Partial Differential Equations in Biology
The boundary element method

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Introduction and notation

The problem:

$$\begin{cases} -\Delta u = f & \text{in } D \subset \mathbb{R}^d \\ u = \varphi & \text{in } \Gamma_D \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_N \, , \end{cases}$$

where $\partial D = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, $\Gamma_D \cap \Gamma_N = \emptyset$ (possibly, $\Gamma_D = \emptyset$ [Neumann problem] or $\Gamma_N = \emptyset$ [Dirichlet problem]).

Given $\boldsymbol{\xi} \in \mathbb{R}^d$

- $K(\cdot, \boldsymbol{\xi})$ denotes the fundamental solution of the Laplace operator:

$$-\Delta K = \delta_{\boldsymbol{\xi}} \, .$$

- $T(\cdot, \boldsymbol{\xi})$ denotes the normal derivative of $K(\cdot, \boldsymbol{\xi})$, defined on $\partial D = \Gamma$ ($\mathbf{n}$ is the unit outward normal vector on Γ):

$$T(\mathbf{x}, \boldsymbol{\xi}) = \nabla_{\mathbf{x}} K(\mathbf{x}, \boldsymbol{\xi}) \cdot \mathbf{n}(\mathbf{x}) \, .$$

Fundamental solutions

If $D \subset \mathbb{R}^2$

$$K(\mathbf{x}, \xi) = \frac{1}{2\pi} \log \frac{1}{|\mathbf{x} - \xi|}$$

$$T(\mathbf{x}, \xi) = -\frac{1}{2\pi} \frac{\mathbf{x} - \xi}{|\mathbf{x} - \xi|^2} \cdot \mathbf{n}(\mathbf{x}) .$$

If $D \subset \mathbb{R}^3$

$$K(\mathbf{x}, \xi) = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \xi|}$$

$$T(\mathbf{x}, \xi) = -\frac{1}{4\pi} \frac{\mathbf{x} - \xi}{|\mathbf{x} - \xi|^3} \cdot \mathbf{n}(\mathbf{x}) .$$

Basic integral equations

Internal points: $\boldsymbol{\xi} \in D$

$$\begin{aligned} u(\boldsymbol{\xi}) \quad &+ \quad \int_\Gamma u(\mathbf{x})\, T(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) - \int_\Gamma \tfrac{\partial u}{\partial n}(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) \\ &= \quad \int_D f(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi})\, d\mathbf{x}\,. \end{aligned} \qquad (1)$$

Boundary points: $\boldsymbol{\xi} \in \Gamma$ (regular boundary)

$$\begin{aligned} \tfrac{1}{2} u(\boldsymbol{\xi}) \quad &+ \quad \int_\Gamma u(\mathbf{x})\, T(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) - \int_\Gamma \tfrac{\partial u}{\partial n}(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) \\ &= \quad \int_D f(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi})\, d\mathbf{x}\,. \end{aligned} \qquad (2)$$

Polyhedral domain and piecewise-polynomial functions

- The domain D is approximated by a polyhedral domain D_h (union of triangles or tetrahedra; here h is the maximum of their diameters).
- The boundary $\partial D = \Gamma$ is therefore approximated by ∂D_h (union of M segments or triangles S_k).
- The approximate solution is piecewise-polynomial (on D_h, if the problem to approximate is defined in D, or on ∂D_h, if the problem to approximate is defined in Γ).

In our case (approximation of the integral equation (2), defined on Γ, by piecewise-constant functions on ∂D_h):

- the unknowns are the values at the mid-point (or baricenter) $\boldsymbol{\xi}_j$ of S_j, $j = 1, \ldots, M$.

The approximate equations: collocation method

Let u_h the approximation of u on the boundary and q_h the approximation of $\frac{\partial u}{\partial n}$ on the boundary (clearly, u_h and q_h are defined on ∂D_h, while u and $\frac{\partial u}{\partial n}$ are defined on Γ). They are identified by their constant values α_k and β_k on the element S_k.
We can start considering an approximate form of (2), valid for $\boldsymbol{\xi} \in \partial D_h$:

$$\begin{aligned} \frac{1}{2} u_h(\boldsymbol{\xi}) \quad &+ \quad \int_{\partial D_h} u_h(\mathbf{x}) T(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) - \int_{\partial D_h} q_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) \\ &= \quad \int_D f(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi})\, d\mathbf{x}\,. \end{aligned} \tag{3}$$

The approximate equations: collocation method (cont'd)

Dirichlet problem. We know $u = \varphi$ on the boundary Γ, therefore we can construct a suitable approximation φ_h on the boundary ∂D_h. We look for q_h, the approximation of $\frac{\partial u}{\partial n}$.
For $j = 1, \ldots, M$

$$\begin{aligned} \tfrac{1}{2}\varphi_h(\boldsymbol{\xi}_j) \quad &+ \quad \int_{\partial D_h} \varphi_h(\mathbf{x}) T(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) \\ &- \quad \int_{\partial D_h} q_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) = \int_D f(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}_j)\, d\mathbf{x}\,. \end{aligned} \tag{4}$$

We can write

$$\begin{aligned} \int_{\partial D_h} q_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) &= \sum_{k=1}^{M} \int_{S_k} q_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) \\ &= \sum_{k=1}^{M} \beta_k \int_{S_k} K(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x})\,. \end{aligned}$$

The approximate equations: collocation method (cont'd)

We have thus obtained the linear system

$$A^{\mathrm{Dir}}\boldsymbol{\beta} = \mathbf{b}^{\mathrm{Dir}}\,,$$

where

$$A^{\mathrm{Dir}}_{jk} = \int_{S_k} K(\mathbf{x},\boldsymbol{\xi}_j)\,ds(\mathbf{x})\,,$$

$$b^{\mathrm{Dir}}_j = \frac{1}{2}\varphi_h(\boldsymbol{\xi}_j) + \int_{\partial D_h} \varphi_h(\mathbf{x})\,T(\mathbf{x},\boldsymbol{\xi}_j)\,ds(\mathbf{x}) - \int_D f(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi}_j)\,d\mathbf{x}\,.$$

The matrix A^{Dir} is not symmetric.

The approximate equations: collocation method (cont'd)

Neumann problem. We know $\frac{\partial u}{\partial n} = g$ on the boundary Γ, therefore we can construct a suitable approximation g_h on the boundary ∂D_h. We look for u_h, the approximation of u.
For $j = 1, \ldots, M$

$$\begin{aligned} \tfrac{1}{2} u_h(\boldsymbol{\xi}_j) &+ \int_{\partial D_h} u_h(\mathbf{x}) T(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) \\ &- \int_{\partial D_h} g_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) = \int_D f(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}_j)\, d\mathbf{x}\,. \end{aligned} \tag{5}$$

We can write

$$\begin{aligned} \int_{\partial D_h} u_h(\mathbf{x}) T(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) &= \sum_{k=1}^{M} \int_{S_k} u_h(\mathbf{x}) T(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) \\ &= \sum_{k=1}^{M} \alpha_k \int_{S_k} T(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x})\,. \end{aligned}$$

The approximate equations: collocation method (cont'd)

We have thus obtained the linear system

$$A^{\mathrm{Neu}} \boldsymbol{\alpha} = \mathbf{b}^{\mathrm{Neu}} \,,$$

where

$$A_{jk}^{\mathrm{Neu}} = \frac{1}{2}\delta_{jk} + \int_{S_k} T(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x})\,,$$

$$b_j^{\mathrm{Neu}} = \int_{\partial D_h} g_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}_j)\, ds(\mathbf{x}) + \int_D f(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}_j)\, d\mathbf{x}\,.$$

The matrix A^{Neu} is not symmetric.
[The matrix A^{Neu} is singular, the kernel being given by constant vectors $c(1, 1, \dots, 1, 1)$: a technical problem that we do not look at in depth.]

The approximate equations: Galerkin method

Another set of approximate equations can be obtained by projecting equation (3) on the subspace of piecewise-constant functions on ∂D_h. This is an example of Galerkin method.

Let us define by $\{\psi_j\}$, $j = 1, \ldots, M$, a set of basis functions of the vector space given by the piecewise-constant functions on ∂D_h.

We can write $u_h(\boldsymbol{\xi}) = \sum_{k=1}^{M} \alpha_k \psi_k(\boldsymbol{\xi})$, $q_h(\boldsymbol{\xi}) = \sum_{k=1}^{M} \beta_k \psi_k(\boldsymbol{\xi})$. [The simplest choice is given by ψ_j equal to 1 in S_j, and equal to 0 in all the other S_l for $l \neq j$.]

The approximate equations: Galerkin method (cont'd)

Multiplying equation (3) by ψ_j and integrating on ∂D_h we find for each $j = 1, \ldots, M$:

$$\begin{aligned}
&\tfrac{1}{2}\int_{\partial D_h} u_h(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\
&\qquad + \int_{\partial D_h}\Big(\int_{\partial D_h} u_h(\mathbf{x})T(\mathbf{x},\boldsymbol{\xi})\,ds(\mathbf{x})\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\
&\qquad - \int_{\partial D_h}\Big(\int_{\partial D_h} q_h(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\,ds(\mathbf{x})\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\
&= \int_{\partial D_h}\Big(\int_D f(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\,d\mathbf{x}\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi})\,.
\end{aligned} \tag{6}$$

The approximate equations: Galerkin method (cont'd)

Dirichlet problem. We know $u = \varphi$ on the boundary Γ, therefore we can construct a suitable approximation φ_h on the boundary ∂D_h. We look for q_h, the approximation of $\frac{\partial u}{\partial n}$.
For $j = 1, \ldots, M$ we have

$$\begin{aligned} \tfrac{1}{2} \textstyle\int_{\partial D_h} & \varphi_h(\boldsymbol{\xi}) \psi_j(\boldsymbol{\xi}) ds(\boldsymbol{\xi}) \\ & + \int_{\partial D_h} \Big(\int_{\partial D_h} \varphi_h(\mathbf{x}) T(\mathbf{x}, \boldsymbol{\xi}) \, ds(\mathbf{x}) \Big) \psi_j(\boldsymbol{\xi}) ds(\boldsymbol{\xi}) \\ & - \int_{\partial D_h} \Big(\int_{\partial D_h} q_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}) \, ds(\mathbf{x}) \Big) \psi_j(\boldsymbol{\xi}) ds(\boldsymbol{\xi}) \\ & = \int_{\partial D_h} \Big(\int_D f(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}) \, d\mathbf{x} \Big) \psi_j(\boldsymbol{\xi}) ds(\boldsymbol{\xi}) \,. \end{aligned} \tag{7}$$

We can write

$$\begin{aligned} \int_{\partial D_h} & \Big(\int_{\partial D_h} q_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi}) \, ds(\mathbf{x}) \Big) \psi_j(\boldsymbol{\xi}) ds(\boldsymbol{\xi}) \\ & = \sum_{k=1}^{M} \beta_k \int_{\partial D_h} \int_{\partial D_h} K(\mathbf{x}, \boldsymbol{\xi}) \psi_k(\mathbf{x}) \psi_j(\boldsymbol{\xi}) \, ds(\mathbf{x}) ds(\boldsymbol{\xi}) \,. \end{aligned}$$

The approximate equations: Galerkin method (cont'd)

We have thus obtained the linear system

$$\widehat{A}^{\mathrm{Dir}}\boldsymbol{\beta} = \widehat{\mathbf{b}}^{\mathrm{Dir}}\,,$$

where

$$\widehat{A}^{\mathrm{Dir}}_{jk} = \int_{\partial D_h}\int_{\partial D_h} K(\mathbf{x},\boldsymbol{\xi})\psi_k(\mathbf{x})\psi_j(\boldsymbol{\xi})\,ds(\mathbf{x})ds(\boldsymbol{\xi})\,,$$

$$\begin{aligned}\widehat{b}^{\mathrm{Dir}}_j = {} & \tfrac{1}{2}\int_{\partial D_h}\varphi_h(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\ & + \int_{\partial D_h}\Big(\int_{\partial D_h}\varphi_h(\mathbf{x})\,T(\mathbf{x},\boldsymbol{\xi})\,ds(\mathbf{x})\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\ & - \int_{\partial D_h}\Big(\int_D f(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\,d\mathbf{x}\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi})\,.\end{aligned}$$

Since $K(\mathbf{x},\boldsymbol{\xi}) = K(\boldsymbol{\xi},\mathbf{x})$, the matrix $\widehat{A}^{\mathrm{Dir}}$ is clearly symmetric.

The approximate equations: Galerkin method (cont'd)

Neumann problem. We know $\frac{\partial u}{\partial n} = g$ on the boundary Γ, therefore we can construct a suitable approximation g_h on the boundary ∂D_h. We look for u_h, the approximation of u.
For $j = 1, \ldots, M$ we have

$$\begin{aligned} &\tfrac{1}{2}\int_{\partial D_h} u_h(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\ &\qquad + \int_{\partial D_h}\Big(\int_{\partial D_h} u_h(\mathbf{x})T(\mathbf{x},\boldsymbol{\xi})\,ds(\mathbf{x})\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\ &\qquad - \int_{\partial D_h}\Big(\int_{\partial D_h} g_h(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\,ds(\mathbf{x})\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\ &= \int_{\partial D_h}\Big(\int_D f(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\,d\mathbf{x}\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi})\,. \end{aligned} \tag{8}$$

The approximate equations: Galerkin method (cont'd)

We can write

$$\int_{\partial D_h} u_h(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) = \sum_{k=1}^{M} \alpha_k \int_{\partial D_h} \psi_k(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi})\,,$$

$$\begin{aligned}\int_{\partial D_h} \Big(\int_{\partial D_h} u_h(\mathbf{x})\, T(\mathbf{x},\boldsymbol{\xi})\, ds(\mathbf{x})\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\ = \textstyle\sum_{k=1}^{M} \alpha_k \int_{\partial D_h}\int_{\partial D_h} T(\mathbf{x},\boldsymbol{\xi})\psi_k(\mathbf{x})\psi_j(\boldsymbol{\xi})\, ds(\mathbf{x})ds(\boldsymbol{\xi})\,.\end{aligned}$$

The approximate equations: Galerkin method (cont'd)

We have thus obtained the linear system

$$\widehat{A}^{\mathrm{Neu}}\boldsymbol{\alpha} = \widehat{\mathbf{b}}^{\mathrm{Neu}}\,,$$

where

$$\widehat{A}^{\mathrm{Neu}}_{jk} = \frac{1}{2}\int_{\partial D_h}\psi_k(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) + \int_{\partial D_h}\int_{\partial D_h} T(\mathbf{x},\boldsymbol{\xi})\psi_k(\mathbf{x})\psi_j(\boldsymbol{\xi})\,ds(\mathbf{x})ds(\boldsymbol{\xi})\,,$$

$$\widehat{b}^{\mathrm{Neu}}_j = \int_{\partial D_h}\Big(\int_{\partial D_h} g_h(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\,ds(\mathbf{x})\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) + \int_{\partial D_h}\Big(\int_D f(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\,d\mathbf{x}\Big)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi})\,.$$

Since

$$T(\mathbf{x},\boldsymbol{\xi}) = \begin{cases} -\frac{1}{2\pi}\frac{\mathbf{x}-\boldsymbol{\xi}}{|\mathbf{x}-\boldsymbol{\xi}|^2}\cdot\mathbf{n}(\mathbf{x}) & \text{for d=2} \\ -\frac{1}{4\pi}\frac{\mathbf{x}-\boldsymbol{\xi}}{|\mathbf{x}-\boldsymbol{\xi}|^3}\cdot\mathbf{n}(\mathbf{x}) & \text{for d=3} \end{cases},$$

we see that $T(\mathbf{x},\boldsymbol{\xi}) \neq T(\boldsymbol{\xi},\mathbf{x})$, and therefore the matrix $\widehat{A}^{\mathrm{Neu}}$ is not symmetric [moreover, as in the collocation case, it is singular...].

The approximate equations: Galerkin method (cont'd)

A symmetric Galerkin formulation for the Neumann problem can be derived by using a different integral equation, that can be proved to hold for $\boldsymbol{\xi} \in \Gamma$:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial u}{\partial n}(\boldsymbol{\xi}) \quad & + \frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_\Gamma u(\mathbf{x})\, T(\mathbf{x},\boldsymbol{\xi})\, ds(\mathbf{x}) \\
& - \frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_\Gamma \frac{\partial u}{\partial n}(\mathbf{x}) K(\mathbf{x},\boldsymbol{\xi})\, ds(\mathbf{x}) \\
& = \frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_D f(\mathbf{x}) K(\mathbf{x},\boldsymbol{\xi})\, d\mathbf{x}\,.
\end{aligned} \tag{9}
$$

Its approximate form for $\boldsymbol{\xi} \in \partial D_h$, in terms of u_h and q_h, is

$$
\begin{aligned}
\frac{1}{2} q_h(\boldsymbol{\xi}) \quad & + \frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_{\partial D_h} u_h(\mathbf{x})\, T(\mathbf{x},\boldsymbol{\xi})\, ds(\mathbf{x}) \\
& - \frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_{\partial D_h} q_h(\mathbf{x}) K(\mathbf{x},\boldsymbol{\xi})\, ds(\mathbf{x}) \\
& = \frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_D f(\mathbf{x}) K(\mathbf{x},\boldsymbol{\xi})\, d\mathbf{x}\,.
\end{aligned} \tag{10}
$$

The approximate equations: Galerkin method (cont'd)

Multiplying equation (10) by ψ_j and integrating on ∂D_h we find for each $j = 1, \ldots, M$:

$$\begin{aligned}
&\tfrac{1}{2} \int_{\partial D_h} q_h(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\
&\qquad + \int_{\partial D_h} \left(\frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_{\partial D_h} u_h(\mathbf{x}) T(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) \right) \psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\
&\qquad - \int_{\partial D_h} \left(\frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_{\partial D_h} q_h(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) \right) \psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\
&\quad = \int_{\partial D_h} \left(\frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_D f(\mathbf{x}) K(\mathbf{x}, \boldsymbol{\xi})\, d\mathbf{x} \right) \psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \,.
\end{aligned} \tag{11}$$

We can write

$$\begin{aligned}
&\int_{\partial D_h} \left(\frac{\partial}{\partial n_{\boldsymbol{\xi}}} \int_{\partial D_h} u_h(\mathbf{x}) T(\mathbf{x}, \boldsymbol{\xi})\, ds(\mathbf{x}) \right) \psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi}) \\
&\qquad = \sum_{k=1}^{M} \alpha_k \int_{\partial D_h} \int_{\partial D_h} \frac{\partial}{\partial n_{\boldsymbol{\xi}}} T(\mathbf{x}, \boldsymbol{\xi}) \psi_k(\mathbf{x}) \psi_j(\boldsymbol{\xi})\, ds(\mathbf{x}) ds(\boldsymbol{\xi}) \,.
\end{aligned}$$

The approximate equations: Galerkin method (cont'd)

We have thus obtained the linear system

$$\widetilde{A}^{\mathrm{Neu}}\boldsymbol{\alpha} = \widetilde{\mathbf{b}}^{\mathrm{Neu}}\,,$$

where

$$\widetilde{A}_{jk}^{\mathrm{Neu}} = \int_{\partial D_h}\int_{\partial D_h} \frac{\partial}{\partial n_{\boldsymbol{\xi}}} T(\mathbf{x},\boldsymbol{\xi})\psi_k(\mathbf{x})\psi_j(\boldsymbol{\xi})\, ds(\mathbf{x})ds(\boldsymbol{\xi})\,,$$

$$\begin{aligned}\widetilde{b}_j^{\mathrm{Neu}} = -\tfrac{1}{2}\int_{\partial D_h} q_h(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi})\\ &+ \int_{\partial D_h}\left(\frac{\partial}{\partial n_{\boldsymbol{\xi}}}\int_{\partial D_h} q_h(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\, ds(\mathbf{x})\right)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi})\\ &+ \int_{\partial D_h}\left(\frac{\partial}{\partial n_{\boldsymbol{\xi}}}\int_{D} f(\mathbf{x})K(\mathbf{x},\boldsymbol{\xi})\, d\mathbf{x}\right)\psi_j(\boldsymbol{\xi})ds(\boldsymbol{\xi})\,.\end{aligned}$$

Since $\frac{\partial}{\partial n_{\boldsymbol{\xi}}} T(\mathbf{x},\boldsymbol{\xi}) = \frac{\partial}{\partial n_{\boldsymbol{\xi}}}\frac{\partial}{\partial n_{\mathbf{x}}} K(\mathbf{x},\boldsymbol{\xi})$ is symmetric in $\mathbf{x}$ and $\boldsymbol{\xi}$, we see that the matrix $\widetilde{A}$ is symmetric.

The approximate equations: Galerkin method (cont'd)

Let us remark that the construction of the matrices $\widehat{A}^{\mathrm{Dir}}$, $\widehat{A}^{\mathrm{Neu}}$ and $\widetilde{A}^{\mathrm{Neu}}$ only requires that we have a basis ψ_k for the space where we look for the approximate solution: namely, we can repeat the same construction for piecewise-linear functions, or piecewise-polynomial functions, once we have a basis for that space of functions.

In the particular case of piecewise-constant functions, we can compute the entries of these matrices in a more explicit way: for instance, since $\psi_k = 1$ in S_k and $\psi_k = 0$ outside S_k, we have

$$\begin{aligned} \widehat{A}^{\mathrm{Dir}}_{jk} &= \int_{\partial D_h} \int_{\partial D_h} K(\mathbf{x}, \boldsymbol{\xi}) \psi_k(\mathbf{x}) \psi_j(\boldsymbol{\xi}) \, ds(\mathbf{x}) ds(\boldsymbol{\xi}) \\ &= \int_{S_k} \int_{S_j} K(\mathbf{x}, \boldsymbol{\xi}) \, ds(\mathbf{x}) ds(\boldsymbol{\xi}) \, . \end{aligned}$$