Il terribile integrale triplo n° 7.

7. Calcolare il volume del corpo limitato dalla superficie

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}.$$

Siccome per $|x|\to+\infty$, $|y|\to+\infty$, $|z|\to+\infty$ il termine di sinistra è maggiore di quello di destra, l'insieme di cui si chiede il volume è

$$K=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2\leq\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}\right\}.$$

Lo possiamo descrivere con coordinate ellittiche tridimensionali, cioè

$$x=a\rho\sin\varphi\cos\theta,\quad y=b\rho\sin\varphi\operatorname{sen}\theta,\quad z=c\rho\cos\varphi.$$

Per quanto riguarda gli estremi, si ha certamente $\theta\in[0,2\pi]$, mentre ρ e φ devono soddisfare la disequazione che definisce K, cioè

$$(\rho^2\operatorname{sen}^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\operatorname{sen}^2\theta+\rho^2\cos^2\varphi)^2\leq$$
$$\leq(\rho^2\operatorname{sen}^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\operatorname{sen}^2\theta-\rho^2\cos^2\varphi),$$

che dà

$$\rho^4\leq\rho^2(\operatorname{sen}^2\varphi-\cos^2\varphi).$$

Questo richiede $\operatorname{sen}^2\varphi\geq\cos^2\varphi$, cioè $\pi/4\leq\varphi\leq3\pi/4$, e poi $\rho^2\leq\sin^2\varphi-\cos^2\varphi$, cioè $\rho\leq\sqrt{\sin^2\varphi-\cos^2\varphi}$.

Dunque bisogna calcolare l'integrale

$$\begin{cases}-\cos\varphi=t\\ \operatorname{sen}\varphi\,d\varphi=dt\end{cases}$$

$$\int_0^{2\pi}d\theta\int_{\pi/4}^{3\pi/4}d\varphi\int_0^{\sqrt{\sin^2\varphi-\cos^2\varphi}}abc\rho^2\sin\varphi\,d\rho=\frac{2}{3}\pi abc\int_{\pi/4}^{3\pi/4}(\sin^2\varphi-\cos^2\varphi)^{3/2}\sin\varphi\,d\varphi=$$

$$=\frac{2}{3}\pi abc\int_{-\sqrt2/2}^{\sqrt2/2}(1-t^2-t^2)^{3/2}dt=\frac{2}{3}\pi abc\int_{-\sqrt2/2}^{\sqrt2/2}(1-2t^2)^{3/2}dt.$$

Resta da calcolare $\displaystyle\int_{-\sqrt2/2}^{\sqrt2/2}(1-2t^2)^{3/2}dt$.

Posto $t = \frac{1}{\sqrt{2}} \sin\varphi$ si ha $dt = \frac{1}{\sqrt{2}} \cos\varphi \, d\varphi$ e dunque:

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} (1-2t^2)^{3/2} dt = \int_{-\pi/2}^{\pi/2} (1-\sin^2\varphi)^{3/2} \frac{1}{\sqrt{2}} \cos\varphi \, d\varphi = \frac{1}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^4\varphi \, d\varphi .$$

Questo integrale vale

$$\int_{-\pi/2}^{\pi/2} \cos^4\varphi \, d\varphi = \int_{-\pi/2}^{\pi/2} \cos^2\varphi (1-\sin^2\varphi) \, d\varphi = \frac{\pi}{2} - \int_{-\pi/2}^{\pi/2} \cos^2\varphi \sin^2\varphi \, d\varphi \overset{\text{per parti}}{=}$$

$$= \frac{\pi}{2} + \frac{\cos^3\varphi \sin\varphi}{3} \Big|_{-\pi/2}^{\pi/2} - \frac{1}{3} \int_{-\pi/2}^{\pi/2} \cos^3\varphi \cos\varphi \, d\varphi ,$$

da cui

$$\frac{4}{3} \int_{-\pi/2}^{\pi/2} \cos^4\varphi \, d\varphi = \frac{\pi}{2} \Rightarrow \int_{-\pi/2}^{\pi/2} \cos^4\varphi \, d\varphi = \frac{3\pi}{8} .$$

In conclusione, il volume vale

$$V = \frac{2}{3} \pi abc \frac{1}{\sqrt{2}} \frac{3\pi}{8} = \frac{1}{4\sqrt{2}} abc \, \pi^2 .$$