  is constituted by edges **all belonging to the spanning tree** (except  $e$ ): hence we have

$$\begin{aligned}
 \int_e \mathbf{A}_h \cdot \boldsymbol{\tau} &= \int_e (\Pi^{N_h} \mathbf{H}^{BS} - \mathbf{grad} \phi_h) \cdot \boldsymbol{\tau} \\
 &= \int_e \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - [\phi_h(v_b) - \phi_h(v_a)] \\
 &= \int_e \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \left[ \int_{C_{v_b}} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} - \int_{C_{v_a}} \mathbf{H}^{BS} \cdot \boldsymbol{\tau} \right] \quad (32) \\
 &= \oint_{D_e} \mathbf{H}^{BS} \cdot d\mathbf{s} \\
 &= \frac{1}{4\pi} \oint_{D_e} \left( \int_{\Omega} \mathbf{B}_h(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y} \right) \cdot d\mathbf{s}(\mathbf{x}).
 \end{aligned}$$

Using (32), we can always **restart** the Webb–Forghani algorithm.

## A basis of the first de Rham cohomology group

The presented approach permits to solve also the problem

$$\left\{ \begin{array}{ll} \mathbf{curl} \mathbf{A}_h = \mathbf{0} & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{A}_h \cdot d\mathbf{s} = \kappa_n & \forall n = 1, \dots, g \\ \int_{e'} \mathbf{A}_h \cdot \boldsymbol{\tau} = 0 & \forall e' \in \mathcal{S}_h, \end{array} \right. \quad (33)$$

for any choice of the constants  $\kappa_n$ .

Taking  $\kappa_n$  equal to  $\ell_{\kappa}(\sigma_n, \hat{\sigma}_j)$ ,  $j = 1, \dots, g$ , ( $\ell_{\kappa}$  denotes the linking number) we find a basis  $\mathbf{T}^{(j)}$  of the **first de Rham cohomology group**, and we have also an explicit formula like (32) for expressing the degrees of freedom of each  $\mathbf{T}^{(j)}$ .

## The linking number

The **linking number** between  $\hat{\sigma}_j$  and another **disjoint** cycle  $\sigma$  is given by:

$$\ell_{\kappa}(\sigma, \hat{\sigma}_j) = \frac{1}{4\pi} \oint_{\sigma} \left( \oint_{\hat{\sigma}_j} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3} \times d\mathbf{s}_y \right) \cdot d\mathbf{s}_x.$$

- The linking number (introduced by Gauss...) is an **integer** that represents the number of times that each cycle **winds** around the other.

The **explicit formula** for determining the basis elements  $\mathbf{T}^{(j)}$  is

$$\int_e \mathbf{T}^{(j)} \cdot \boldsymbol{\tau} = \ell_{\kappa}(D_e, \hat{\sigma}_j) \quad (34)$$

(where  $\hat{\sigma}_j$  has been chosen inside  $\mathbb{R}^3 \setminus \bar{\Omega}$ , namely, not intersecting  $D_e$ ).



## A basis of the second de Rham cohomology group

It can be also noted that the solutions  $\mathbf{W}^{(s)}$ ,  $s = 1, \dots, p$ , of the problems

$$\begin{cases} \operatorname{div} \mathbf{v}_h = 0 & \text{in } \Omega \\ \int_{(\partial\Omega)_r} \mathbf{v}_h \cdot \mathbf{n} = \delta_{r,s} & \forall r = 1, \dots, p \\ \int_f \mathbf{v}_h \cdot \boldsymbol{\nu}_f = 0 & \forall f \notin \mathcal{M}_h \end{cases} \quad (35)$$

furnish a **basis** of the **second de Rham cohomology group** of  $\Omega$ .